Title: Classical Physics Lecture - 091823

Speakers: Aldo Riello

Collection: Classical Physics 2023/24

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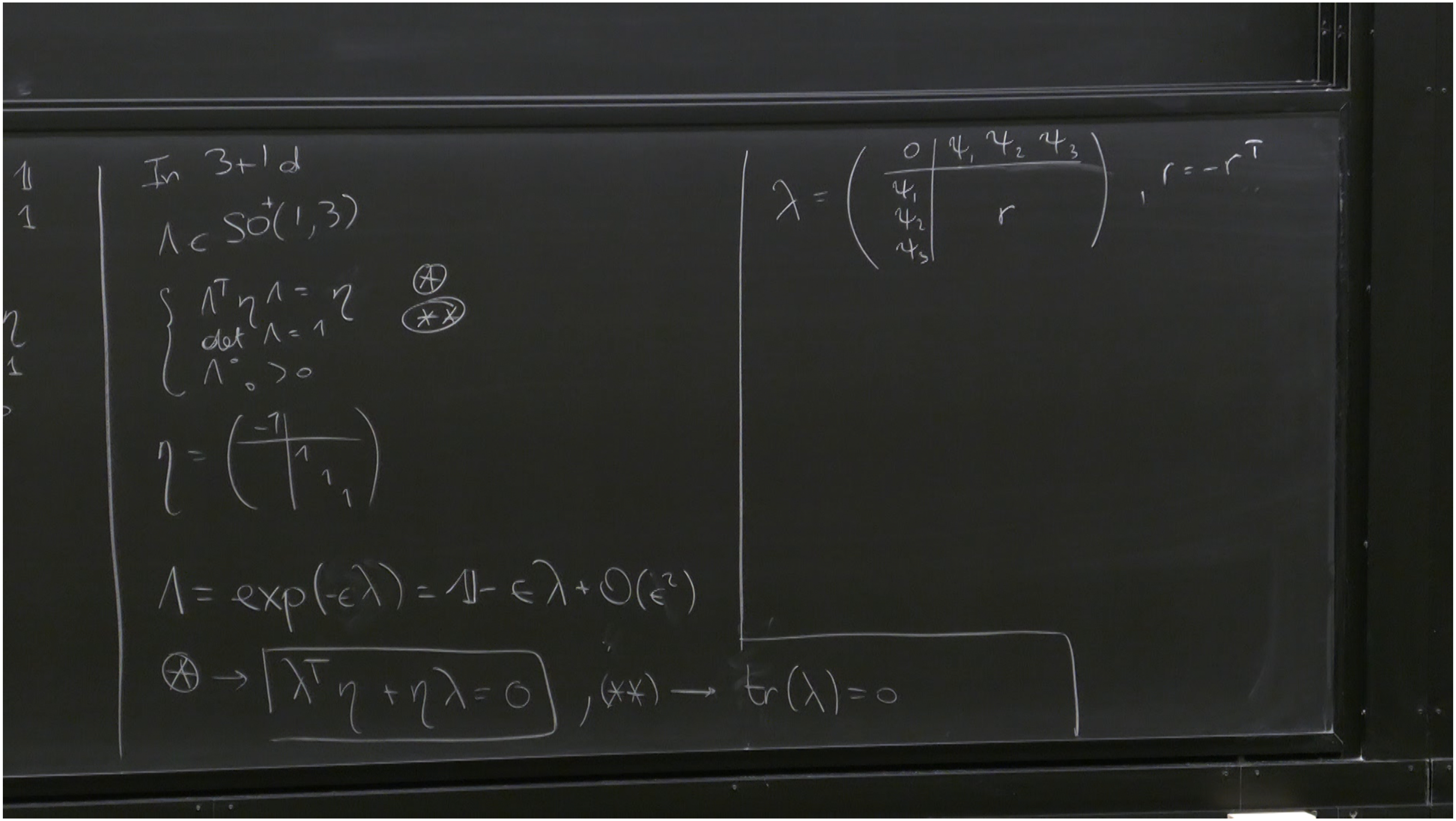


$\mathbb{1}$

1

η

1

In 3+1 d

$\Lambda \in SO^+(1,3)$

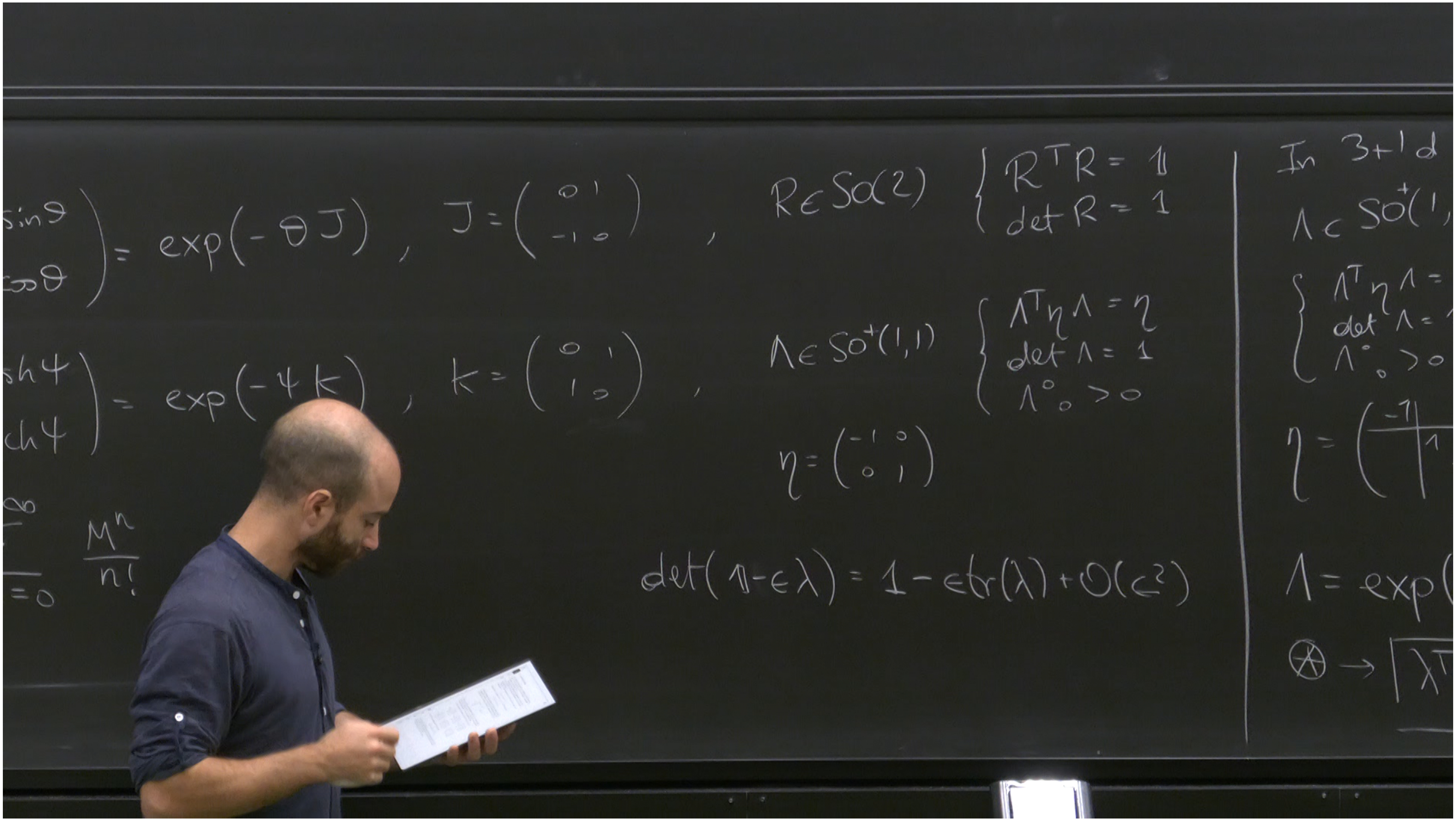
$$\begin{cases} \Lambda^T \eta \Lambda = \eta & \circledast \\ \det \Lambda = 1 & (\ast\ast) \\ \Lambda^0{}_0 > 0 \end{cases}$$

$$\eta = \left(\begin{array}{c|ccc} -1 & & & \\ \hline & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right)$$

$$\Lambda = \exp(-\epsilon\lambda) = \mathbb{1} - \epsilon\lambda + O(\epsilon^2)$$

$$\circledast \to \boxed{\lambda^T \eta + \eta\lambda = 0}, \quad (\ast\ast) \longrightarrow \mathrm{tr}(\lambda) = 0$$

$$\lambda = \left(\begin{array}{c|ccc} 0 & \psi_1 & \psi_2 & \psi_3 \\ \hline \psi_1 & & & \\ \psi_2 & & r & \\ \psi_3 & & & \end{array}\right), \quad r = -r^T$$



$$\begin{pmatrix} \cdots & \sin\theta \\ \cdots & \cos\theta \end{pmatrix} = \exp(-\theta J), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad R \in SO(2) \begin{cases} R^T R = \mathbb{1} \\ \det R = 1 \end{cases}$$

$$\begin{pmatrix} \cdots & \operatorname{sh}\psi \\ \cdots & \operatorname{ch}\psi \end{pmatrix} = \exp(-\psi K), \quad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Lambda \in SO^+(1,1) \begin{cases} \Lambda^T \eta \Lambda = \eta \\ \det \Lambda = 1 \\ \Lambda^0{}_0 > 0 \end{cases}$$

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sum_{n=0}^{\infty} \frac{M^n}{n!}$$

$$\det(\mathbb{1} - \epsilon\lambda) = 1 - \epsilon \operatorname{tr}(\lambda) + O(\epsilon^2)$$

In 3+1 d

$\Lambda \in SO^+(1,$

$$\begin{cases} \Lambda^T \eta \Lambda = \\ \det \Lambda = \\ \Lambda^0{}_0 > 0 \end{cases}$$

$$\eta = \left(\begin{array}{c|c} -1 & \\ \hline & 1 \end{array} \right.$$

$\Lambda = \exp($

$⊛ \rightarrow \boxed{\lambda^T}$

In 3+1 d

$\Lambda \in SO^{+}(1,3)$

$$\begin{cases} \Lambda^T \eta \Lambda = \eta & \circledast \\ \det \Lambda = 1 & (**) \\ \Lambda^0{}_0 > 0 & \end{cases}$$

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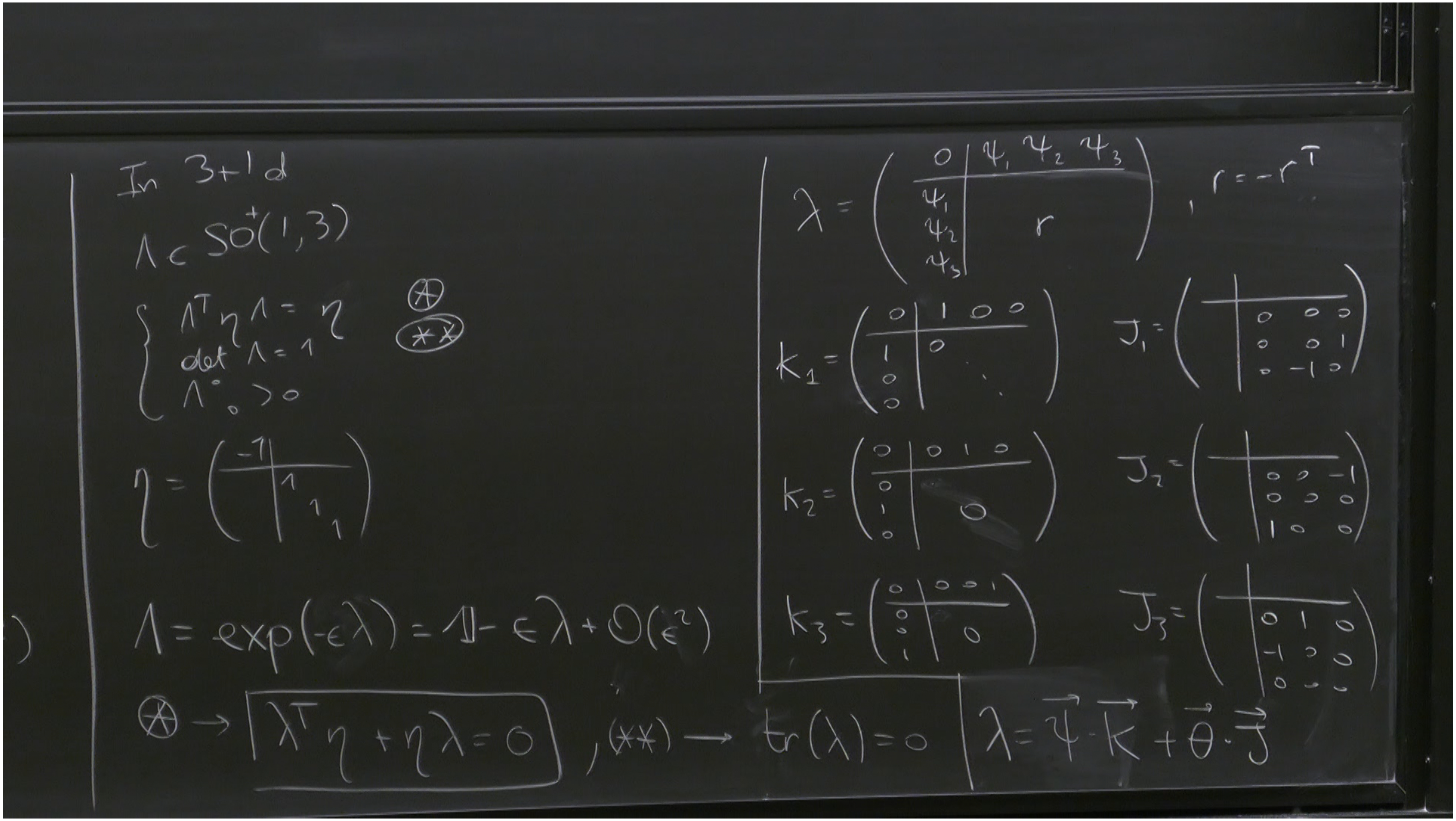
$$K_1 = \left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & & \\ 0 & & \ddots & \\ 0 & & & \end{array}\right) \qquad J_1 = \left(\begin{array}{c|ccc} & & & \\ \hline & 0 & 0 & 0 \\ & 0 & 0 & 1 \\ & 0 & -1 & 0 \end{array}\right)$$

$$K_2 = \left(\begin{array}{c|ccc} 0 & 0 & 1 & 0 \\ \hline 0 & & & \\ 1 & & 0 & \\ 0 & & & \end{array}\right) \qquad J_2 = \left(\begin{array}{c|ccc} & & & \\ \hline & 0 & 0 & -1 \\ & 0 & 0 & 0 \\ & 1 & 0 & 0 \end{array}\right)$$

$$K_3 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & & & \\ 0 & & 0 & \\ 1 & & & \end{array}\right) \qquad J_3 = \left(\begin{array}{c|ccc} & & & \\ \hline & 0 & 1 & 0 \\ & -1 & 0 & 0 \\ & 0 & 0 & 0 \end{array}\right)$$



In 3+1 d

$\Lambda \in SO^{+}(1,3)$

$$\begin{cases} \Lambda^T \eta \Lambda = \eta & \textcircled{*} \\ \det \Lambda = 1 & \textcircled{**} \\ \Lambda^0{}_0 > 0 \end{cases}$$

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$$\Lambda = \exp(-\epsilon\lambda) = \mathbb{1} - \epsilon\lambda + \mathcal{O}(\epsilon^2)$$

