Title: Classical Physics Lecture - 091323

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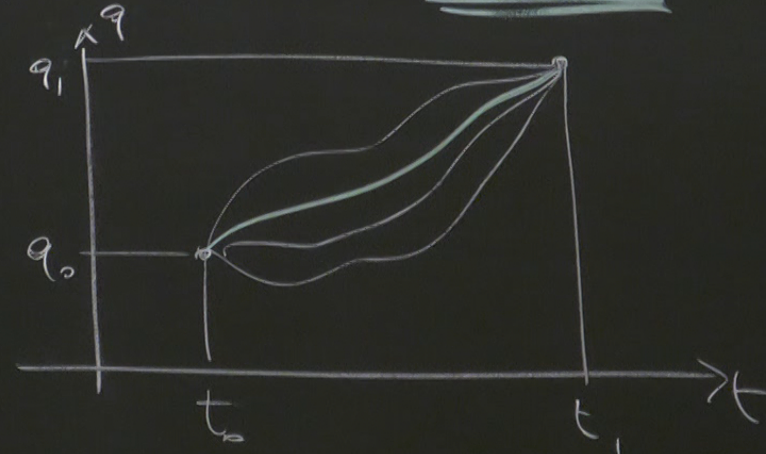
HAMILTON JACOBI

Action principle

① $(t_0, q_0, t_1, q_1) \to \bar{\gamma}_{(t_0 q_0 t_1 q_1)} : \delta S|_{\bar{\gamma}} = 0$



$\ldots q_1]$

<u>Last time</u>

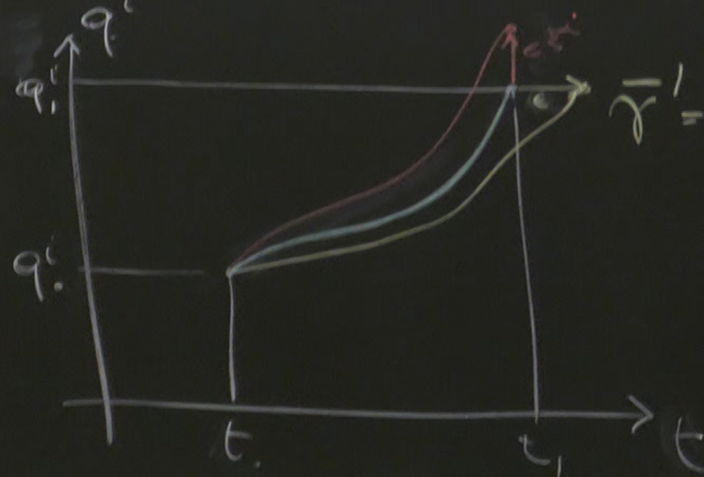
$$\sum_i \xi^i \frac{\partial}{\partial q_1^i} S = \ldots = \sum_i \xi^i \underbrace{\frac{\partial L}{\partial \dot{q}^i}(t_1)}_{\bar{p}_1 \equiv \bar{p}(t_1)} \Big|_{\bar{\gamma}_{q_0 t_0 q_1 t_1}}$$

$$\begin{cases} \left.\dfrac{\partial S}{\partial q_1^i}\right|_{\bar{\gamma}} = \bar{p}_1 \\[2ex] \left.\dfrac{\partial S}{\partial q_0^i}\right|_{\bar{\gamma}} = -\bar{p}_0 \end{cases}$$

$S|_{\bar\gamma} = 0$

② Reinsert $\bar\gamma_{t_0 q_0 t_1 q_1}$ into S to define Hamilton's principal f.

$$\mathcal{S}(t_0, q_0; t_1, q_1) = \int_{t_0}^{t_1} dt\, L|_{\bar\gamma_{t_0 q_0 t_1 q_1}} = S[\bar\gamma_{t_0 q_0 t_1 q_1}]$$



$$\bar\gamma' = \bar\gamma + \epsilon\,\delta\bar\gamma$$

$$\begin{cases} \bar\gamma'(t_0) = \bar\gamma(t_0) \\ \bar\gamma'(t_1 + \epsilon) = \bar\gamma(t_1) \end{cases}$$

