Title: Classical Physics Lecture - 091523

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Collection: Classical Physics 2023/24

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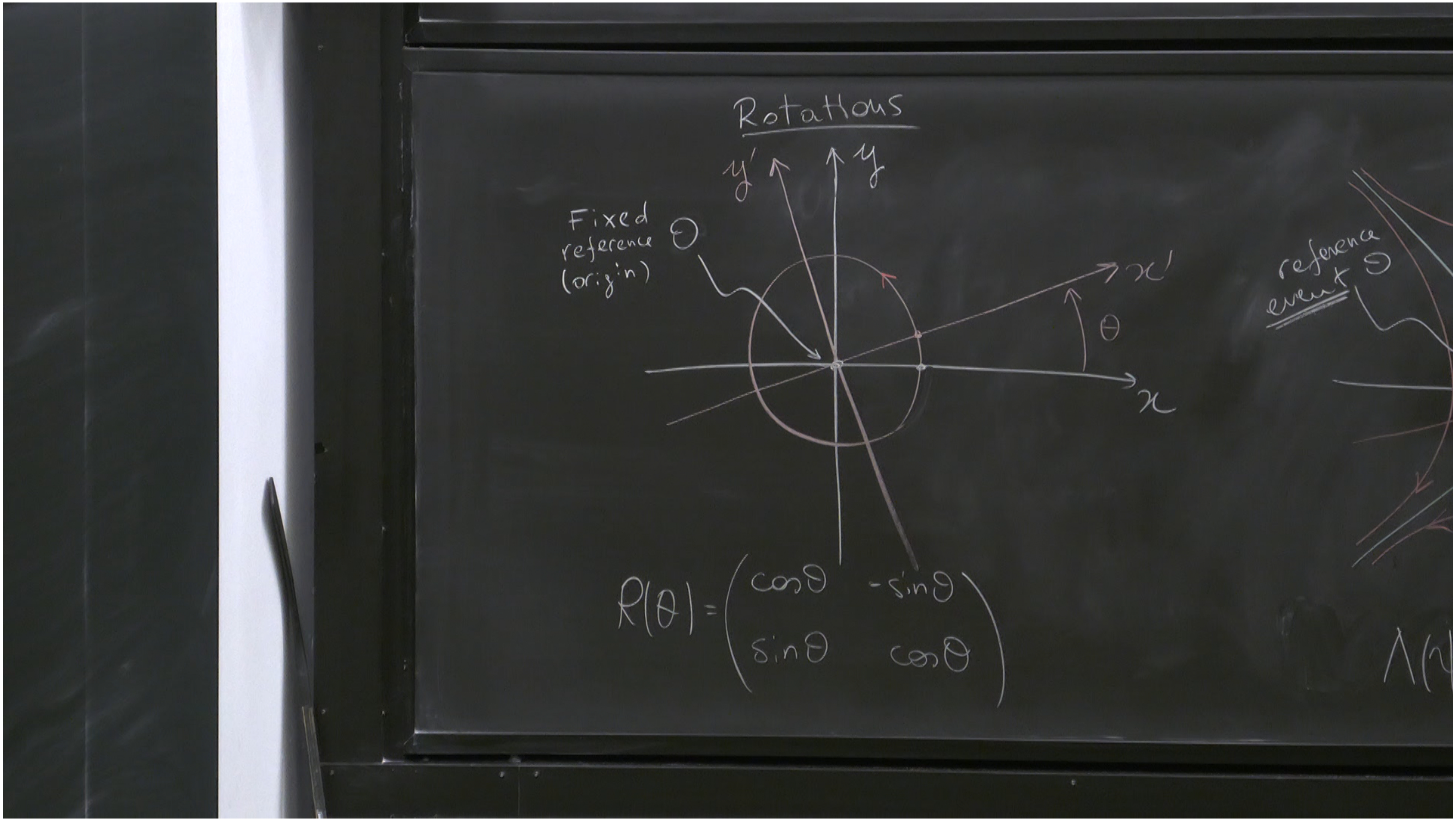
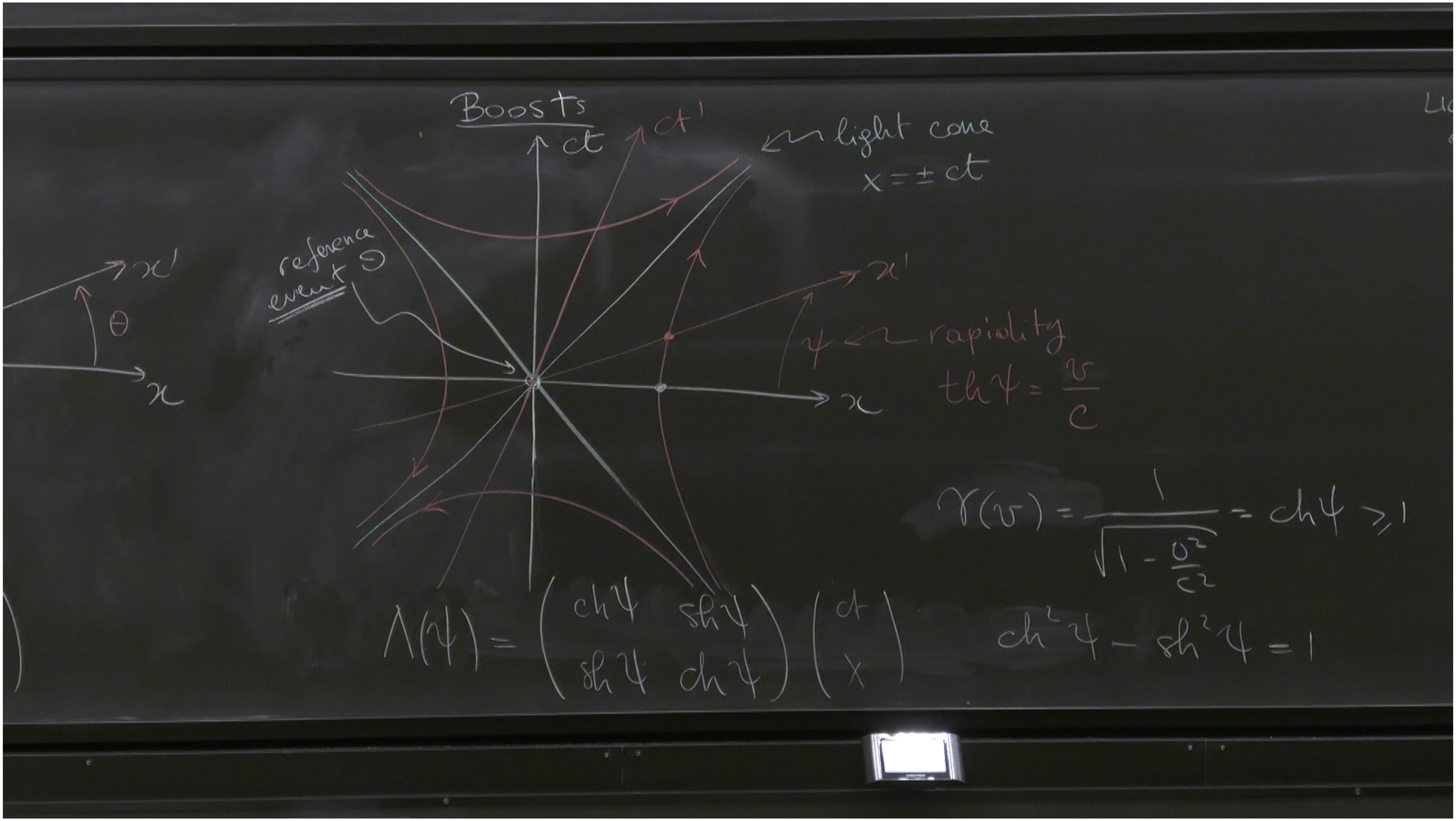
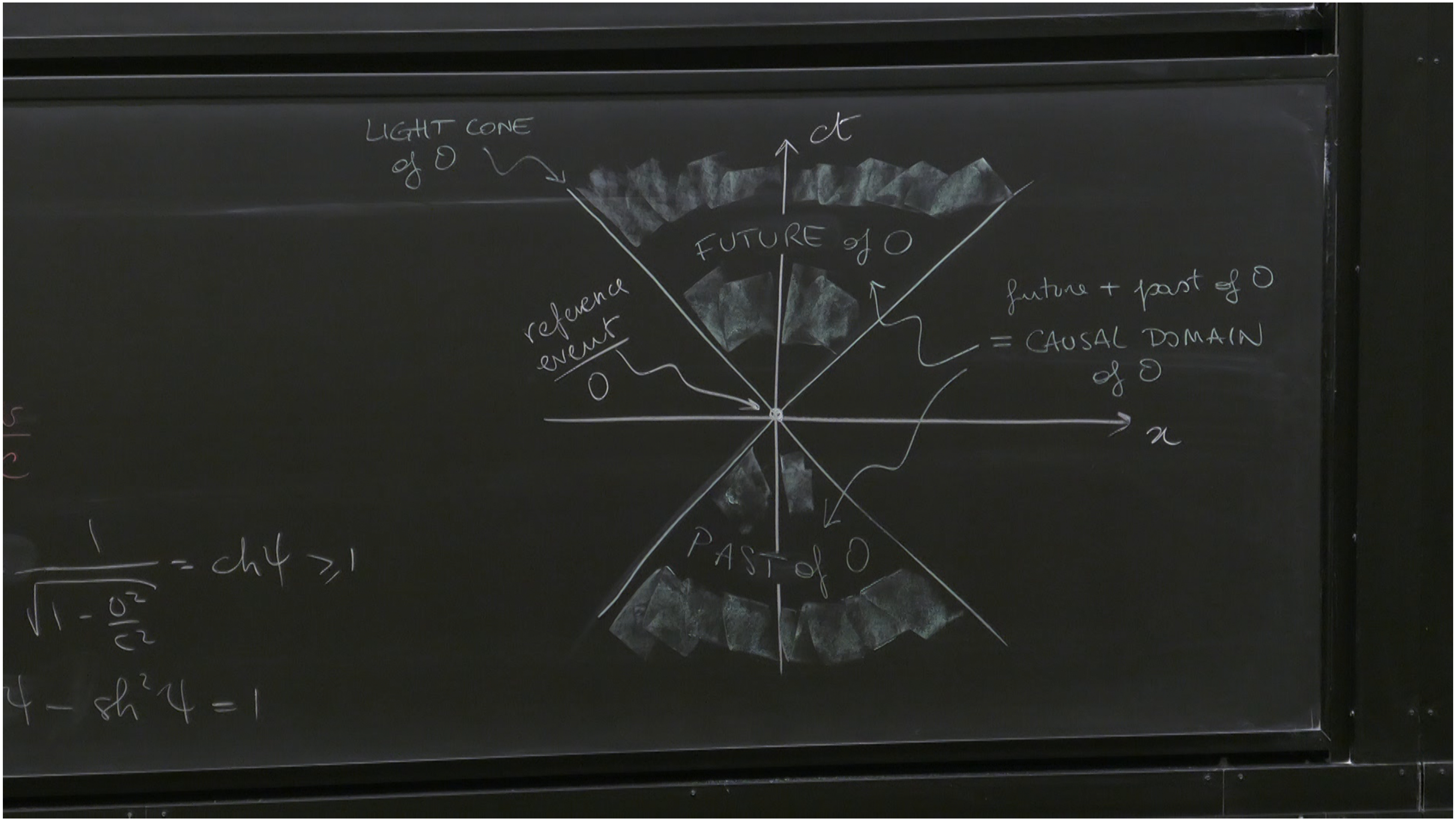