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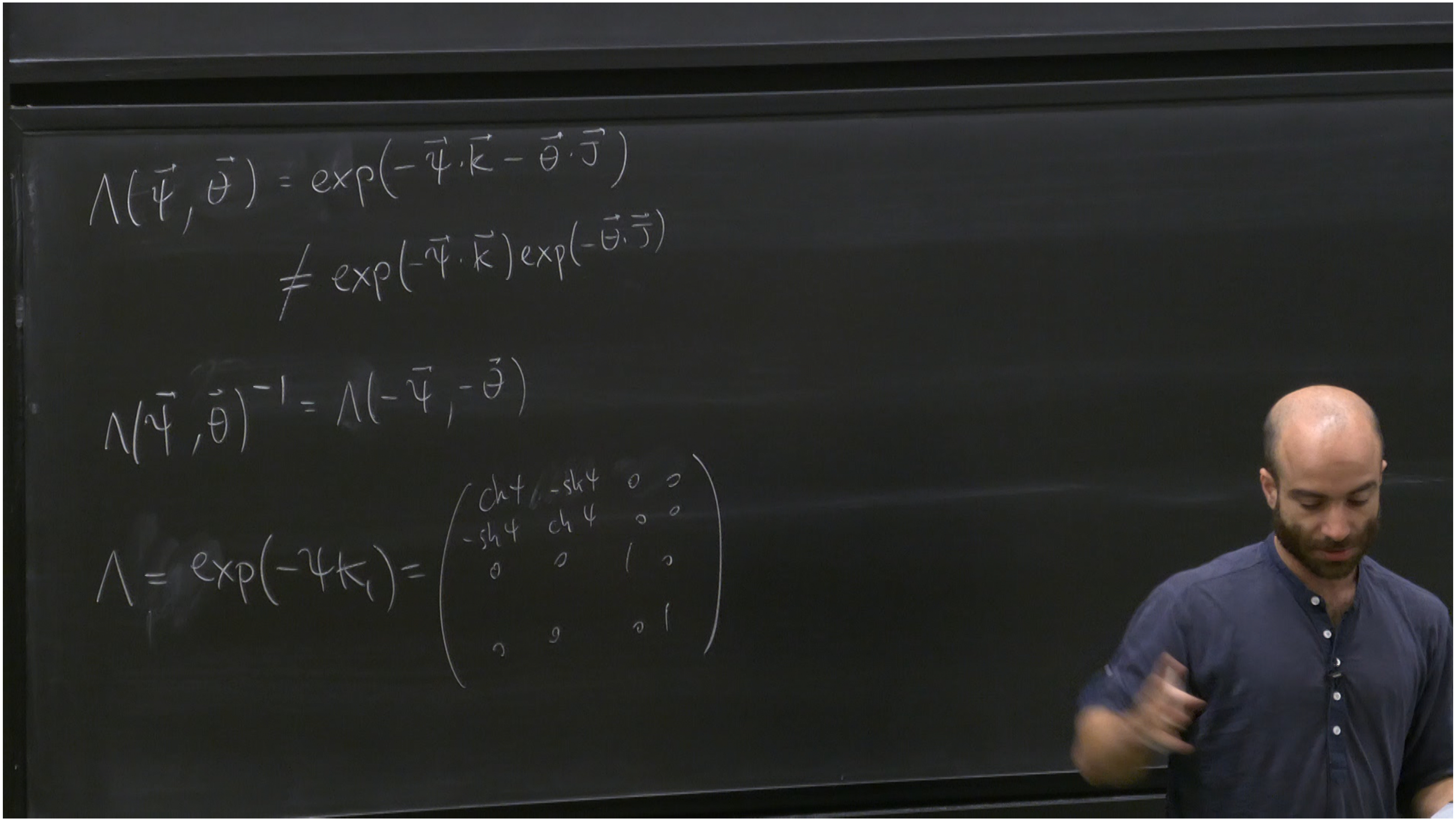
$$k_3 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & & & \\ 0 & & 0 & \\ 1 & & & \end{array}\right) \qquad J_3 = \left(\begin{array}{c|ccc} & & & \\ \hline & 0 & 1 & 0 \\ & -1 & 0 & 0 \\ & 0 & 0 & 0 \end{array}\right)$$

$$\lambda = \vec{\psi} \cdot \vec{K} + \vec{\theta} \cdot \vec{J}$$

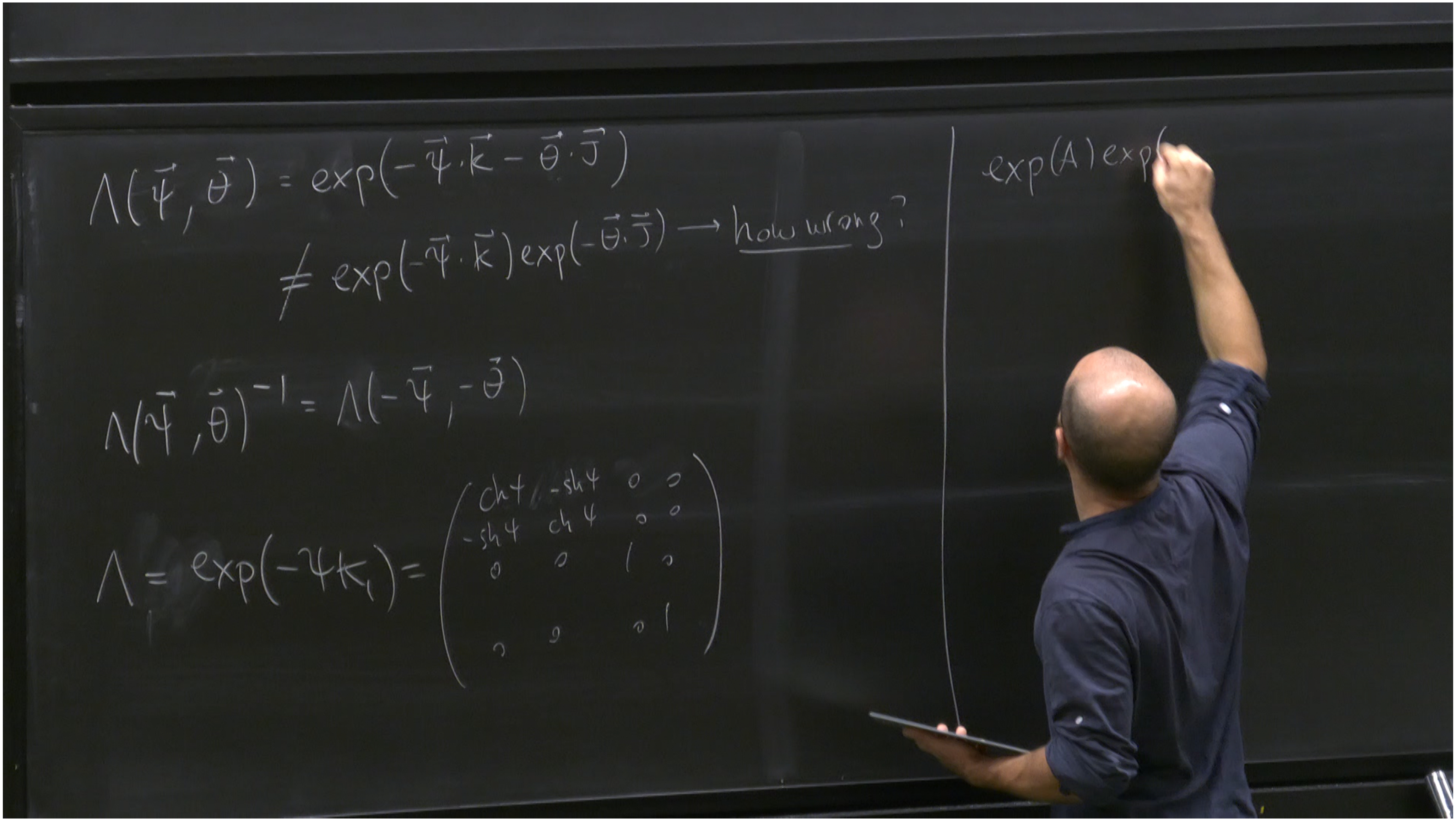


$$\Lambda(\vec{\psi},\vec{\theta}) = \exp(-\vec{\psi}\cdot\vec{k} - \vec{\theta}\cdot\vec{J})$$

$$\neq \exp(-\vec{\psi}\cdot\vec{k})\exp(-\vec{\theta}\cdot\vec{J})$$

$$\Lambda(\vec{\psi},\vec{\theta})^{-1} = \Lambda(-\vec{\psi},-\vec{\theta})$$

$$\Lambda = \exp(-\psi k_1) = \begin{pmatrix} \mathrm{ch}\,\psi & -\mathrm{sh}\,\psi & 0 & 0 \\ -\mathrm{sh}\,\psi & \mathrm{ch}\,\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



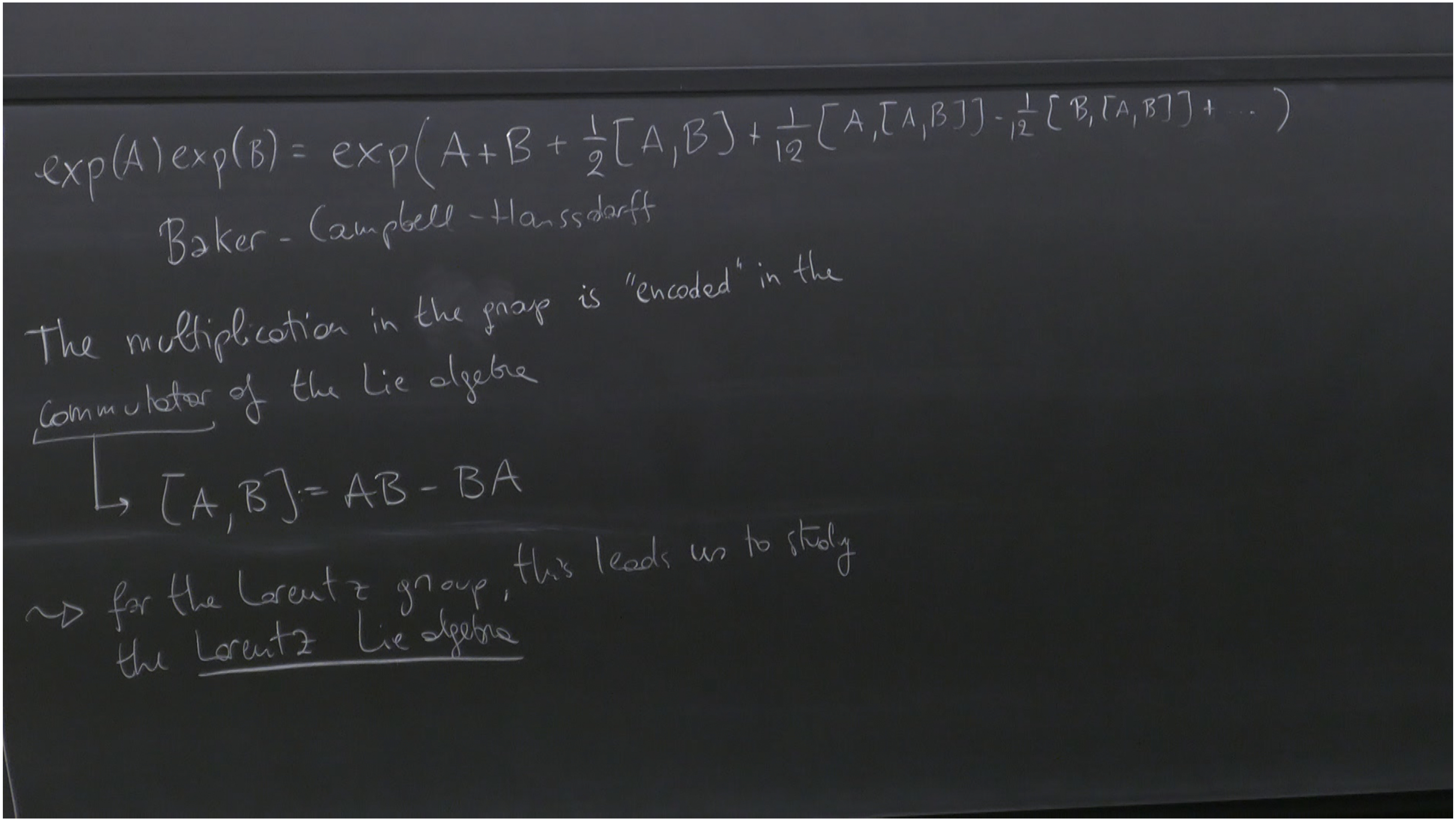
$$\Lambda(\vec{\psi},\vec{\theta}) = \exp(-\vec{\psi}\cdot\vec{K} - \vec{\theta}\cdot\vec{J})$$

$$\neq \exp(-\vec{\psi}\cdot\vec{K})\exp(-\vec{\theta}\cdot\vec{J}) \longrightarrow \underline{\text{how wrong}}?$$

$$\Lambda(\vec{\psi},\vec{\theta})^{-1} = \Lambda(-\vec{\psi},-\vec{\theta})$$

$$\Lambda = \exp(-\psi K_1) = \begin{pmatrix} \mathrm{ch}\,\psi & -\mathrm{sh}\,\psi & 0 & 0 \\ -\mathrm{sh}\,\psi & \mathrm{ch}\,\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\exp(A)\exp($



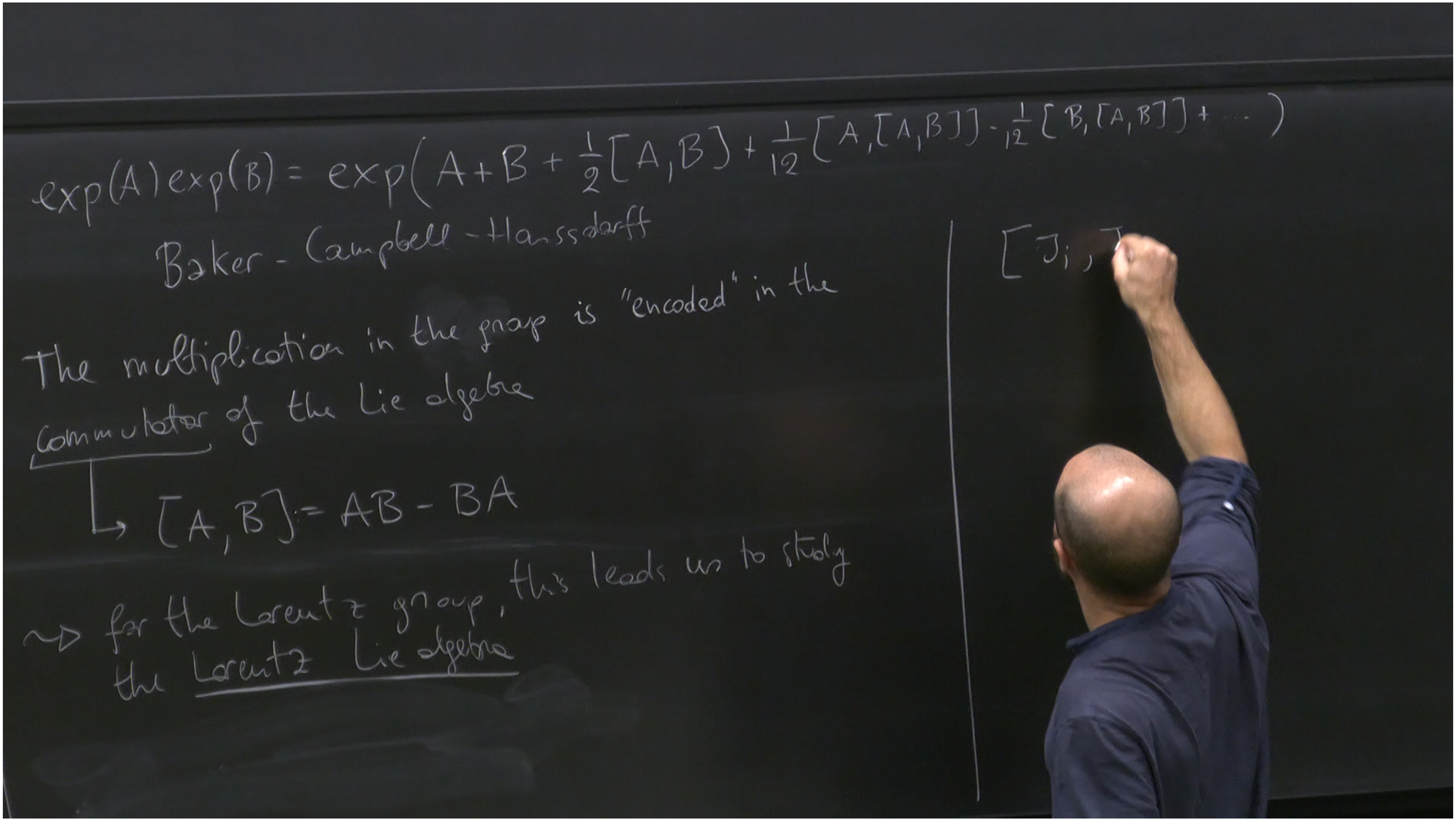
$$\exp(A)\exp(B) = \exp\left(A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[A,B]]+\dots\right)$$

Baker - Campbell - Haussdorff

The multiplication in the group is "encoded" in the commutator of the Lie algebra

↳ $[A,B] = AB - BA$

⇝ for the Lorentz group, this leads us to study the Lorentz Lie algebra



$$\exp(A)\exp(B) = \exp\left(A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[A,B]]+\dots\right)$$

Baker - Campbell - Haussdorff

The multiplication in the group is "encoded" in the <u>commutator</u> of the Lie algebra