Last

$\sum_i \xi^i \frac{\partial}{\partial}$

$$\begin{cases} \frac{\partial S}{\partial q} \\ \frac{\partial S}{\partial q} \end{cases}$$

Variation wrt time

$$\bar\gamma' := \bar\gamma + \epsilon\,\delta\bar\gamma$$

$$\text{with}\quad \begin{cases} \bar\gamma'(t_0) = \bar\gamma(t_0) \longrightarrow \\ \bar\gamma'(t_1+\epsilon) \overset{(*)}{=} \bar\gamma(t_1) \longrightarrow \end{cases} \boxed{\begin{aligned} &\delta\bar\gamma(t_0) = 0 \\ &\delta\bar\gamma(t_1) = -\dot{\bar\gamma}(t_1) \end{aligned}}$$

$$\begin{aligned} \bar\gamma'(t_1+\epsilon) &:= \bar\gamma(t_1+\epsilon) + \epsilon\,\delta\bar\gamma(t_1+\epsilon) \\ &= \left(\bar\gamma(t_1) + \epsilon\,\dot{\bar\gamma}(t_1) + O(\epsilon^2)\right) + \left(\epsilon\,\delta\bar\gamma(t_1) + O(\epsilon^2)\right) \\ &\overset{(*)}{=} \bar\gamma(t_1) \end{aligned}$$

$$\Rightarrow \delta\bar\gamma(t_1) = -\dot{\bar\gamma}(t_1) + O(\epsilon)$$

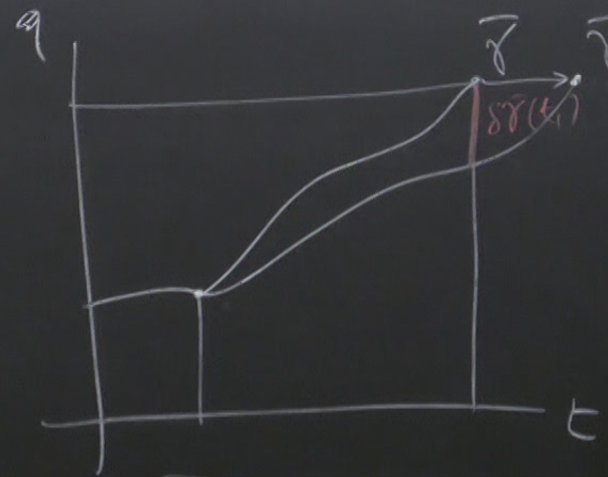
$\bar{\gamma}(t_0) = 0$

$\delta\bar{\gamma}(t_1) = -\dot{\bar{\gamma}}(t_1)$

$\bar{\gamma}(t_1+\epsilon)$

$) + O(\epsilon^2)\Big) + \Big(\epsilon\,\delta\bar{\gamma}(t_1) + O(\epsilon^2)\Big)$

(ϵ)



$$\int_{t_1}^{t_1+\epsilon} L|_{\bar{\gamma}'}\, dt$$
$$= \epsilon\, L|_{\bar{\gamma}'}(t_1)$$
$$= \epsilon\big(L|_{\bar{\gamma}} + O(\epsilon)\big)(t_1)$$

$$\frac{\partial}{\partial t_1} S = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\Big[\int_{t_0}^{t_1+\epsilon} dt$$
$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\Big[\int_{t_1}^{t_1+\epsilon} L$$
$$= L|_{\bar{\gamma}}(t_1) + \int_{t}$$
$$= L|_{\bar{\gamma}}(t_1) + \int_{t}$$

$$\frac{\partial}{\partial t_1} S = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\int_{t_0}^{t_1+\epsilon} dt \, L\big|_{\bar\gamma'} - \int_{t_0}^{t_1} dt \, L\big|_{\bar\gamma} \right]$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\underbrace{\int_{t_1}^{t_1+\epsilon} L\big|_{\bar\gamma'}} + \int_{t_0}^{t_1} \left(L\big|_{\bar\gamma'} - L\big|_{\bar\gamma} \right) \right]$$