Rotations
y'
y
Fixed reference (origin)
x'
θ
x
reference event
R(θ) = (cosθ -sinθ sinθ cosθ)


Boosts
ct
ct'
light cone
x = ± ct
reference event
x'
θ
x
x'
ψ rapidity
th ψ = v/c
x
γ(v) = 1/√(1 − v²/c²) = ch ψ ≥ 1
Λ(ψ) = (ch ψ sh ψ ; sh ψ ch ψ)(ct ; x)
ch²ψ − sh²ψ = 1


LIGHT CONE
of O
ct
FUTURE of O
future + past of O
= CAUSAL DOMAIN
of O
reference
event
O
x
PAST of O
1
√1 - v²/c²
= chψ ≥ 1
ψ - sh²ψ = 1

Addition of velocities

3 reference frames:

$$\begin{tikzcd} K \arrow[r, "v_1"'] \arrow[rr, bend right, "v_3?"'] & K' \arrow[r, "v_2"'] & K'' \end{tikzcd}$$

$$\Lambda(\psi_3) := \Lambda(\psi_2)\,\Lambda(\psi_1) = \Lambda(\psi_1 + \psi_2)$$

$$\frac{v_3}{c} = \operatorname{th}\psi_3 = \operatorname{th}(\psi_1+\psi_2) = \frac{\operatorname{sh}(\psi_1+\psi_2)}{\operatorname{ch}(\psi_1+\psi_2)}$$

$$= \frac{\operatorname{sh}\psi_1 \operatorname{ch}\psi_2 + \operatorname{ch}\psi_1 \operatorname{sh}\psi_2}{\operatorname{sh}\psi_1 \operatorname{sh}\psi_2 + \operatorname{ch}\psi_1 \operatorname{ch}\psi_2}$$

$$= \frac{\operatorname{th}\psi_1 + \operatorname{th}\psi_2}{1 + \operatorname{th}\psi_1 \operatorname{th}\psi_2}$$

$$= \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

An object moving at speed u' in K' is seen moving at speed

$$u = \frac{u' + v}{1 + \frac{uv}{c^2}}$$

in K, if K' moves at speed v wrt K.