↳ $[A,B] = AB - BA$

⇝ for the Lorentz group, this leads us to study the <u>Lorentz Lie algebra</u>

$[J_i,$

$$\frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[A,B]] + \dots)$$

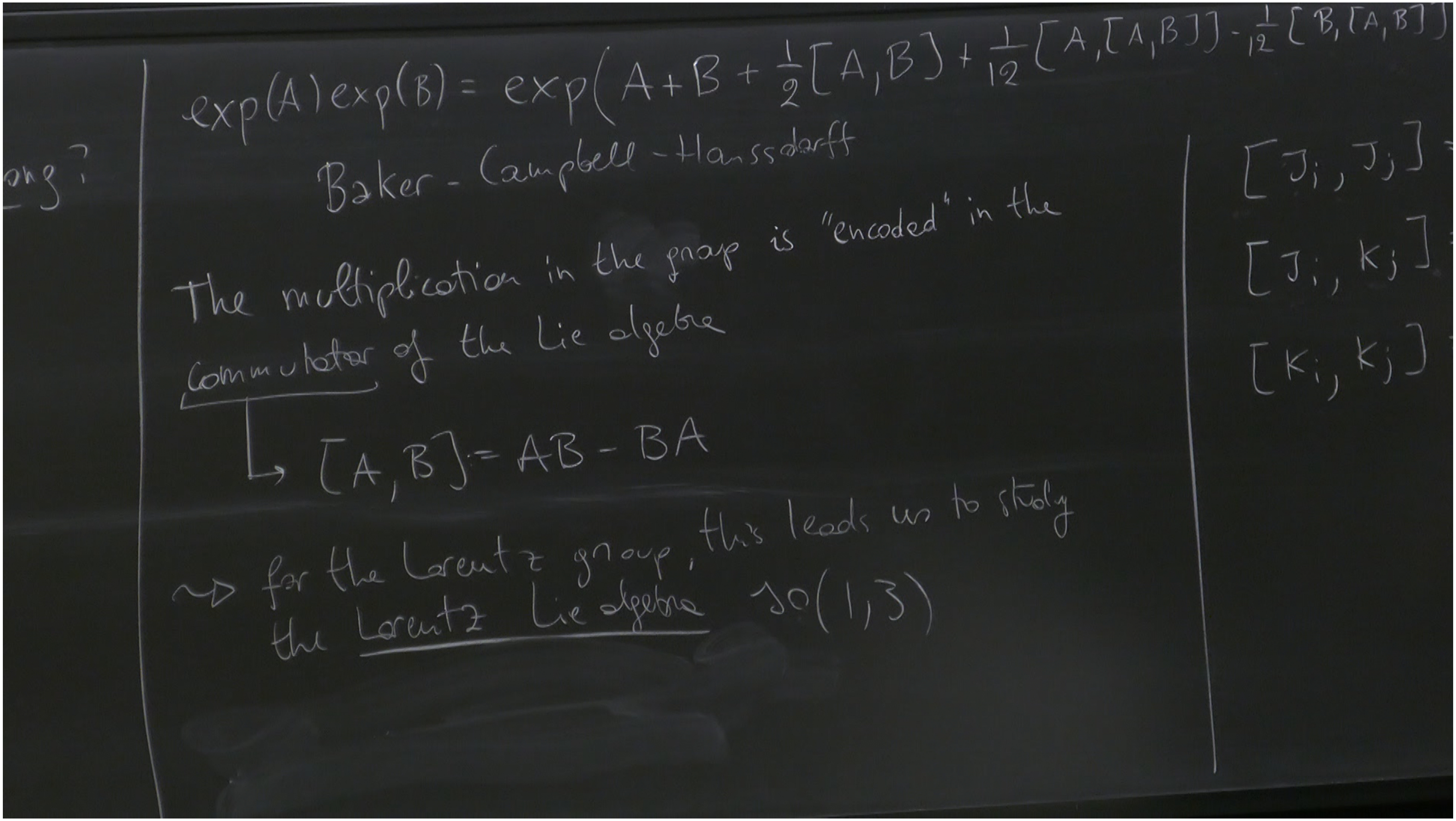
the

study

$$[J_i, J_j] = -\sum_k \epsilon_{ijk} J_k \longleftarrow$$ rotations form a subalgebra of $so(1,3)$

$$[J_i, K_j] = -\sum_k \epsilon_{ijk} K_k$$

$$[K_i, K_j] = +\sum_k \epsilon_{ijk} J_k$$



ong?

$$\exp(A)\exp(B) = \exp\Big(A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[A,B]]$$

Baker - Campbell - Haussdorff

The multiplication in the group is "encoded" in the commutator of the Lie algebra

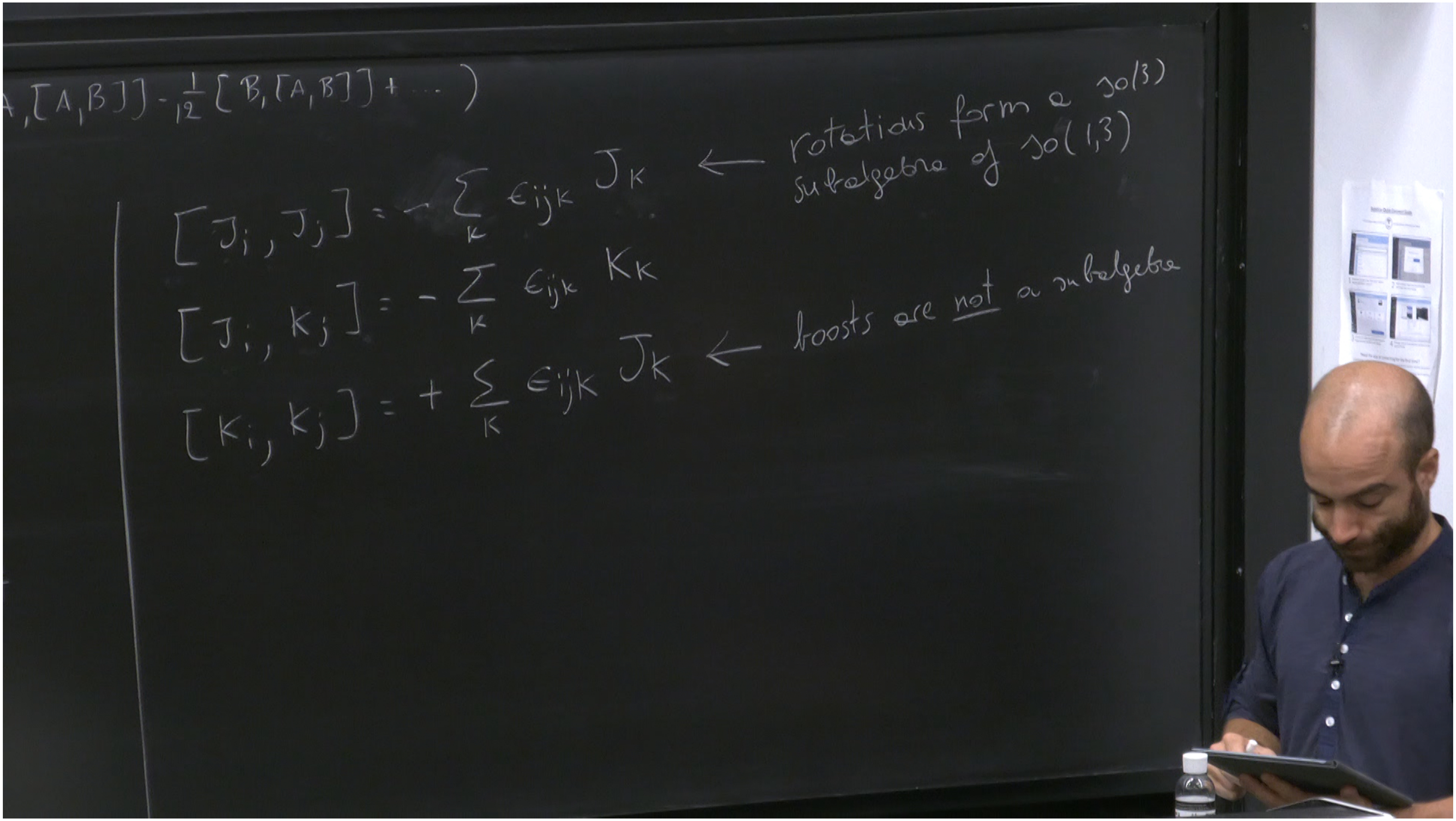
$$\hookrightarrow [A,B] := AB - BA$$

⤳ for the Lorentz group, this leads us to study the Lorentz Lie algebra $\mathfrak{so}(1,3)$

$$[J_i, J_j] =$$

$$[J_i, K_j] =$$

$$[K_i, K_j] =$$



$$A,[A,B]] - \frac{1}{12}[B,[A,B]] + \ldots)$$

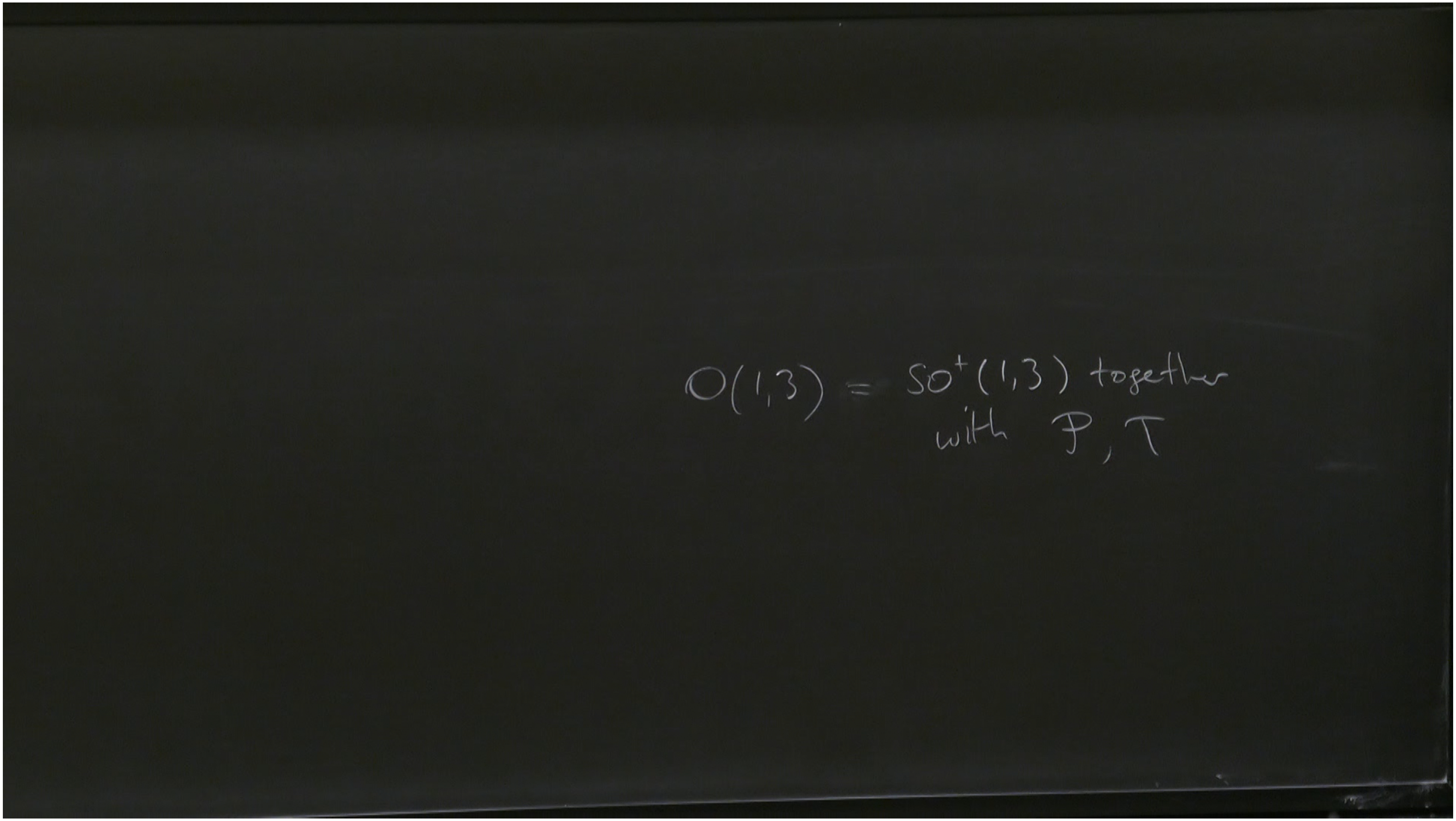
$$[J_i, J_j] = -\sum_k \epsilon_{ijk} J_k \quad \leftarrow$$

rotations form a $so(3)$ subalgebra of $so(1,3)$

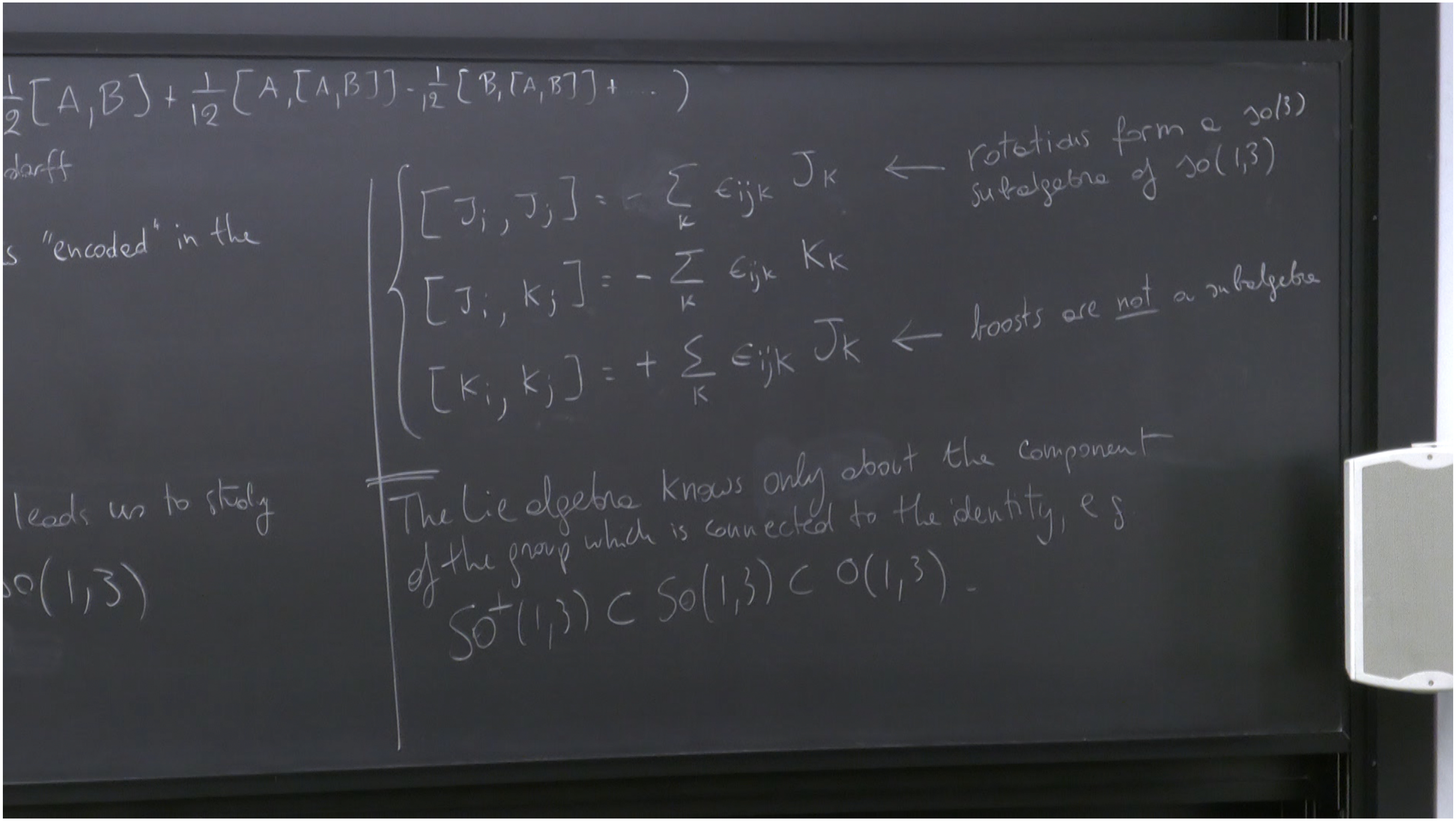
$$[J_i, K_j] = -\sum_k \epsilon_{ijk} K_k$$

$$[K_i, K_j] = +\sum_k \epsilon_{ijk} J_k \quad \leftarrow$$

boosts are <u>not</u> a subalgebra



$O(1,3) = SO^+(1,3)$ together with $\mathcal{P}, T$



$$\tfrac{1}{2}[A,B] + \tfrac{1}{12}[A,[A,B]] - \tfrac{1}{12}[B,[A,B]] + \dots)$$

dorff

s "encoded" in the

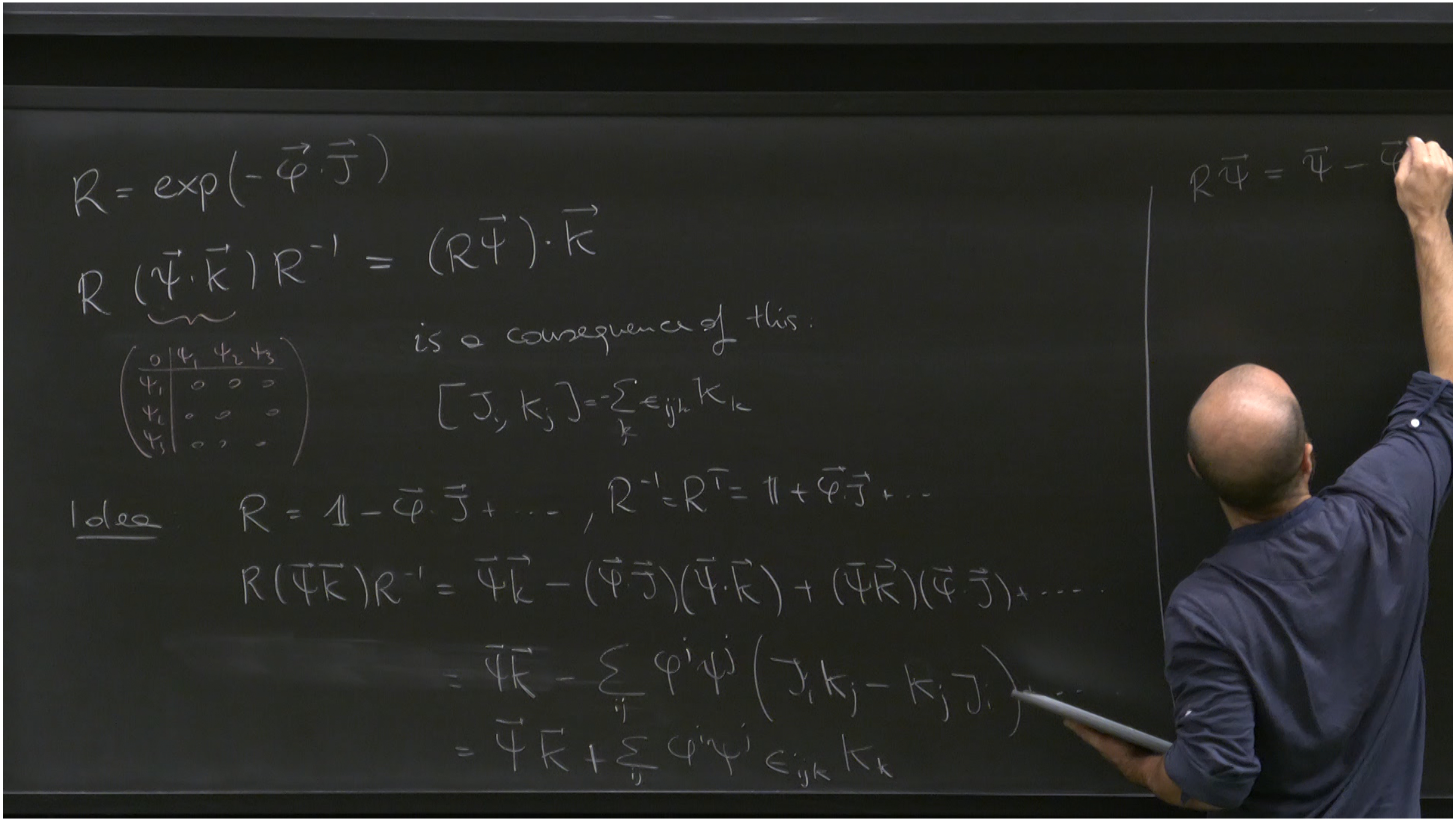
leads us to study

$o(1,3)$

$$\begin{cases} [J_i, J_j] = - \sum_k \epsilon_{ijk} J_k & \leftarrow \text{rotations form a } so(3) \text{ subalgebra of } so(1,3) \\ [J_i, K_j] = - \sum_k \epsilon_{ijk} K_k & \\ [K_i, K_j] = + \sum_k \epsilon_{ijk} J_k & \leftarrow \text{boosts are } \underline{\text{not}} \text{ a subalgebra} \end{cases}$$