$$= L\big|_{\bar\gamma}(t_1) + \int_{t_0}^{t_1} \delta L\big|_{\bar\gamma}$$

$$= L\big|_{\bar\gamma}(t_1) + \int_{t_0}^{t_1} \sum_i \boxed{\left(\frac{\partial L}{\partial \gamma^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot\gamma^i} \right)\bigg|_{\bar\gamma}} \delta\gamma^i + \left[\sum_i \frac{\partial L}{\partial \dot\gamma^i} \delta\gamma^i \right]_{t_0}^{t_1} \qquad \equiv 0 \ (\bar\gamma \text{ is on shell!})$$

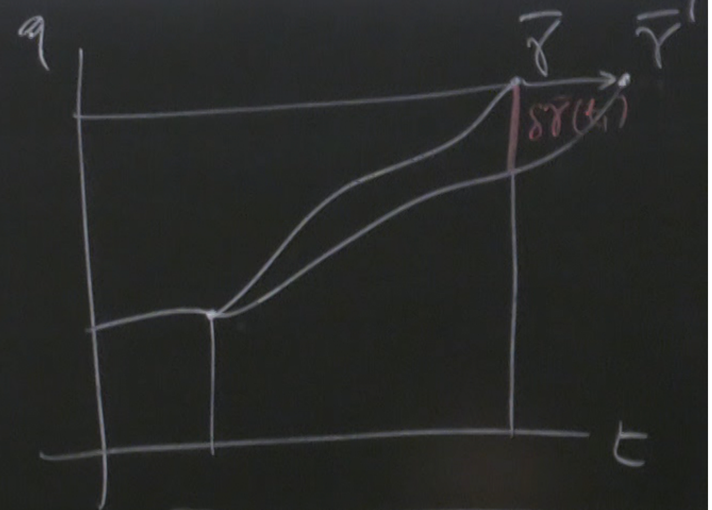
$$= \left(L - \sum_i \frac{\partial L}{\partial \dot\gamma} \dot\gamma \right)\bigg|_{\bar\gamma} (t_1)$$

time

$\delta\bar{\gamma}$

$$(t_0) = \bar{\gamma}(t_0) \longrightarrow \boxed{\begin{array}{l} \delta\bar{\gamma}(t_0) = 0 \\ \delta\bar{\gamma}(t_1) = -\dot{\bar{\gamma}}(t_1) \end{array}}$$

$$(t_1+\epsilon) \overset{(*)}{=} \bar{\gamma}(t_1) \longrightarrow$$



$$:= \bar{\gamma}(t_1+\epsilon) + \epsilon\,\delta\bar{\gamma}(t_1+\epsilon)$$

$$= \left(\bar{\gamma}(t_1) + \epsilon\,\dot{\bar{\gamma}}(t_1) + O(\epsilon^2)\right) + \left(\epsilon\,\delta\bar{\gamma}(t_1) + O(\epsilon^2)\right)$$

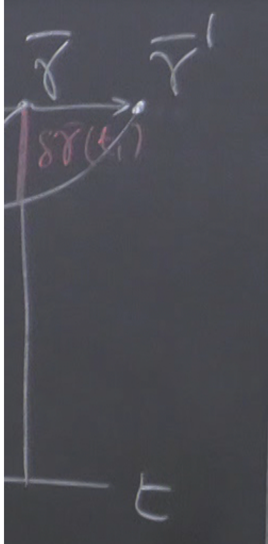
$$\overset{(*)}{=} \bar{\gamma}(t_1)$$

$$(t_1) = -\dot{\bar{\gamma}}(t_1) + O(\epsilon)$$

$$\int_{t_1}^{t_1+\epsilon} L|_{\bar{\gamma}'}\,dt$$

$$= \epsilon\, L|_{\bar{\gamma}'}(t_1)$$

$$= \epsilon\left(L|_{\bar{\gamma}} + O(\epsilon)\right)(t$$



dt

(t_1)