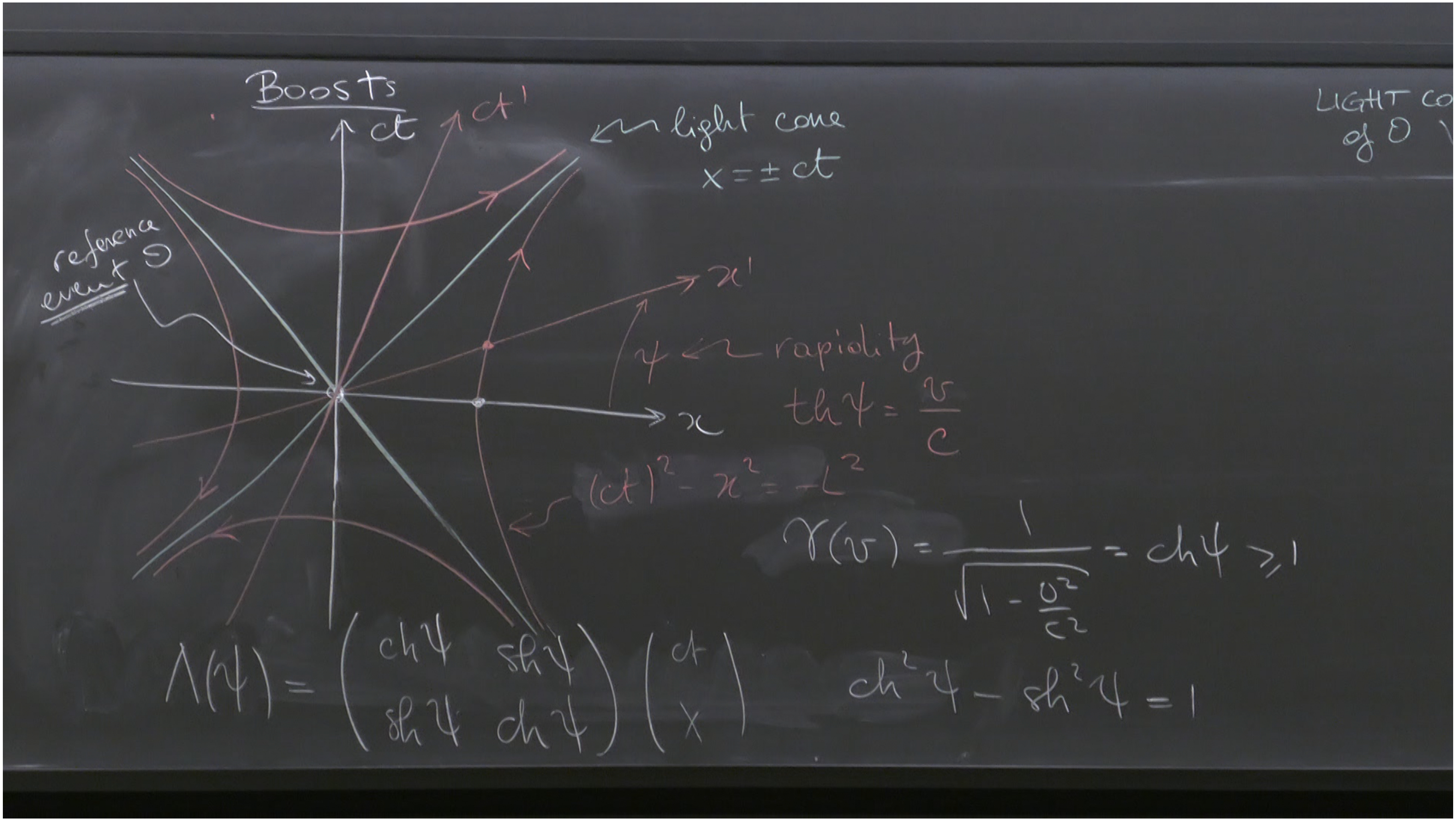
$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