The Lie algebra knows only about the component of the group which is connected to the identity, e.g.

$$SO^+(1,3) \subset SO(1,3) \subset O(1,3).$$



$$R = \exp(-\vec{\varphi}\cdot\vec{J})$$

$$R\,(\underbrace{\vec{\psi}\cdot\vec{k}})\,R^{-1} = (R\vec{\psi})\cdot\vec{k}$$

$$\left(\begin{array}{c|ccc} 0 & \psi_1 & \psi_2 & \psi_3 \\ \hline \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \end{array}\right)$$

is a consequence of this:

$$[J_i, k_j] = -\sum_k \epsilon_{ijk} k_k$$

<u>Idea</u>

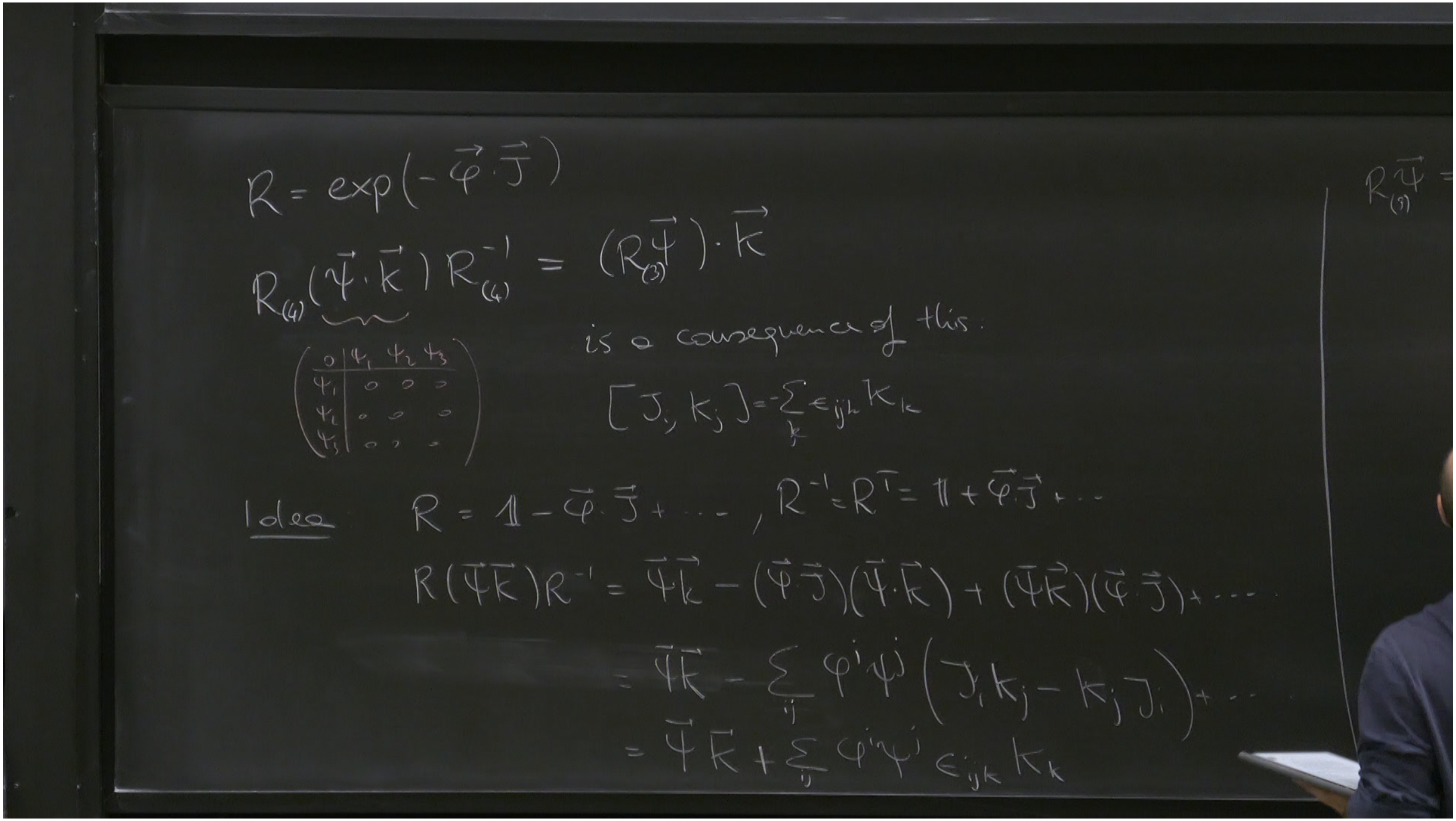
$$R = 1\!\!1 - \vec{\varphi}\cdot\vec{J} + \dots\,, \quad R^{-1} = R^T = 1\!\!1 + \vec{\varphi}\cdot\vec{J} + \dots$$

$$R(\vec{\psi}\vec{k})R^{-1} = \vec{\psi}\vec{k} - (\vec{\varphi}\cdot\vec{J})(\vec{\psi}\cdot\vec{k}) + (\vec{\psi}\vec{k})(\vec{\varphi}\cdot\vec{J}) + \dots$$

$$= \vec{\psi}\vec{k} - \sum_{ij} \varphi^i \psi^j \left(J_i k_j - k_j J_i\right) + \dots$$

$$= \vec{\psi}\vec{k} + \sum_{ij} \varphi^i \psi^j \epsilon_{ijk} k_k$$

$$R\vec{\psi} = \vec{\psi} - \vec{\varphi}$$



$$R = \exp(-\vec{\varphi}\cdot\vec{J})$$

$$\underbrace{R_{(4)}(\vec{\psi}\cdot\vec{K})R_{(4)}^{-1}} = (R_{(3)}\vec{\psi})\cdot\vec{K}$$

$$\left(\begin{array}{c|ccc} 0 & \psi_1 & \psi_2 & \psi_3 \\ \hline \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \end{array}\right)$$

is a consequence of this:

$$[J_i, K_j] = -\sum_k \epsilon_{ijk} K_k$$

$$R_{(3)}\vec{\psi} =$$

<u>Idea</u>

$$R = \mathbb{1} - \vec{\varphi}\cdot\vec{J} + \ldots, \quad R^{-1} = R^T = \mathbb{1} + \vec{\varphi}\cdot\vec{J} + \ldots$$

$$R(\vec{\psi}\vec{K})R^{-1} = \vec{\psi}\vec{K} - (\vec{\varphi}\cdot\vec{J})(\vec{\psi}\cdot\vec{K}) + (\vec{\psi}\vec{K})(\vec{\varphi}\cdot\vec{J}) + \ldots$$

$$= \vec{\psi}\vec{K} - \sum_{ij}\varphi^i\psi^j\left(J_iK_j - K_jJ_i\right) + \ldots$$

$$= \vec{\psi}\vec{K} + \sum_{ij}\varphi^i\psi^j\,\epsilon_{ijk}K_k$$

$$R_{(3)}\vec{\psi} = \vec{\psi} - (\vec{\varphi}\cdot\vec{J})_{(3)}\vec{\psi} + \cdots$$

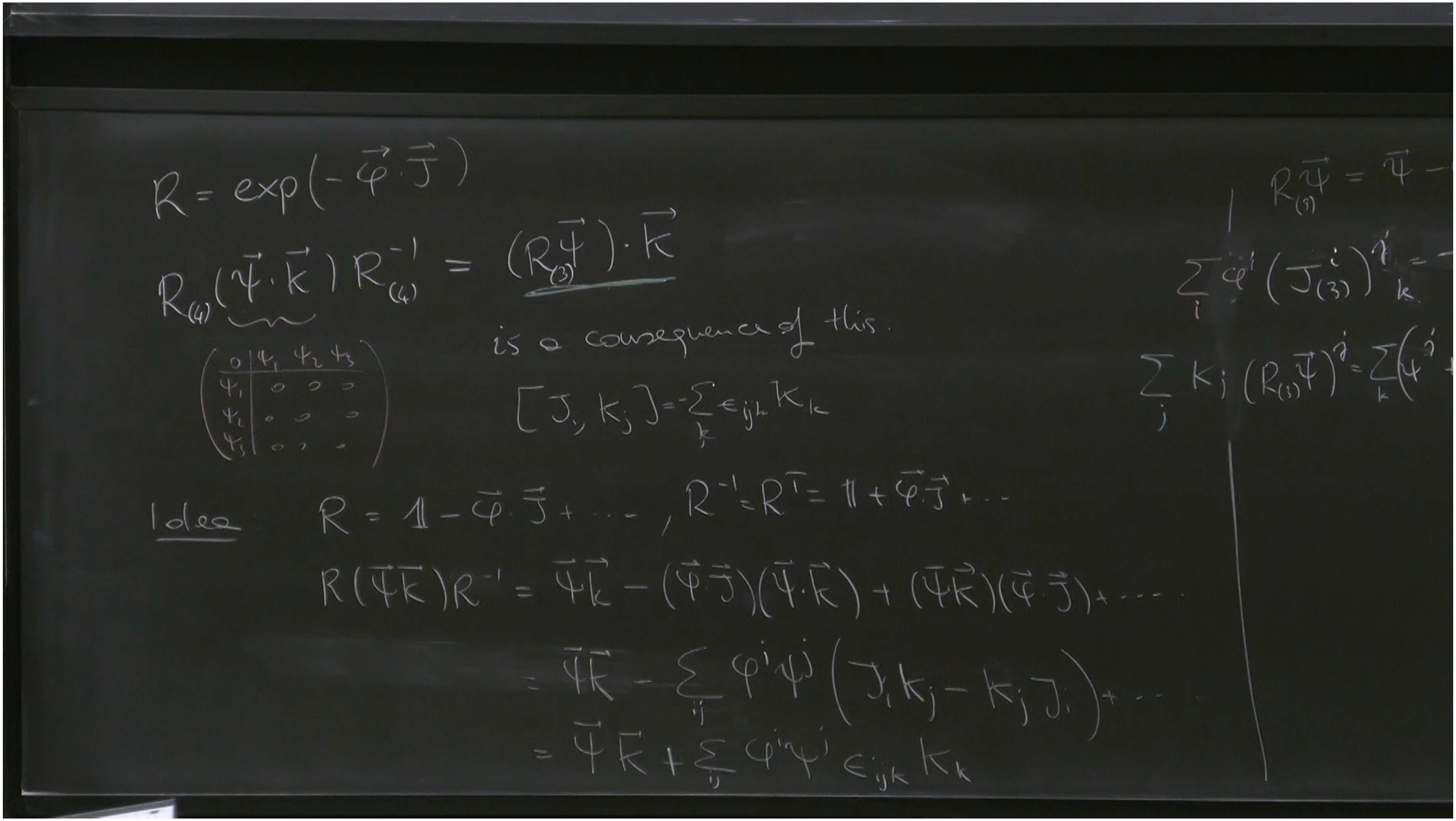
$$\left(J^{i}_{(3)}\right)^{j}{}_{k} = -\epsilon_{ijk}$$

$$\left(R_{(3)}\vec{\psi}\right)^{j} = \psi^{j} + \sum_{i} \varphi^{i}\epsilon_{ijk}\,\psi^{k}$$

$$R_{(3)}\vec{\psi} = \vec{\psi} - \underline{(\vec{\varphi}\cdot\vec{J})_{(3)}}\,\vec{\psi} + \cdots$$

$$\sum_i \varphi^i \left(J^i_{(3)}\right)^j{}_k = -\sum_i \varphi^i \epsilon_{ijk}$$

$$\left(R_{(3)}\vec{\psi}\right)^j = \psi^j + \sum_i \varphi^i \epsilon_{ijk}\, \psi^k$$



$$R = \exp(-\vec{\varphi}\cdot\vec{J})$$

$$R_{(4)}\underbrace{(\vec{\psi}\cdot\vec{k})}\, R_{(4)}^{-1} = \underline{(R_{(3)}\vec{\psi})\cdot\vec{k}}$$

$$\left(\begin{array}{c|ccc} 0 & \psi_1 & \psi_2 & \psi_3 \\ \hline \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \end{array}\right)$$

is a consequence of this.

$$[J_i, k_j] = -\sum_k \epsilon_{ijk} k_k$$

<u>Idea</u>: $R = 1\!\!1 - \vec{\varphi}\cdot\vec{J} + \dots\,,\; R^{-1} = R^T = 1\!\!1 + \vec{\varphi}\cdot\vec{J} + \dots$

$$R(\vec{\psi}\vec{k})R^{-1} = \vec{\psi}\vec{k} - (\vec{\varphi}\cdot\vec{J})(\vec{\psi}\cdot\vec{k}) + (\vec{\psi}\vec{k})(\vec{\varphi}\cdot\vec{J}) + \dots$$