$+O(\epsilon))(t_1)$

$$\frac{\partial}{\partial t_1} S = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\int_{t_0}^{t_1+\epsilon} dt\, L|_{\bar{\gamma}'} - \int_{t_0}^{t_1} dt\, L|_{\bar{\gamma}}\right]$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\underbrace{\int_{t_1}^{t_1+\epsilon} L|_{\bar{\gamma}'}} + \int_{t_0}^{t_1}\left(L|_{\bar{\gamma}'} - L|_{\bar{\gamma}}\right)\right]$$

$$= L|_{\bar{\gamma}}(t_1) + \int_{t_0}^{t_1} \delta L|_{\bar{\gamma}}$$

$$= L|_{\bar{\gamma}}(t_1) + \int_{t_0}^{t_1} \sum_i \boxed{\left(\frac{\partial L}{\partial \gamma^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\gamma}^i}\right)\Big|_{\bar{\gamma}}} \delta\gamma^i + \left[\sum_i \frac{\partial L}{\partial \dot{\gamma}^i}\delta\gamma^i\right]_{t_0}^{t_1}$$

$\equiv 0$ ($\bar{\gamma}$ is on shell!)

$$(\star) \quad = \left(L - \sum_i \frac{\partial L}{\partial \dot{\gamma}^i}\dot{\gamma}^i\right)\Big|_{\bar{\gamma}}(t_1) \equiv E|_{\bar{\gamma}}(t_1)$$

$$\Rightarrow \begin{cases} \dfrac{\partial S}{\partial t_1} = -E(t_1) = -\sum_i \bar{p}_{1i}\,\dot{\bar{q}}^{i}_1 - L|_\gamma(t_1) \equiv -H(\bar{q}_1, \bar{p}_1, t_1) \qquad \dfrac{\partial S}{\partial q_1} = \bar{p}_1 \\ \dfrac{\partial S}{\partial t_0} = E(t_0) \end{cases}$$

$\bar{q}_1 \approx q_1$ the boundary condition! (argument of S)

To conclude, putting the two results together:

$$\frac{\partial S}{\partial t_1} + H\left(q_1, \frac{\partial S}{\partial q_1}, t_1\right) = 0$$

Hamilton-Jacobi

$$= -\sum_i \bar{p}_{1i}\,\dot{\bar{q}}^1_i - L|_{\bar\gamma}(t_1) \equiv -H(\bar{q}_1, \bar{p}_1, t_1) \qquad \frac{\partial S}{\partial q_1} = \bar{p}_1$$

q_1 the boundary condition!

(argument of S)

$\circ)$

putting the two results together:

$$H\left(q_1, \frac{\partial S}{\partial q_1}, t_1\right) = 0$$

lton-Jacobi

equivalent formulation of mechanics, but fully encoded in boundary condition

...ulation of
...ut fully
...ndary condition

Schrödinger eq

$$-i\hbar \frac{\partial \psi}{\partial t} + H(\hat{q}, \hat{p}, t)\psi = 0$$

In position representation:

$$-i\hbar \frac{\partial \psi(q)}{\partial t} + H\left(q, -i\hbar \frac{\partial}{\partial q}, t\right)\psi = 0 \quad (✿)$$

$$\psi = \rho\, e^{+\frac{i}{\hbar}\varphi}$$

functions of (q, t)

$\longrightarrow$ plug in (✿) and keep only leading order in $\hbar$ (i.e. $O(\hbar^0) = O(1)$)