What about $u' = c$

$\Rightarrow u = c$

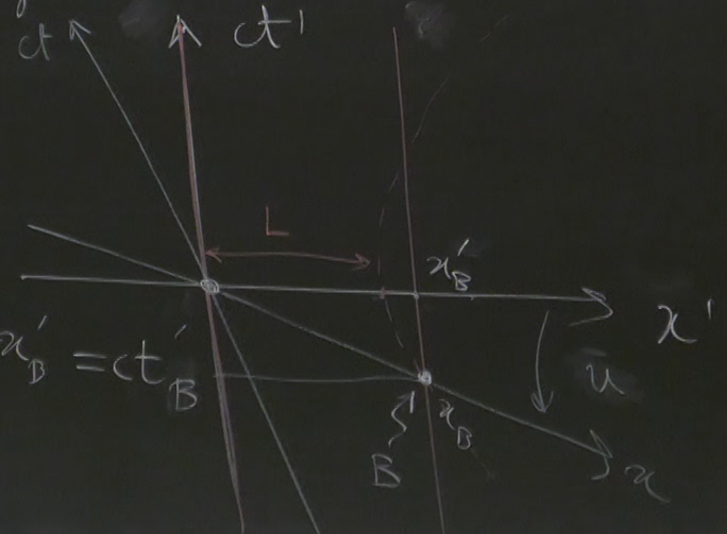
$\Rightarrow u = c$ is a fixed of this formula

$\Rightarrow c$ is a universal velocity


Boosts
ct
ct'
light cone
x = ± ct
LIGHT CO
of O
reference event O
x'
ψ rapidity
th ψ = v/c
x
(ct)² - x² = -L²
γ(v) = 1/√(1 - v²/c²) = ch ψ ≥ 1
Λ(ψ) = (ch ψ sh ψ; sh ψ ch ψ) (ct; x)
ch² ψ - sh² ψ = 1

Length contraction

- Rod at rest in reference K'
- K' moves at velocity u wrt K.
- Rod length in K' is $L' = \ell \equiv$ proper length $\equiv$ (in the rest frame)



- that $x_B' = -\frac{u}{c} x_B' = ct_B'$

$), \quad ch^2\psi - sh^2\psi = 1$

$$L^2 = (1 - th^2\psi)\,\ell^2 = \frac{1}{ch^2\psi}\,\ell^2 = \frac{\ell^2}{\gamma(u)^2}$$