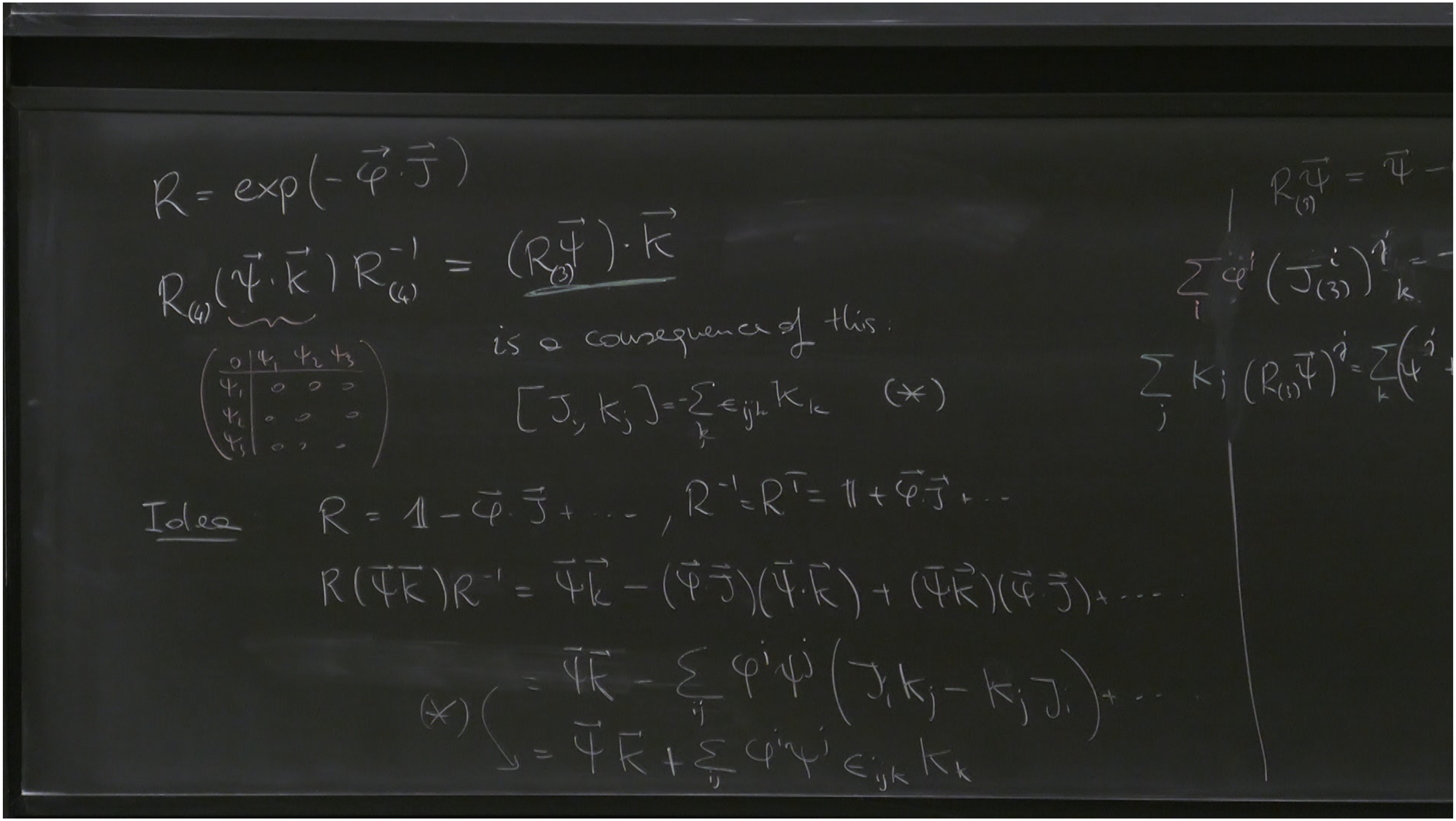
$$= \vec{\psi}\vec{k} - \sum_{ij} \varphi^i \psi^j \left(J_i k_j - k_j J_i\right) + \dots$$

$$= \vec{\psi}\vec{k} + \sum_{ij} \varphi^i \psi^j \epsilon_{ijk} k_k$$

$$R_{(3)}\vec{\psi} = \vec{\psi} -$$

$$\sum_i \varphi^i \left(J_{(3)}^i\right)^j{}_k = -$$

$$\sum_j k_j \left(R_{(3)}\vec{\psi}\right)^j = \sum_k \left(\psi^j +\right.$$



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is a consequence of this:

$$[J_i, k_j] = -\sum_k \epsilon_{ijk} k_k \qquad (*)$$

Idea: $R = 1\!\!1 - \vec{\varphi}\cdot\vec{J} + \dots$, $R^{-1} = R^T = 1\!\!1 + \vec{\varphi}\cdot\vec{J} + \dots$

$$R(\vec{\psi}\vec{k})R^{-1} = \vec{\psi}\vec{k} - (\vec{\varphi}\vec{J})(\vec{\psi}\cdot\vec{k}) + (\vec{\psi}\vec{k})(\vec{\varphi}\vec{J}) + \dots$$

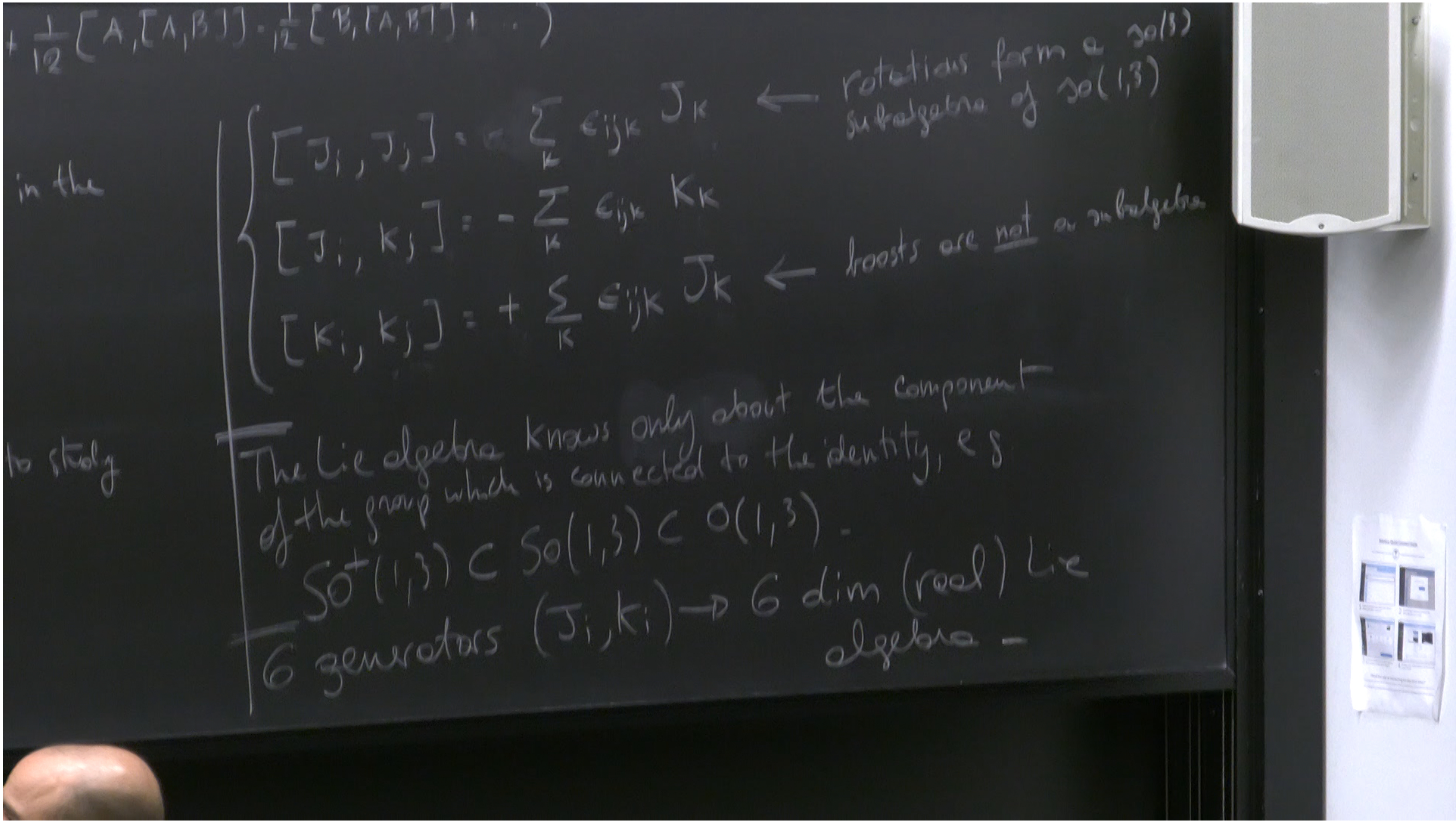
$$= \vec{\psi}\vec{k} - \sum_{ij} \varphi^i \psi^j (J_i k_j - k_j J_i) + \dots$$

$$\xrightarrow{(*)} = \vec{\psi}\vec{k} + \sum_{ij} \varphi^i \psi^j \epsilon_{ijk} k_k$$

$$R_{(3)}\vec{\psi} = \vec{\psi} -$$

$$\sum_i \varphi^i (J_{(3)}^i)^j{}_k = -$$

$$\sum_j k_j (R_{(3)}\vec{\psi})^j = \sum_k (\psi^j$$



$$+\frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[A,B]] + \dots)$$

in the

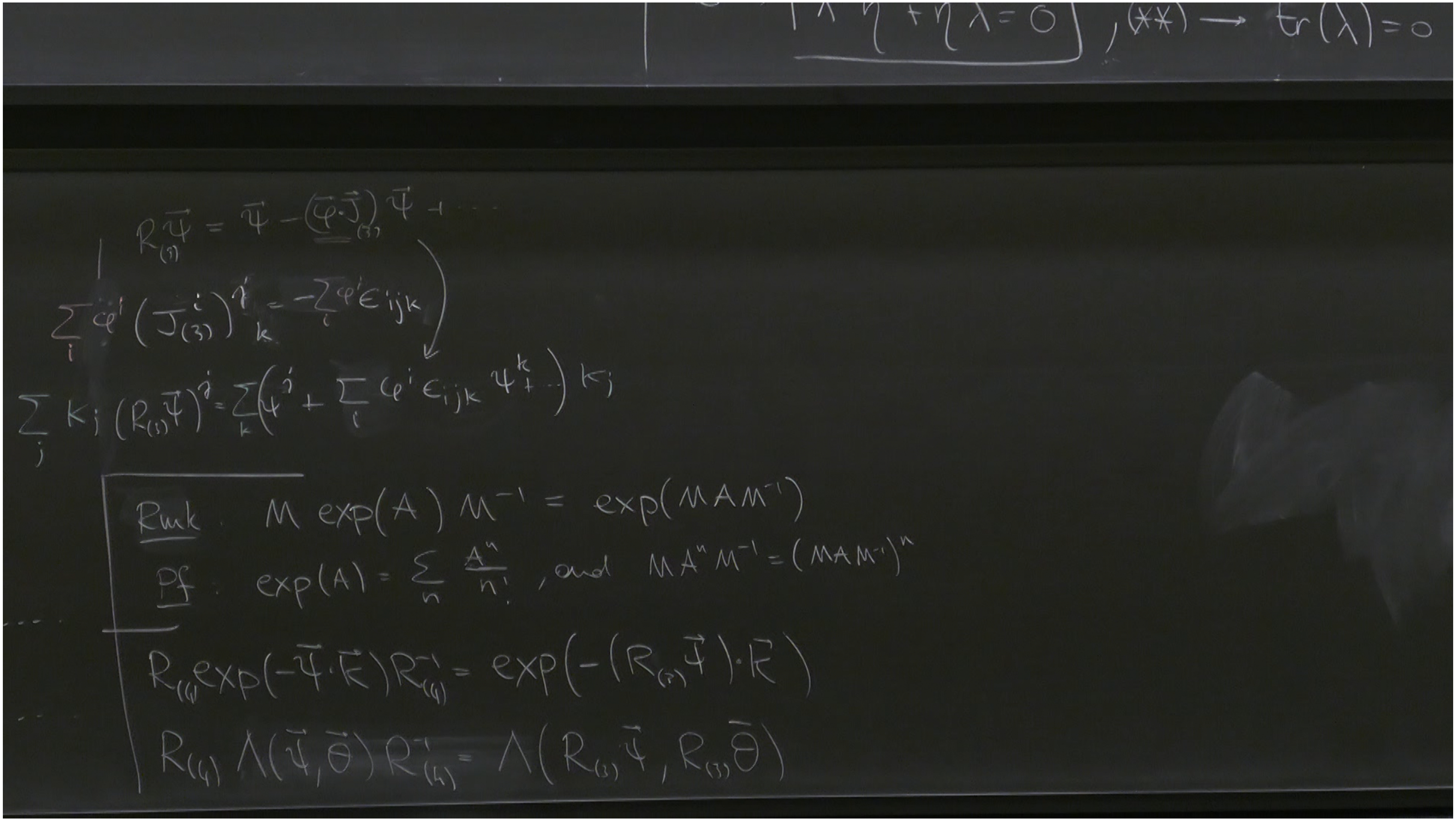
to study

$$\begin{cases} [J_i, J_j] = -\sum_k \epsilon_{ijk} J_k & \leftarrow \text{rotations form a } so(3) \text{ subalgebra of } so(1,3) \\ [J_i, K_j] = -\sum_k \epsilon_{ijk} K_k & \\ [K_i, K_j] = +\sum_k \epsilon_{ijk} J_k & \leftarrow \text{boosts are \underline{not} a subalgebra} \end{cases}$$

The Lie algebra knows only about the component of the group which is connected to the identity, e.g.

$$SO^+(1,3) \subset SO(1,3) \subset O(1,3).$$

6 generators $(J_i, K_i) \rightarrow$ 6 dim (real) Lie algebra.



$$\lambda\,\eta+\eta\lambda=0\ ,\ (**)\longrightarrow \mathrm{tr}(\lambda)=0$$

$$R_{(3)}\vec{\psi}=\vec{\psi}-\underline{(\vec{\varphi}\cdot\vec{J})}_{(3)}\vec{\psi}+\cdots$$

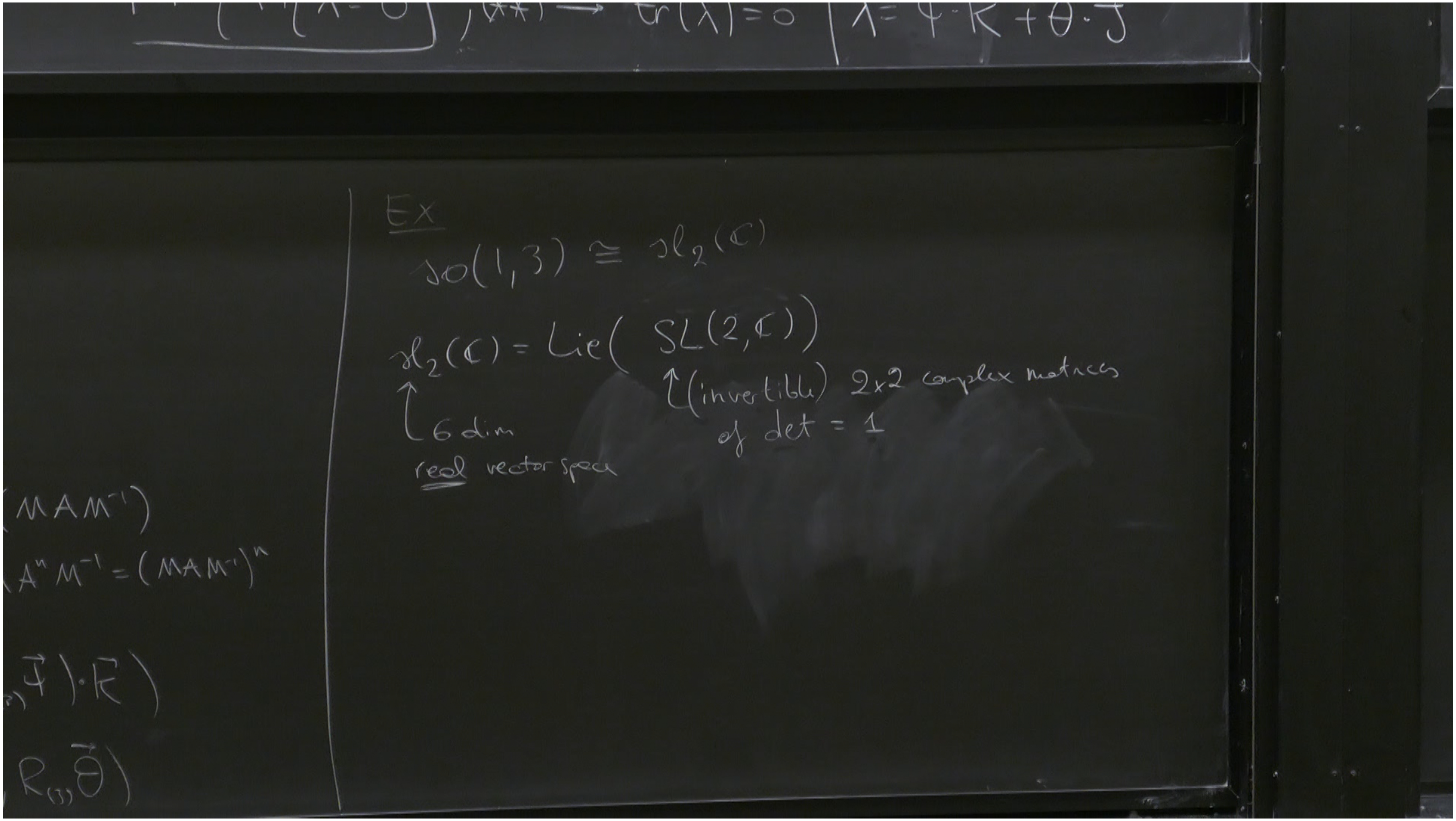
$$\sum_i \varphi^i\left(J^i_{(3)}\right)^j{}_k=-\sum_i\varphi^i\epsilon_{ijk}$$

$$\sum_j k_j\left(R_{(3)}\vec{\psi}\right)^j=\sum_k\Big(\psi^j+\sum_i\varphi^i\epsilon_{ijk}\psi^k+\cdots\Big)k_j$$

$$\boxed{\begin{array}{ll}\underline{\text{Rmk}}. & M\exp(A)M^{-1}=\exp(MAM^{-1})\\ \underline{\text{Pf}}: & \exp(A)=\sum_n\frac{A^n}{n!}\ \text{, and }\ MA^nM^{-1}=(MAM^{-1})^n\end{array}}$$

$$R_{(4)}\exp(-\vec{\psi}\cdot\vec{K})R^{-1}_{(4)}=\exp\left(-(R_{(3)}\vec{\psi})\cdot\vec{K}\right)$$

$$R_{(4)}\Lambda(\vec{\psi},\vec{\theta})R^{-1}_{(4)}=\Lambda\left(R_{(3)}\vec{\psi},R_{(3)}\vec{\theta}\right)$$



$\mathrm{tr}(\lambda)=0 \quad \lambda = \vec{\varphi}\cdot\vec{K} + \vec{\theta}\cdot\vec{J}$

(MAM^{-1})