d t

$$\boxed{L = \frac{\ell}{\gamma(u)}}$$

Lorentz-Fitzgerald contraction of lengths

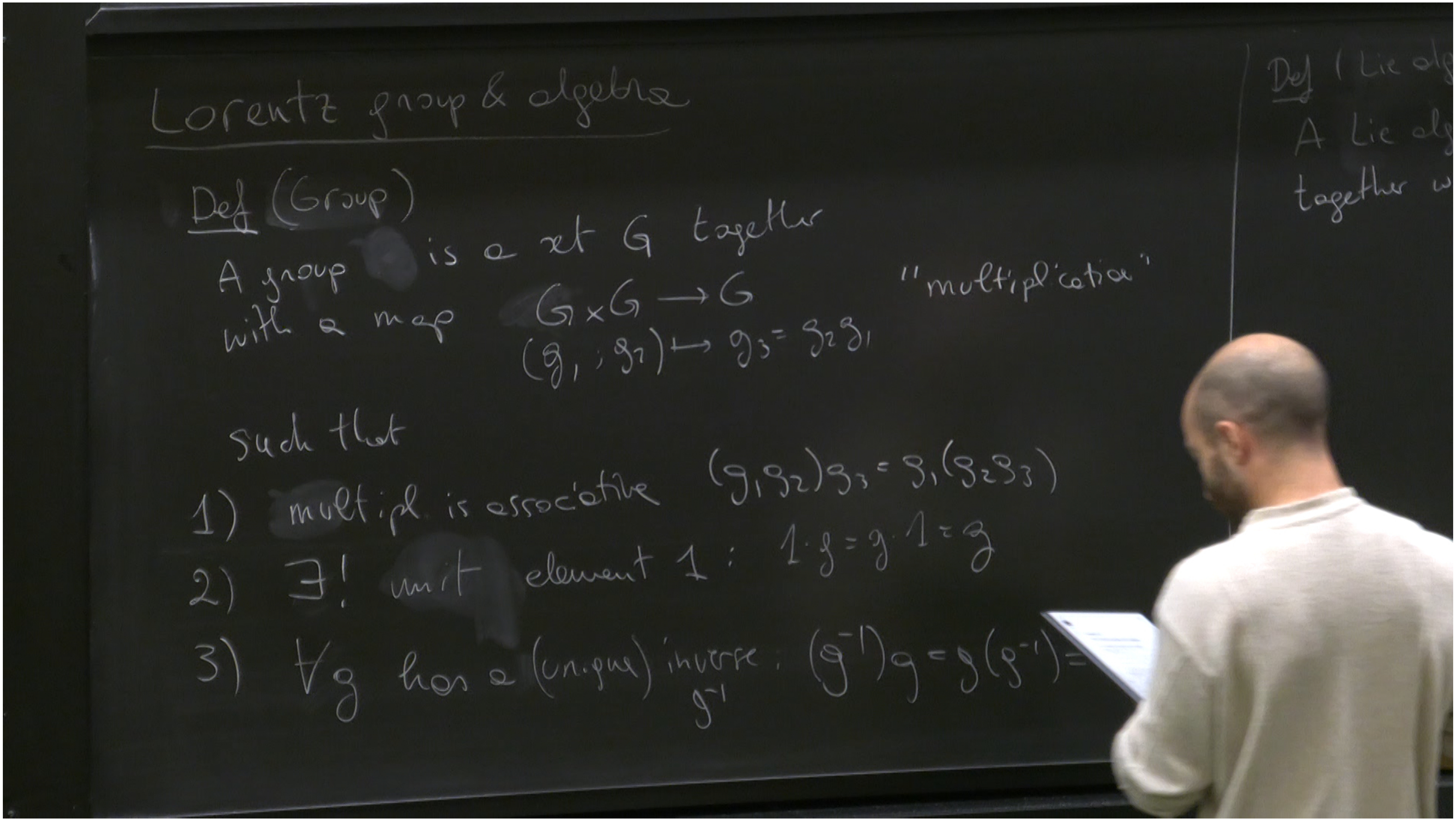
$$= (ct'_B)^2 - (x'_B)^2$$

$$(th\psi)^2\ell^2 - \ell^2$$

$$T = \gamma(u)\,\tau$$

the proper time interval

Time dilation formula


Lorentz group & algebra
Def (Group)
A group is a set G together
with a map G × G → G "multiplication"
(g1 ; g2) ↦ g3 = g2 g1
such that
1) multipl. is associative (g1 g2) g3 = g1 (g2 g3)
2) ∃! unit element 1 : 1·g = g·1 = g
3) ∀g has a (unique) inverse g⁻¹ : (g⁻¹) g = g (g⁻¹) =
Def (Lie alg
A Lie alg
together w

A Lie algebra is a vect space A

together with $[\ ,\ ] : A \times A \to A$

$$(a_1, a_2) \mapsto a_3 = [a_1, a_2]$$

such that

1) $[\ ,\ ]$ is bilinear (in $\mathbb{R}$)

2) skew sym : $[a_1, a_2] + [a_2, a_1] = 0$

3) satisfies Jacobi : $[[a_1, a_2], a_3] + \text{cycl} = 0$

multiplication"

$(g_2 g_3)$

$= g$

$= g(g^{-1}) = 1$

• 1+1 d boost form a grp

$G = SO^{\uparrow}(1,1)$

defined as the grp of linear transf. of $\mathbb{R}^2$ such that

M1) leave the Minkowski norm inv.

$$S^2 \equiv (ct')^2 - (x')^2 = (ct)^2 - x^2$$

M2) have unit determinant

M3) do not flip the direction of time

$$\Lambda^t{}_t = \Lambda^0{}_0 = \left(\begin{array}{c|c} \bullet & \\ \hline & \end{array}\right) > 0$$

- 2d rotations form a group

$G = SO(2)$

defined as the group of linear transf of $\mathbb{R}^2$

such that

E1) leave the Euclidean norm invariant

$$(x')^2 + (y')^2 = (x)^2 + (y)^2$$

E2) have unit determinant, i.e

they preserve the orientation of $\mathbb{R}^2$

- 1+1 d

$G = S$

defined

such th

M1) leave

$S^2 \equiv$

M2) have

M3) do

form a group

the group of linear transf of $\mathbb{R}^2$

the Euclidean norm invariant

$(x')^2 + (y')^2 = (x)^2 + (y)^2$

unit determinant, ie

preserve the orientation of $\mathbb{R}^2$

• 1+1 d boost form a grp

$G = SO^+(1,1)$

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(M1) leave the Minkowskian norm inv.

$S^2 \equiv (ct')^2 - (x')^2 = (ct)^2 - x^2$

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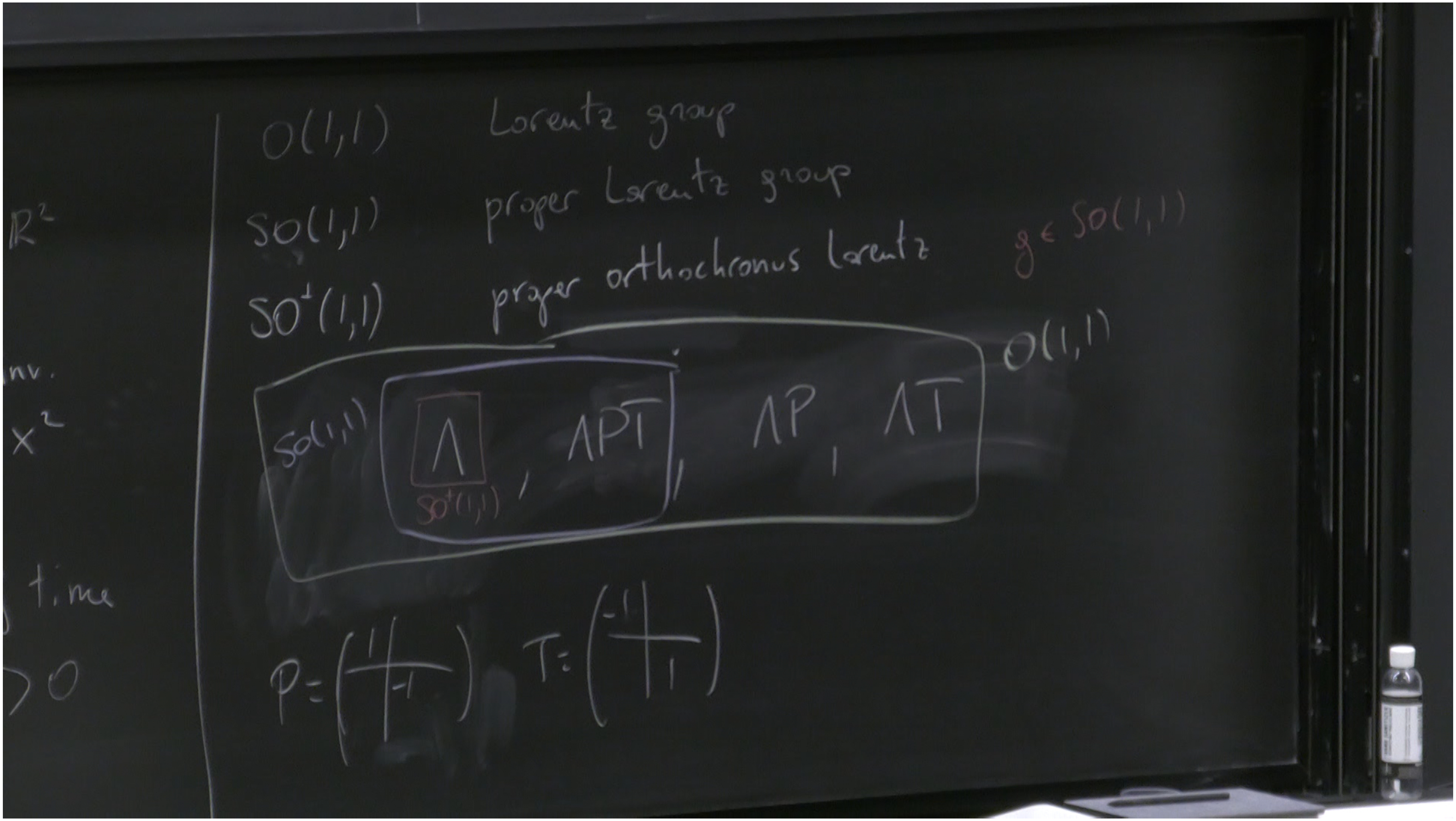
$\Lambda^t{}_t = \Lambda^0{}_0 = \begin{pmatrix} \bullet & \\ & \end{pmatrix} > 0$


$\mathbb{R}^2$
inv.
x^2
time
> 0
$O(1,1)$ Lorentz group
$SO(1,1)$ proper Lorentz group
$SO^{\uparrow}(1,1)$ proper orthochronus lorentz
$g \in SO(1,1)$
$O(1,1)$
$SO(1,1)$
Λ, ΛPT, ΛP, ΛT
$SO^{\uparrow}(1,1)$
$P = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ $T = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$

In 1+3d

$$\eta = \left(\begin{array}{c|ccc} -1 & & & \\ \hline & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right)$$

Minkowski metric on $\mathbb{R}^4$

$$(*) \qquad s^2 = X^T \eta X$$

Minkowski "norm"

$$= -(ct)^2 + (\vec{x})^2 \qquad X = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}$$

$\mathbb{R}^4$

$$X = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}$$

Proper orthochronous
Lorentz grp as before
with M1 replaced by (*)

$$\rightsquigarrow \boxed{\Lambda^T \eta \Lambda = \eta} \quad (M1)$$

(generalization of $R^T = R^{-1}$ for rotations)

M2) $\det \Lambda = 1$

M3) $\Lambda^0{}_0 > 0$

1)

rotations)

<u>Def</u>

X is said spacelike / timelike / null/lightlike if $X^T \eta X$ is > 0 / < 0 / $= 0$

Rmk

$$\Lambda = \left(\begin{array}{c|c} 1 & \\ \hline & R \end{array}\right)$$ are Lorentz transf.