$A^n M^{-1} = (MAM^{-1})^n$

$\vec{\varphi})\cdot\vec{K})$

$R_{(3)}, \vec{\theta})$

Ex

$\mathfrak{so}(1,3) \cong \mathfrak{sl}_2(\mathbb{C})$

$\mathfrak{sl}_2(\mathbb{C}) = \mathrm{Lie}(SL(2,\mathbb{C}))$

6 dim <u>real</u> vector space

(invertible) 2x2 complex matrices of det = 1

$(\ast\ast) \longrightarrow \operatorname{tr}(\lambda) = 0$ | $\lambda = \varphi \cdot K + \theta \cdot J$

$(M A M^{-1})$

$A^n M^{-1} = (M A M^{-1})^n$

$\vec{\varphi}) \cdot \vec{K})$

$R_{(3)}, \vec{\theta})$

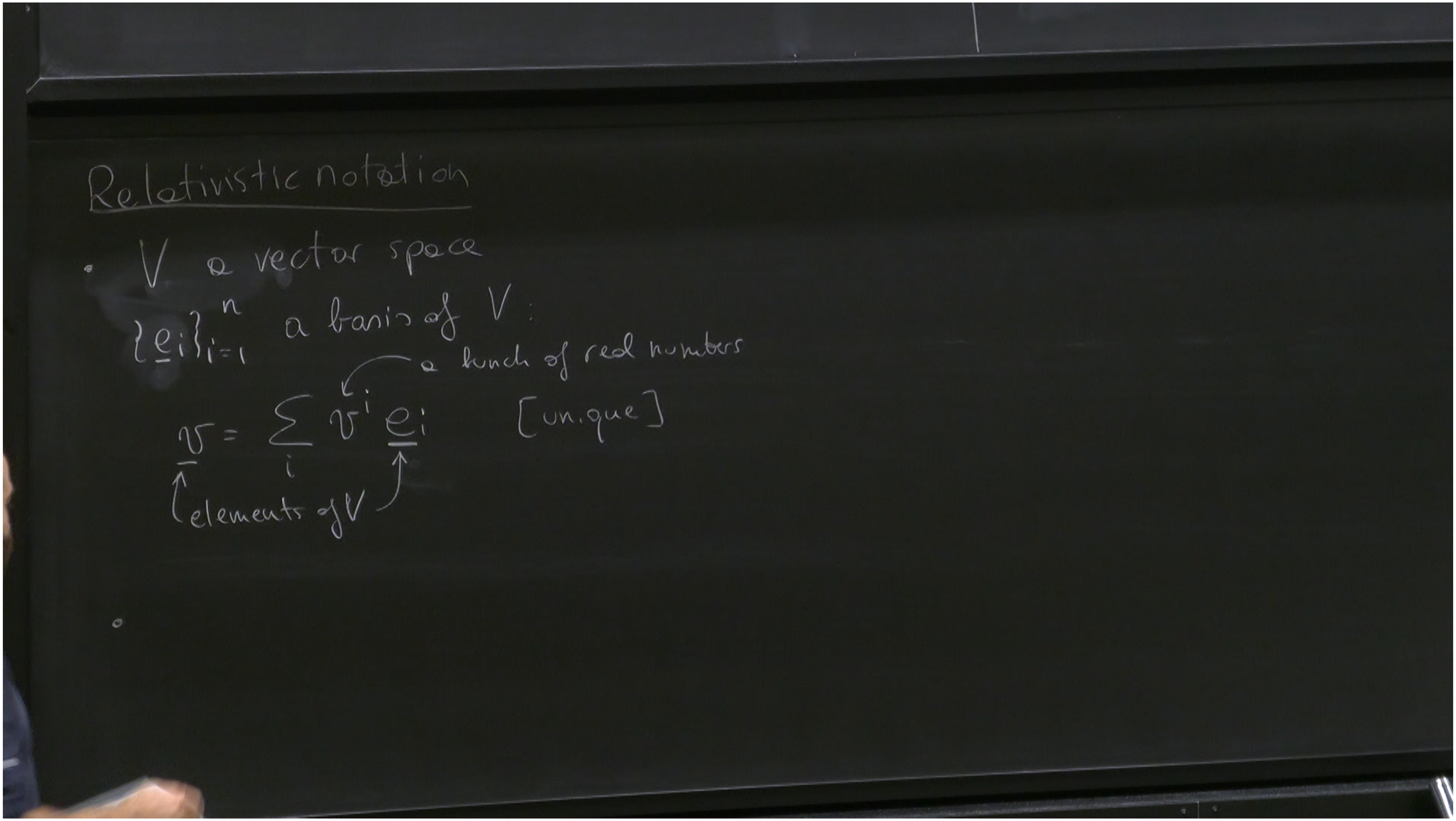
Ex

$\mathfrak{so}(1,3) \cong \mathfrak{sl}_2(\mathbb{C})$

$\mathfrak{sl}_2(\mathbb{C}) = \mathrm{Lie}(\,SL(2,\mathbb{C})\,)$

$\uparrow$ 6 dim real vector space

$\uparrow$ (invertible) 2x2 complex matrices of det = 1



Relativistic notation

- V a vector space

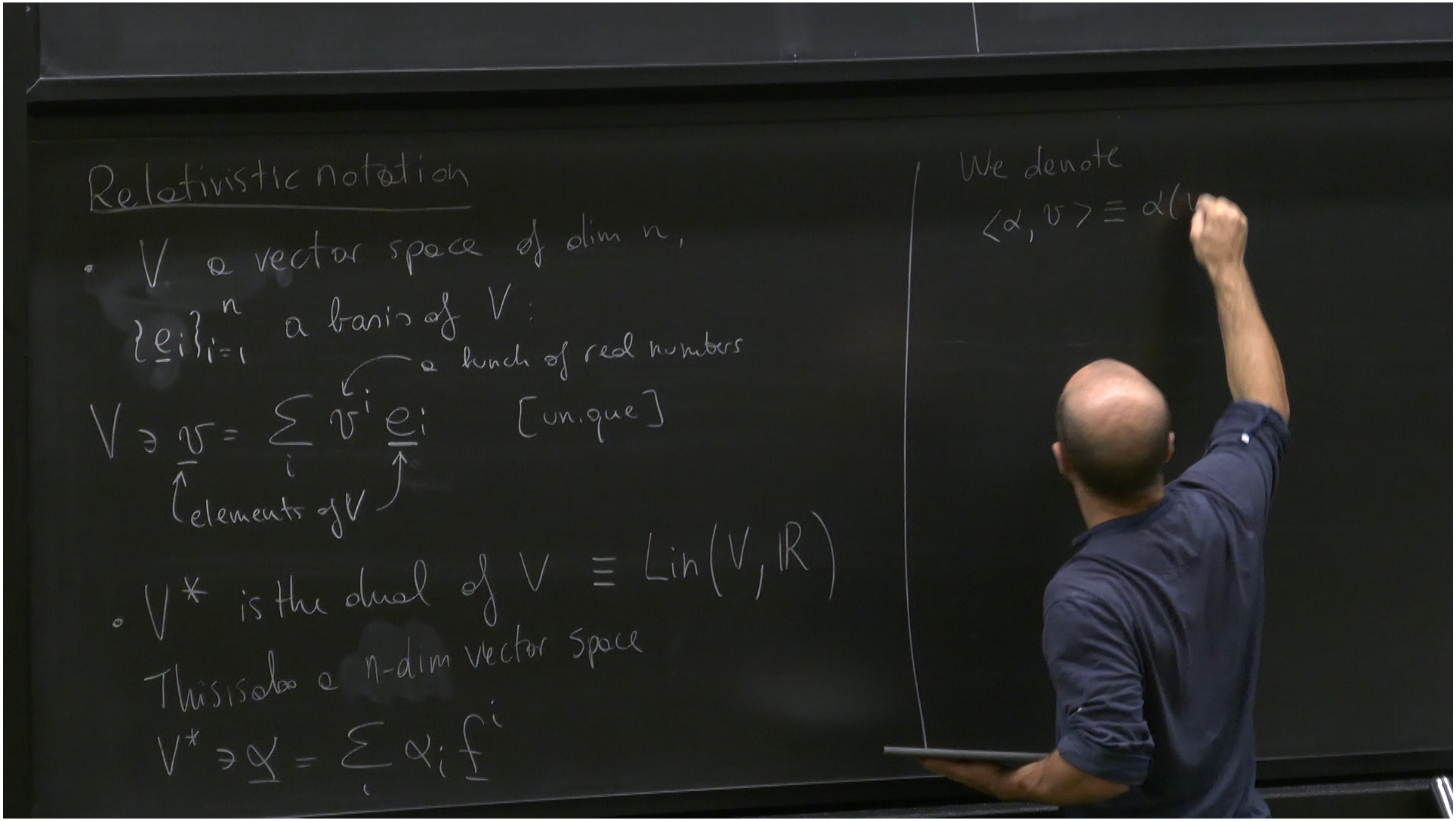
$\{\underline{e}_i\}_{i=1}^{n}$ a basis of V:

a bunch of real numbers

$$\underline{v} = \sum_i v^i \, \underline{e}_i \qquad [\text{unique}]$$

elements of V

-



Relativistic notation

- V a vector space of dim n,

$\{\underline{e}_i\}_{i=1}^{n}$ a basis of V:

a bunch of real numbers

$$V \ni \underline{v} = \sum_i v^i \underline{e}_i \qquad [\text{unique}]$$

elements of V

- V^* is the dual of V $\equiv \mathrm{Lin}(V, \mathbb{R})$

This is also a n-dim vector space

$$V^* \ni \underline{\alpha} = \sum_i \alpha_i \underline{f}^i$$

We denote

$\langle \alpha, v \rangle \equiv \alpha($

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- Given $\{\underline{e}_i\}$, I can choose a dual basis of V^* such that

$$\langle \underline{f}^i, \underline{e}_j \rangle = \delta^i_j$$

$$\langle \alpha, v \rangle = \sum_{i,j} \alpha^i v_j \langle \underline{f}_i, \underline{e}_j \rangle$$

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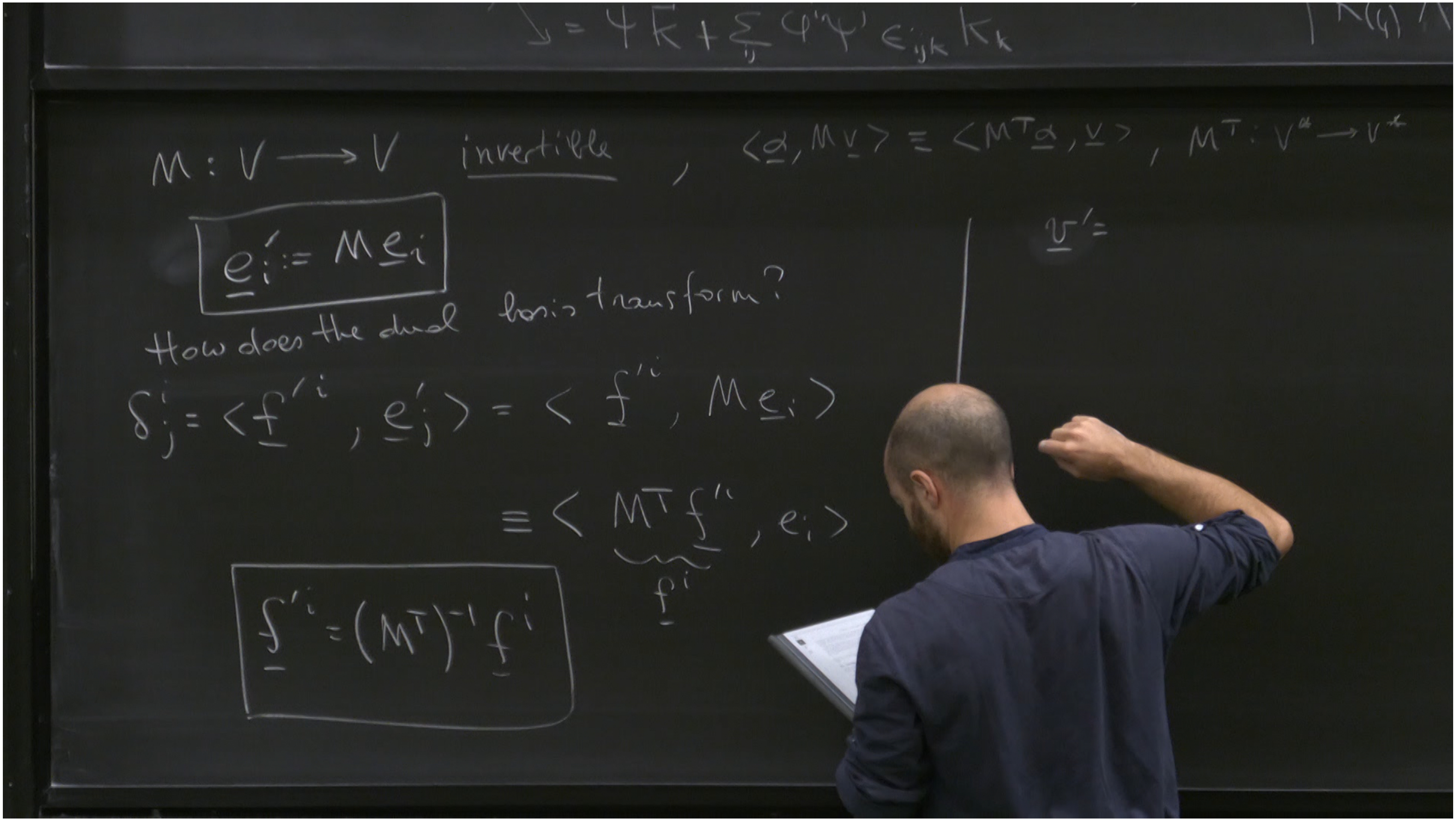
$v = v^i \underline{e}_i$

$\alpha = \alpha_i \underline{f}^i$



$= \psi \vec{K} + \sum_{ij} \varphi^i \varphi^j \epsilon_{ijk} K_k$

$K(\varphi) \wedge$

$M : V \longrightarrow V$ invertible , $\langle \underline{\alpha}, M\underline{v} \rangle \equiv \langle M^T \underline{\alpha}, \underline{v} \rangle$, $M^T : V^* \longrightarrow V^*$

$$\boxed{\underline{e}'_i := M\underline{e}_i}$$

How does the dual basis transform?

$$\delta^i_j = \langle \underline{f}'^i, \underline{e}'_j \rangle = \langle \underline{f}'^i, M\underline{e}_i \rangle$$

$$\equiv \langle \underbrace{M^T \underline{f}'^i}_{\underline{f}^i}, e_i \rangle$$

$$\boxed{\underline{f}'^i = (M^T)^{-1} \underline{f}^i}$$

$\underline{v}' =$

$\epsilon_{ijk} K_k$

$R_{(4)} \Lambda(\psi, \theta) R_{(4)} = \Lambda(R_{(3)} \psi, R_{(3)} \theta)$

$> \equiv \langle M^T \underline{\alpha}, \underline{v} \rangle \,, \quad M^T : V^* \longrightarrow V^*$

New basis developed on the old one:

$$\underline{e}'_i = (?)^j{}_i \, \underline{e}_j$$

$$\langle \underline{f}^j, M \underline{e}_i \rangle \equiv M^j{}_i$$

$\epsilon_{ijk} K_k$ | $R_{(4)} \Lambda(\psi, \theta) R^{-1}_{(4)} = \Lambda(R_{(3)}\psi, R_{(3)}\theta)$

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The components of $\underline{v}$ in the new basis

$$\underline{v} = \sum v^i \underline{e}_i = \sum v'^i \underline{e}'_i$$

$$v'^i = \langle \underline{f}'^i, \underline{v} \rangle = \langle (M^T)^{-1} \underline{f}^i, \underline{v} \rangle$$