R: rotation in 3d

$SO(3) \subset SO^+(1,3)$

$$\{\rho_\Lambda(R) = \Lambda \rho(R) \Lambda^{-1} \text{ for } \Lambda \text{ fixed}\} \subset SO^+(1,3)$$

$$\cong$$

$$SO(3)$$

$$\rho_\Lambda(R_1)\rho_\Lambda(R_2) = \Lambda \rho(R_1)\Lambda^{-1}\Lambda\rho(R_2)\Lambda^{-1}$$

$$= \Lambda \rho(R_1)\rho(R_2)\Lambda^{-1}$$

$$= \Lambda \rho(R_1 R_2)\Lambda^{-1}$$

$$= \rho_\Lambda(R_1 R_2)$$

Minkowski:
metric on $\mathbb{R}^4$

X Minkowski:
"norm"

$\ldots^2 + (\vec{X})^2$

$$X = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix} = \begin{pmatrix} X^0 \\ \vec{X} \end{pmatrix}$$

$$\Lambda = \left(\begin{array}{c|ccc} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \hline \Lambda^1{}_0 & & & \\ \vdots & & \Lambda^i{}_j & \\ \vdots & & & \end{array}\right)$$

Proper orthochronous
Lorentz grp as before
with M1 replaced by (*)

$$\rightsquigarrow \boxed{\Lambda^T \eta \Lambda = \eta} \quad (M1)$$

(generalization of $R^T = R^{-1}$ for rotations

M2) $\det \Lambda = 1$

M3) $\Lambda^0{}_0 > 0$

$$\{ \rho_\Lambda(R) = \Lambda \rho(R) \Lambda^{-1} \text{ for } \Lambda \text{ fixed} \} \subset SO^+(1,3)$$

$$\cong SO(3)$$

$$\begin{aligned}
\rho_\Lambda(R_1)\,\rho_\Lambda(R_2) &= \Lambda \rho(R_1) \Lambda^{-1} \Lambda \rho(R_2) \Lambda^{-1} \\
&= \Lambda \rho(R_1) \rho(R_2) \Lambda^{-1} \\
&= \Lambda \rho(R_1 R_2) \Lambda^{-1} \\
&= \rho_\Lambda(R_1 R_2)
\end{aligned}$$

A rotation subgroup for every reference frame.

Observation

$\Lambda(\psi) = ($

Observation

$$\Lambda(\psi) = \begin{pmatrix} \operatorname{ch}\psi & \operatorname{sh}\psi \\ \operatorname{sh}\psi & \operatorname{ch}\psi \end{pmatrix} \underset{\substack{\uparrow \\ \text{Taylor} \\ \text{in } \psi}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \psi \\ \psi & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2!}\psi^2 & 0 \\ 0 & \frac{1}{2!}\psi^2 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{3!}\psi^3 \\ \frac{1}{3!}\psi^3 & 0 \end{pmatrix} + \ldots$$

$$K := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$K^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Lambda(\psi) = \sum_{n=0}^{\infty} \frac{\psi^n}{n!} K^n \equiv e^{\psi K}$$

Rotations

$$R(\theta) = \exp(\theta J) \simeq 1 + \theta J + \dots$$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

In higher dim:

$$\Lambda = e^{\epsilon\lambda} \simeq 1 + \epsilon\lambda + \frac{\epsilon^2}{2!}\lambda^2 + \dots$$

Now, we want to study $\Lambda^T \eta \Lambda = \eta$ at order $O(\epsilon)$

and

the

$1 + \theta J + \dots$

$\epsilon\lambda + \frac{\epsilon^2}{2!}\lambda^2 + \dots$

$\boxed{\Lambda^T \eta \Lambda = \eta}$ at order $O(\epsilon)$

and thereby understand the form of λ in analogy with K & J from previous 2d examples

$$(1+\epsilon\lambda)^T \eta (1+\epsilon\lambda) = \eta + O(\epsilon^2)$$

$$\boxed{\lambda^T \eta + \eta \lambda = 0}$$ (M1) infinitesimal