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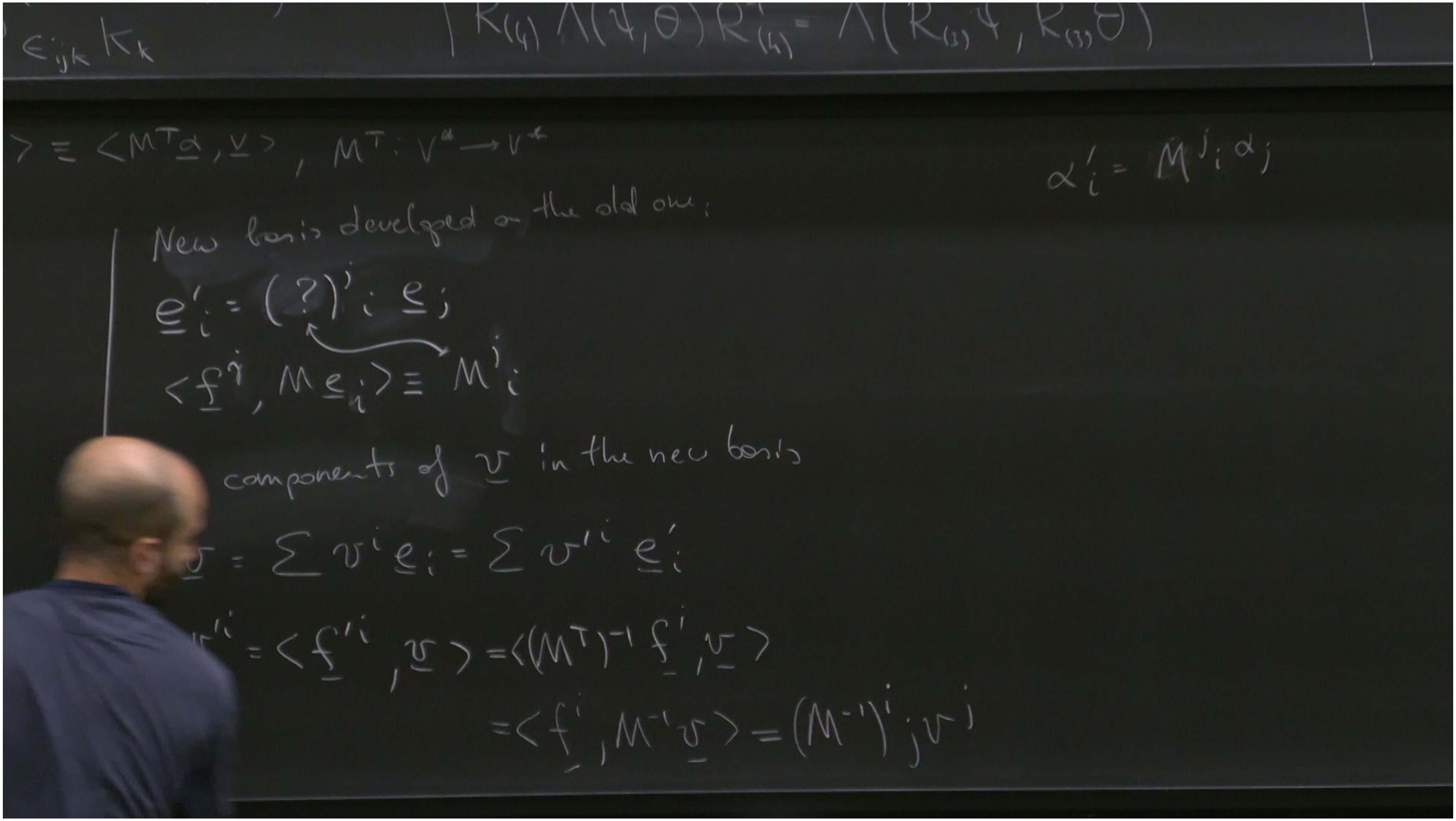

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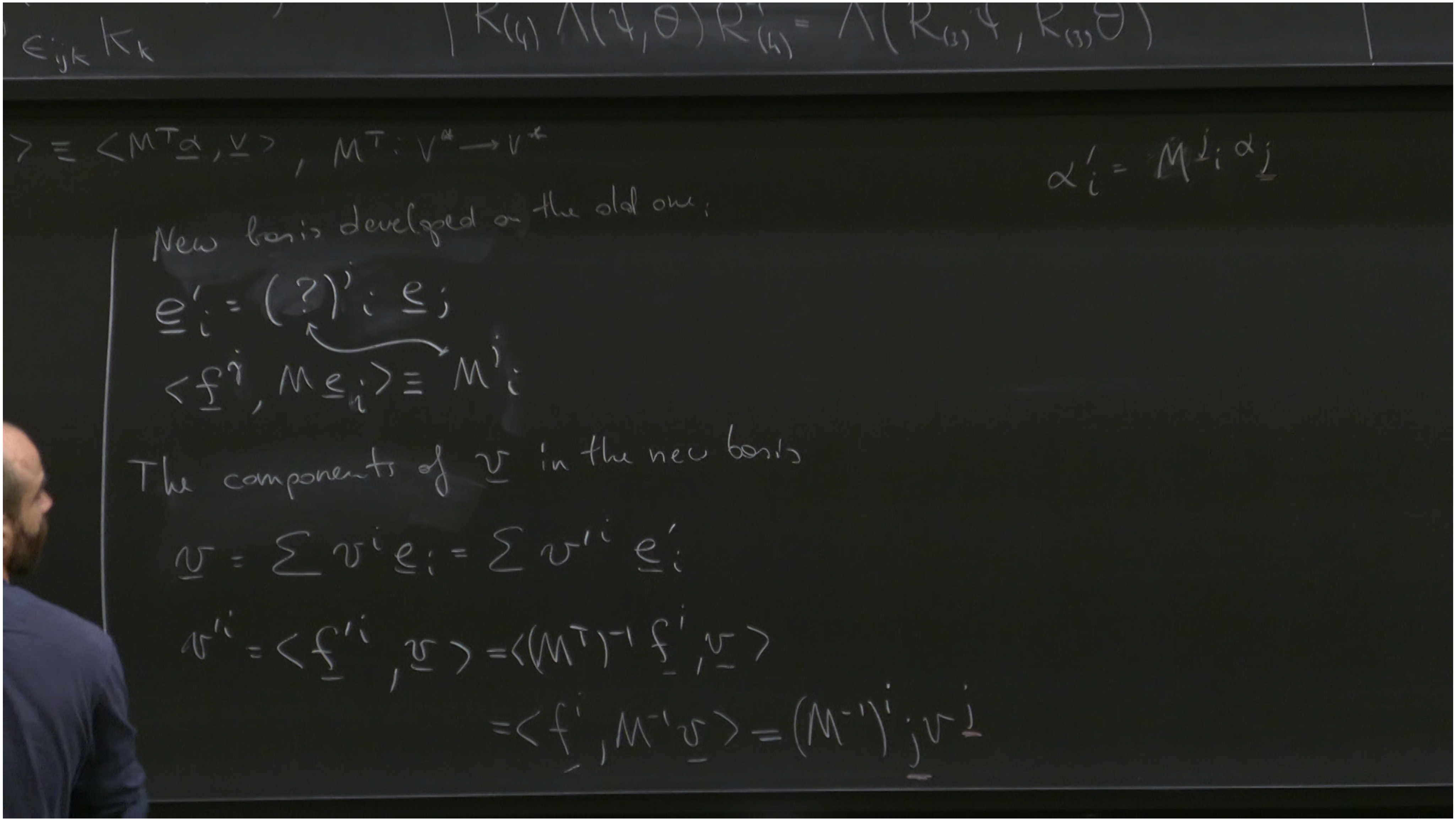

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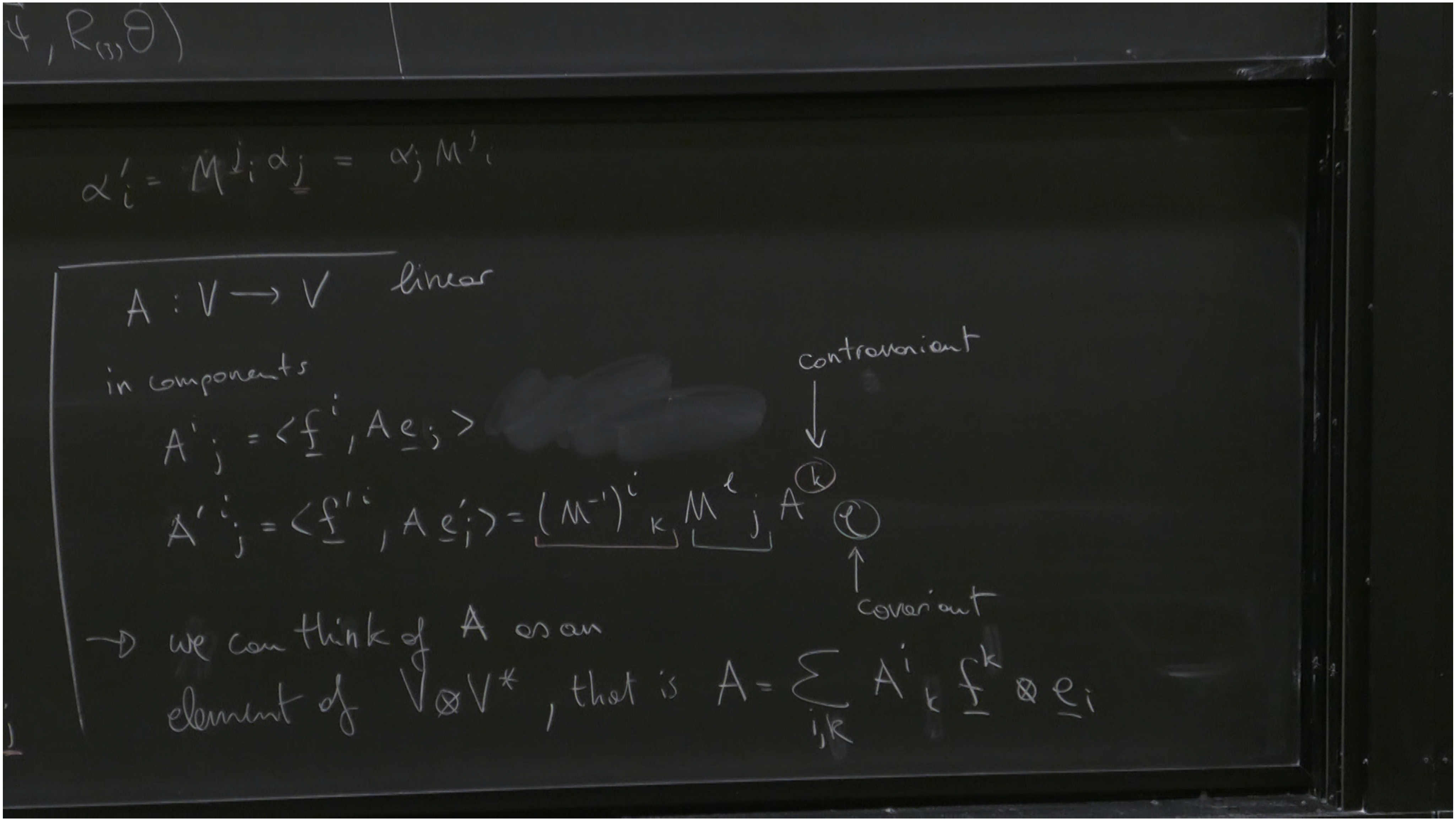


$\psi, R_{(3)}, \theta)$

$$\alpha'_i = M^j{}_i \alpha_{\underline{j}} = \alpha_j M^j{}_i$$

$A : V \longrightarrow V$ linear

in components

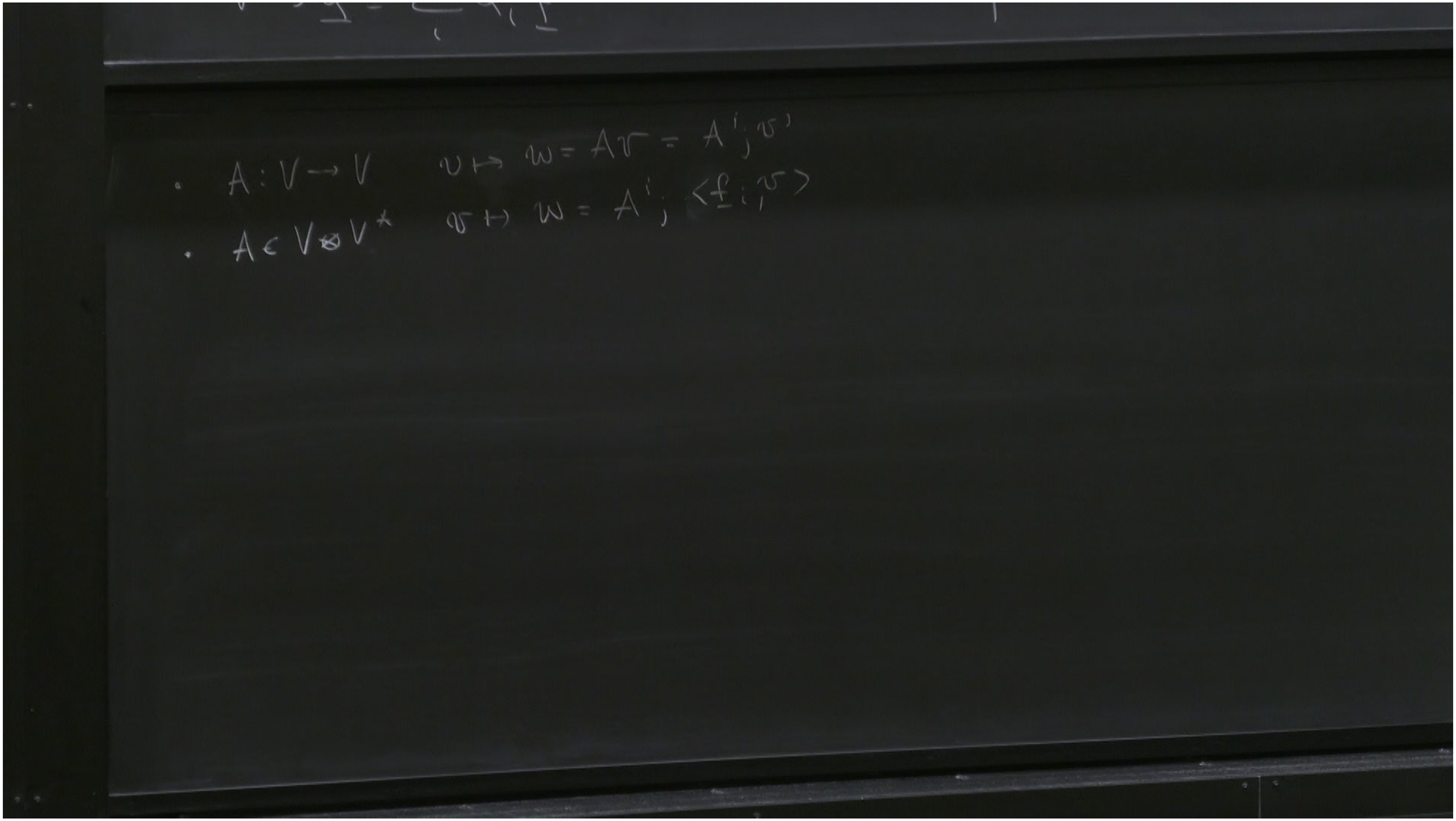
$$A^i{}_j = \langle \underline{f}^i, A \underline{e}_j \rangle$$

contravariant

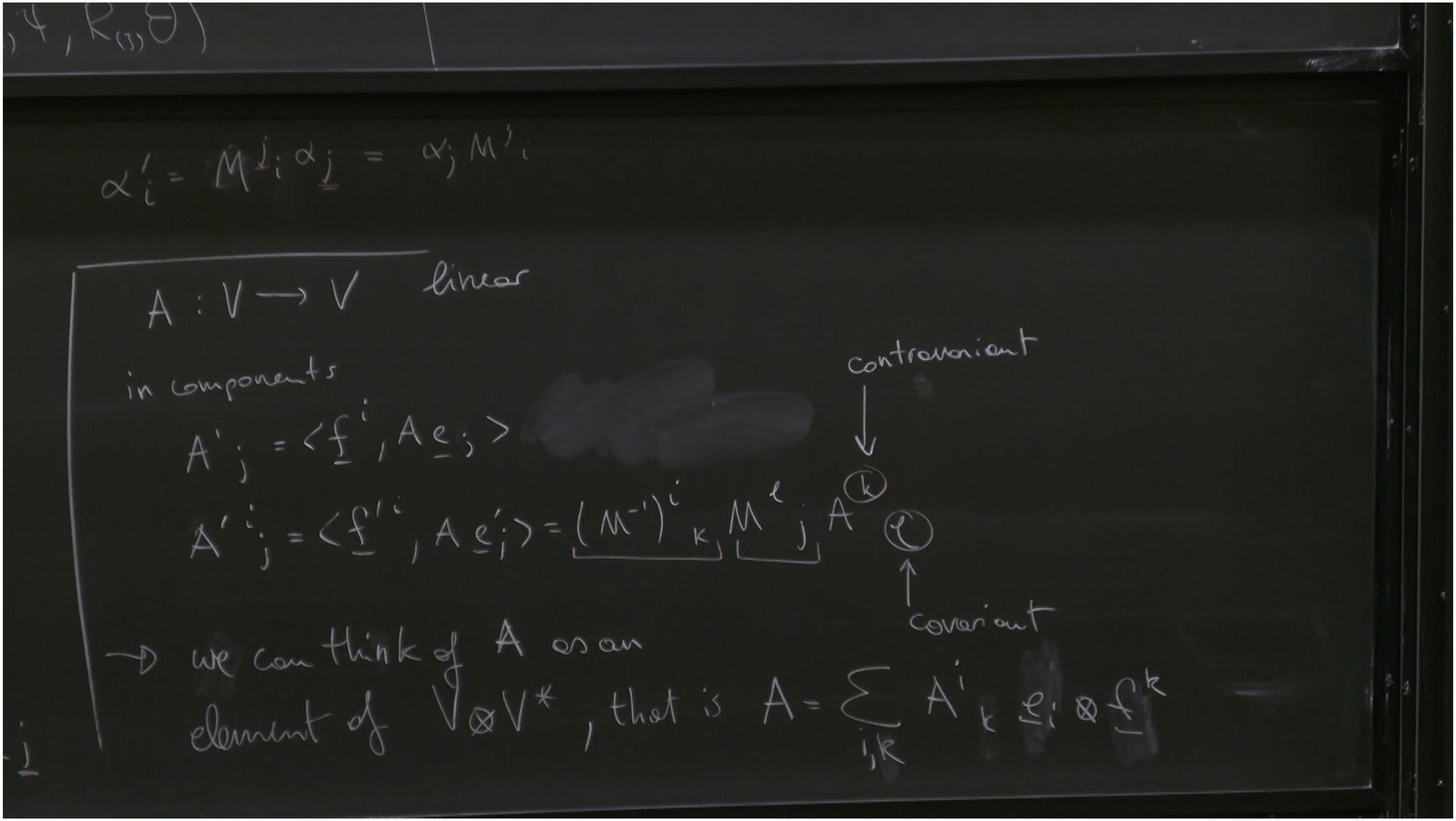
$$A'^i{}_j = \langle \underline{f}'^i, A \underline{e}'_j \rangle = \underbrace{(M^{-1})^i{}_k}_{} \underbrace{M^l{}_j}_{} A^{\textcircled{k}}{}_{\textcircled{l}}$$

covariant

→ we can think of A as an element of $V \otimes V^*$, that is $A = \sum_{i,k} A^i{}_k \underline{f}^k \otimes \underline{e}_i$



- $A : V \longrightarrow V \qquad v \mapsto w = Av = A^i{}_j v^j$
- $A \in V \otimes V^* \qquad v \mapsto w = A^i{}_j \langle \underline{f}^j , v \rangle$



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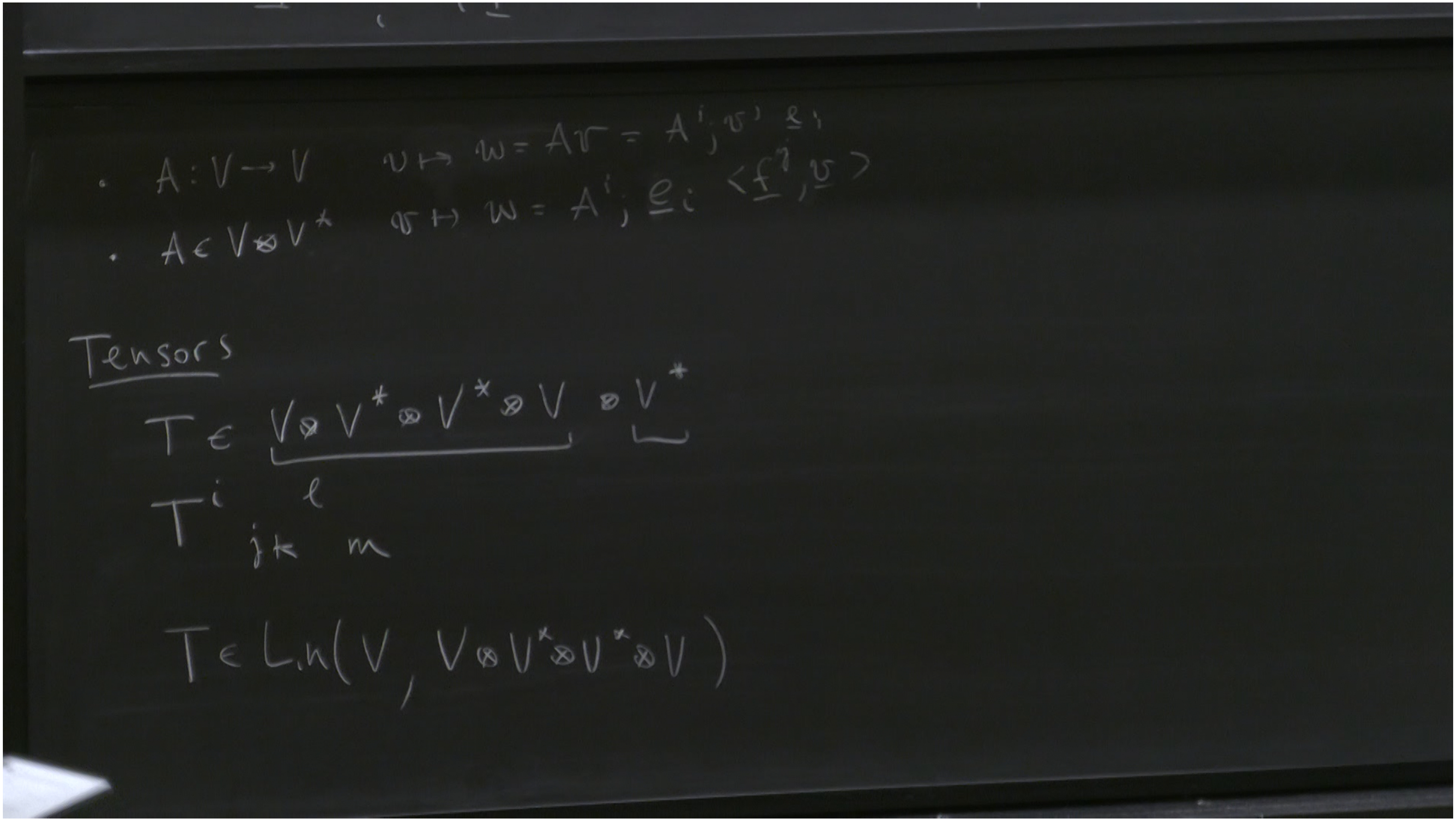

- $A : V \to V \qquad v \mapsto w = Av = A^i{}_j v^j \underline{e}_i$
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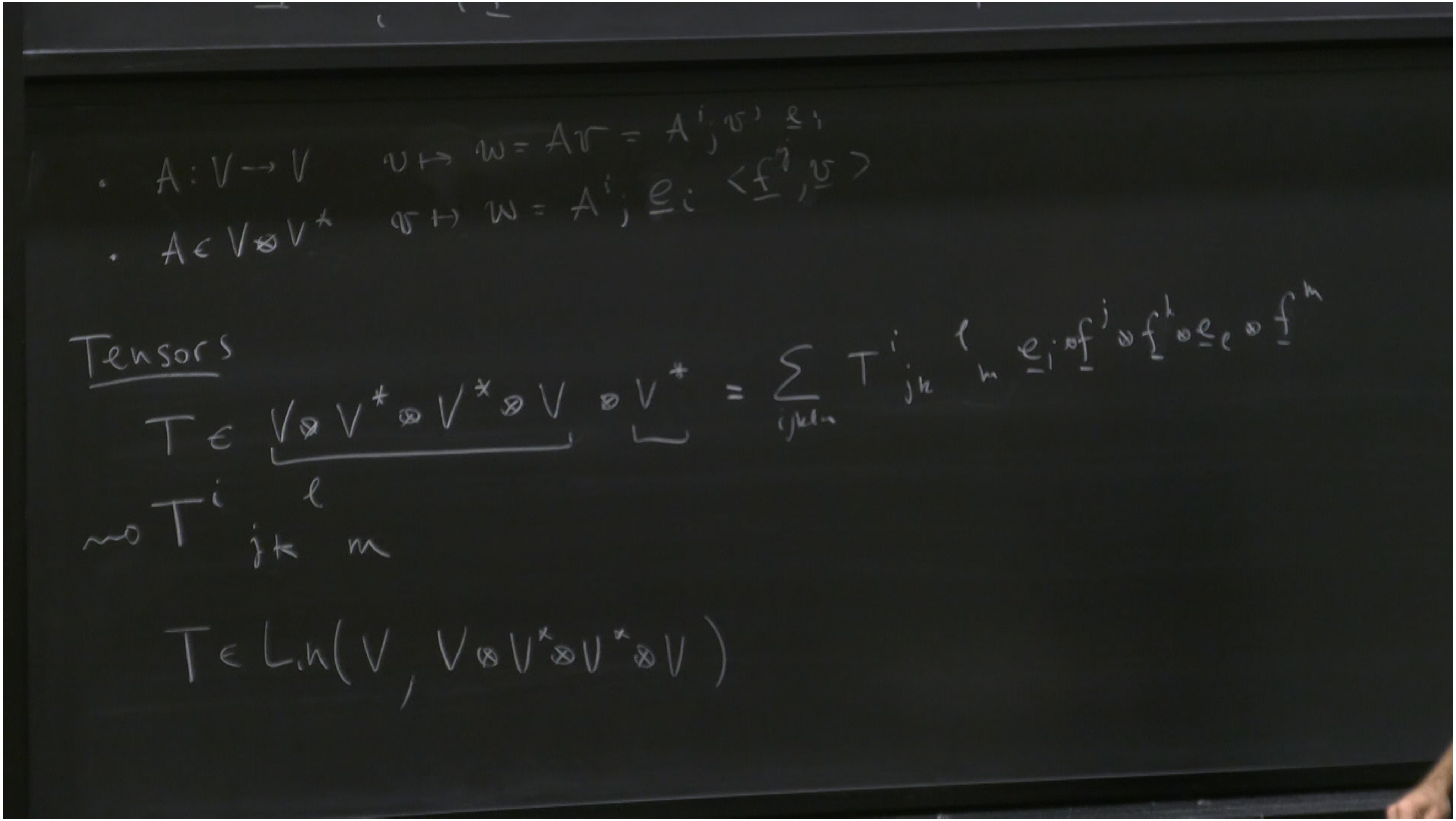
<u>Tensors</u>

$$T \in \underbrace{V \otimes V^* \otimes V^* \otimes V}_{} \otimes \underbrace{V^*}_{}$$

$$T^i{}_{jk}{}^l{}_m$$

$$T \in \mathrm{Lin}(V, V \otimes V^* \otimes V^* \otimes V)$$



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Tensors

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$$\rightsquigarrow T^i{}_{jk}{}^l{}_m$$

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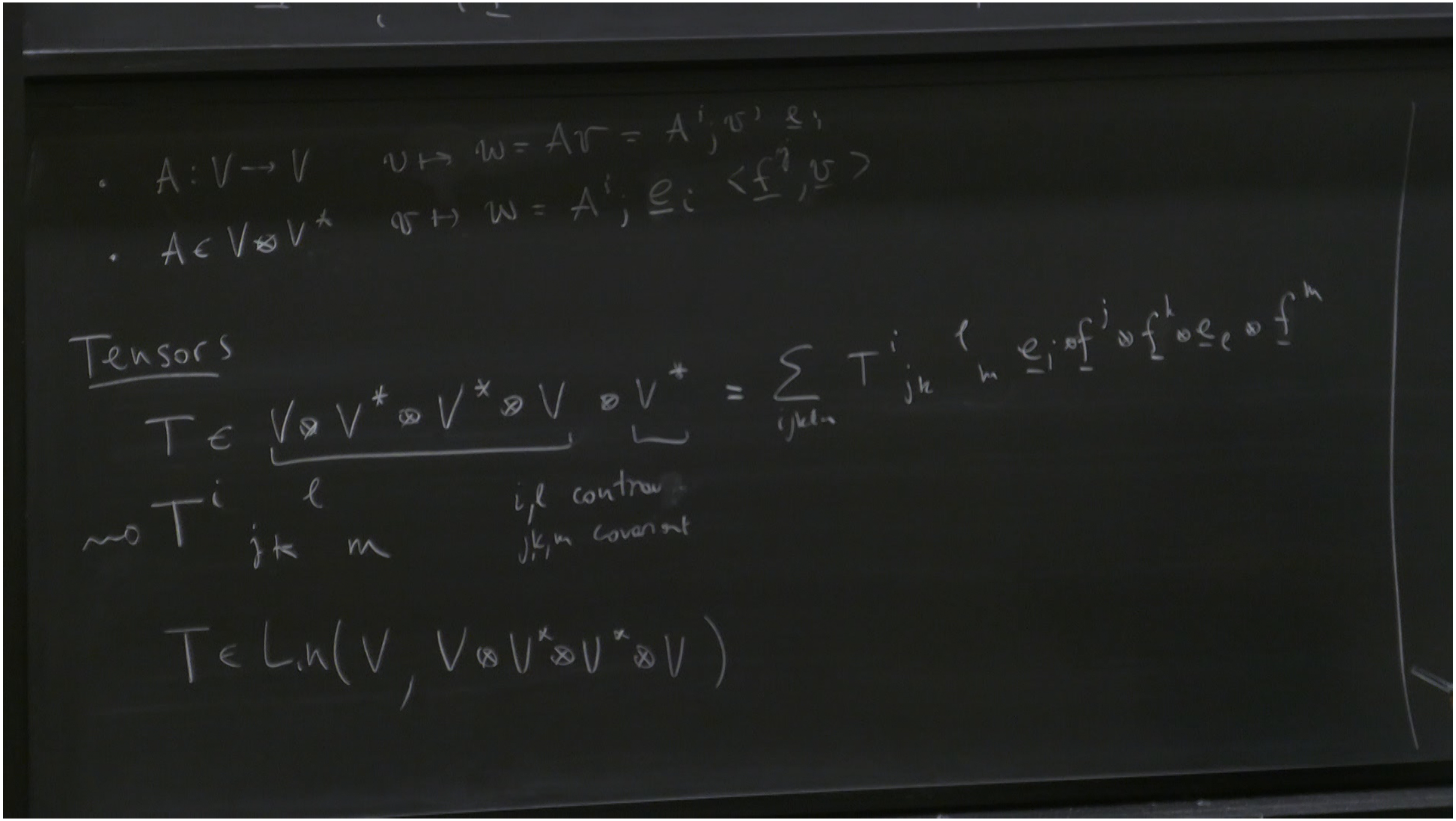

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$\rightsquigarrow T^i{}_{jk}{}^l{}_m$ $\quad$ i, l contrav

j, k, m covariant

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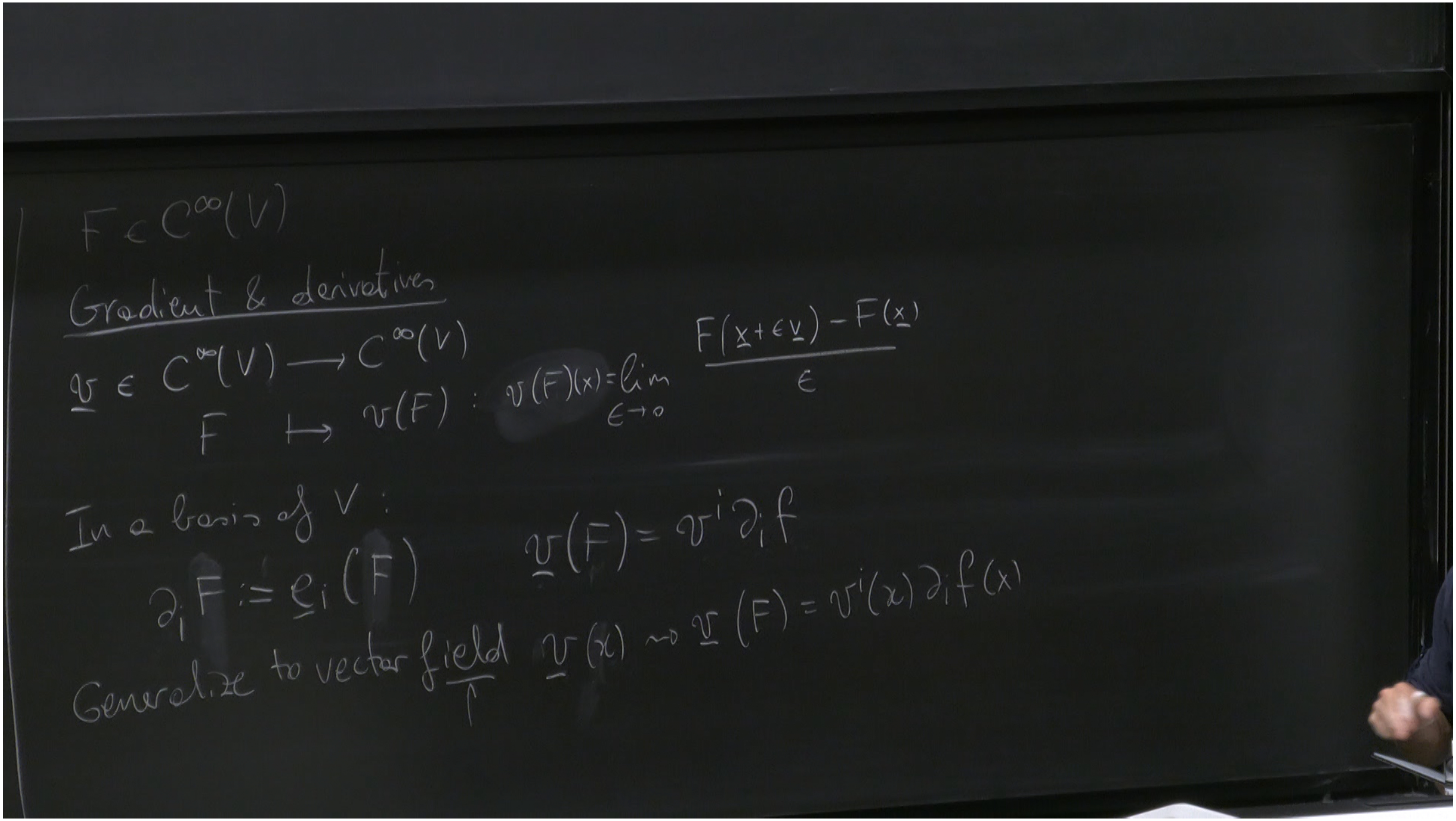

$F \in C^\infty(V)$

Gradient & derivatives

$$\begin{aligned}\underline{v} \in\ & C^\infty(V) \longrightarrow C^\infty(V)\\ & \quad F \longmapsto v(F) : \quad v(F)(x) = \lim_{\epsilon\to 0}\frac{F(\underline{x}+\epsilon\underline{v})-F(\underline{x})}{\epsilon}\end{aligned}$$

In a basis of V :

$$\partial_i F := \underline{e}_i(F) \qquad \underline{v}(F) = v^i \partial_i f$$

Generalize to vector $\underset{\uparrow}{\underline{\text{field}}}$ $\underline{v}(x) \rightsquigarrow \underline{v}(F) = v^i(x)\,\partial_i f(x)$