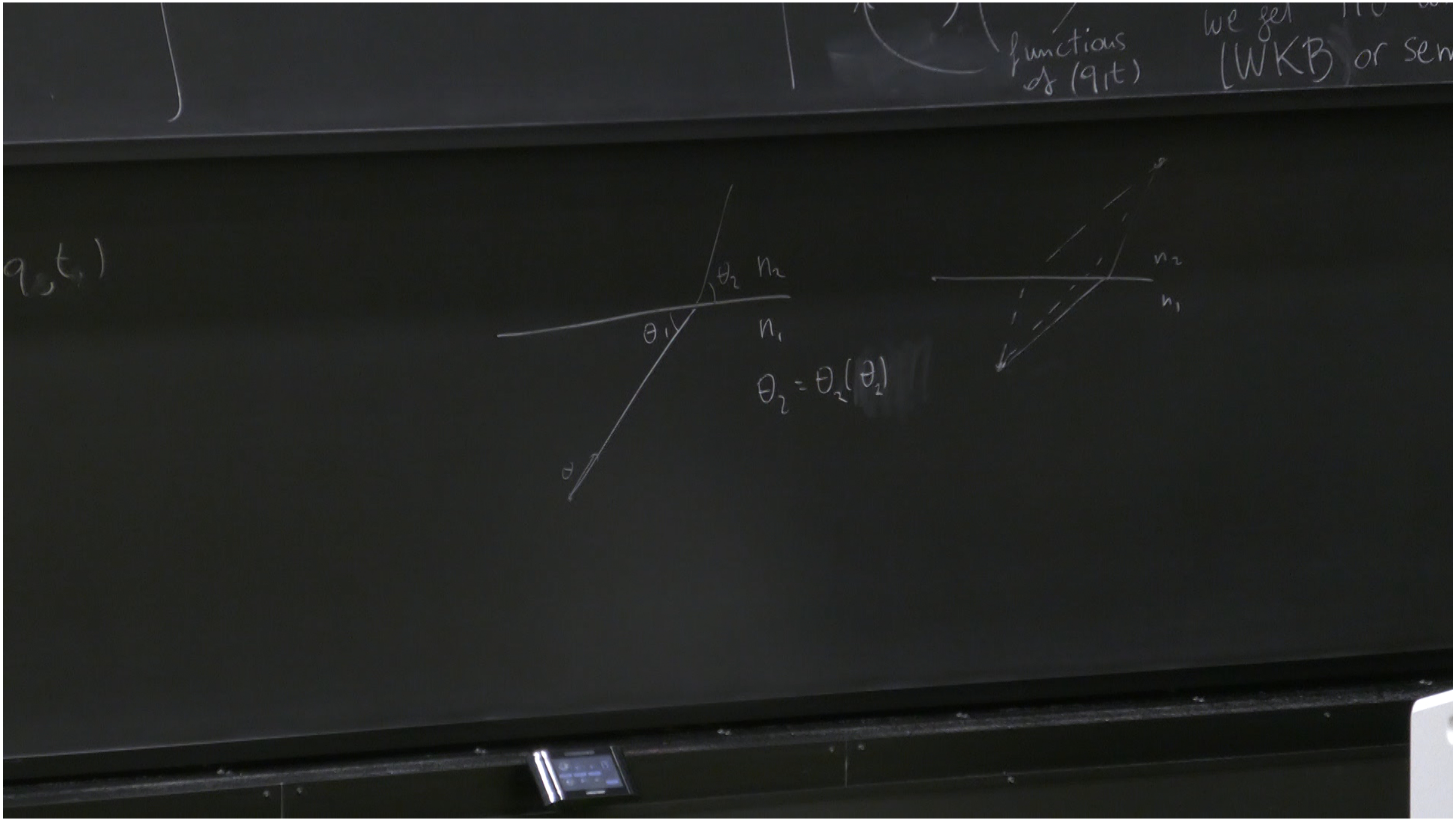
we get HJ with $S \to \varphi$

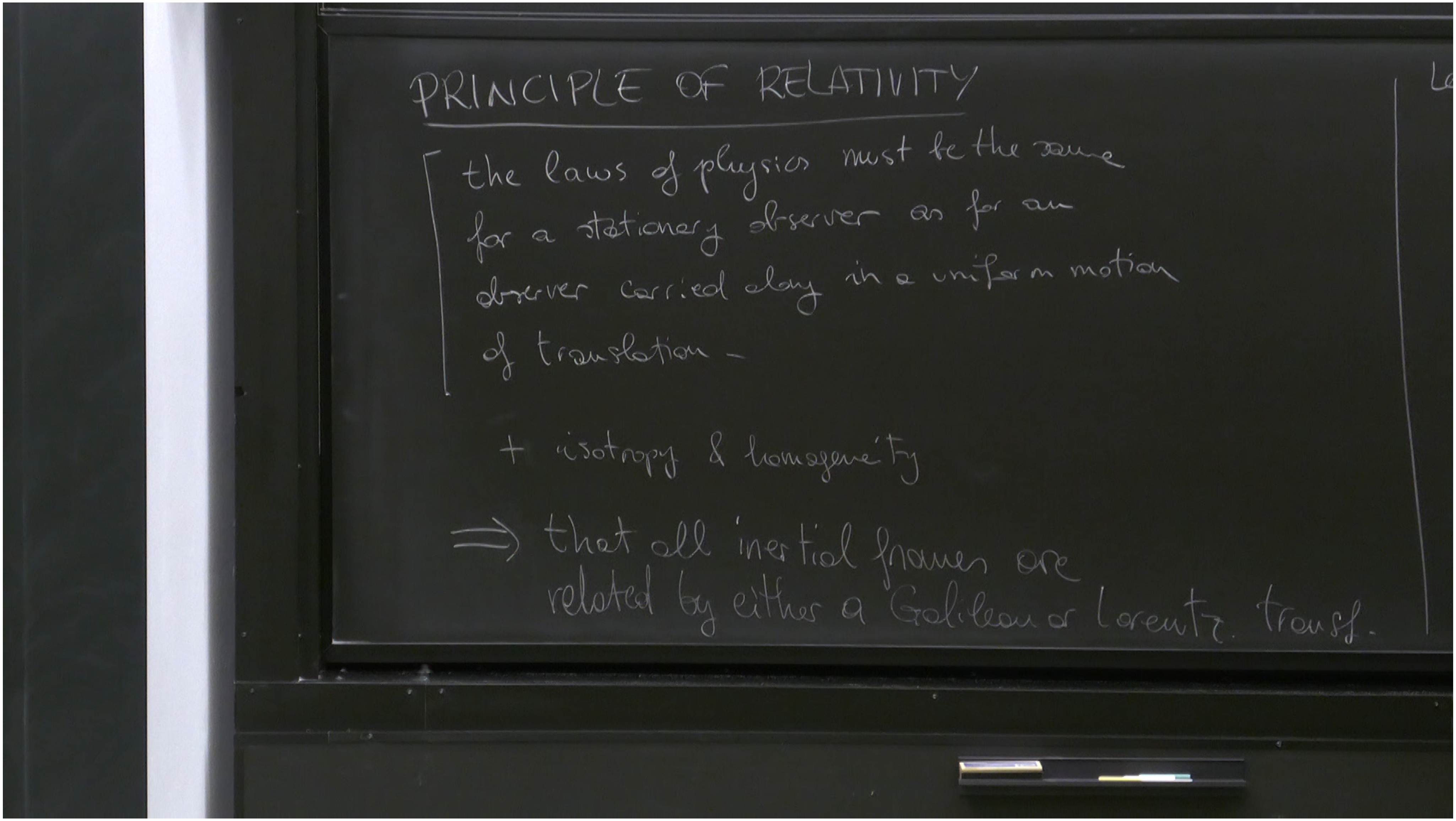
(WKB or semiclassical approx)

Hamilton-Jacobi

- fix q_0 & t_0
- find $S(q, t, q_0, t_0)$ using HJ in (q, t)

$\Rightarrow$ then deduce $p(t)$ by $\frac{\partial S}{\partial q}$


functions
of (q,t)
we get
[WKB or sem
q,t)
θ2 n2
θ1 n1
θ2 = θ2(θ2)
n2
n1


PRINCIPLE OF RELATIVITY
the laws of physics must be the same
for a stationary observer as for an
observer carried along in a uniform motion
of translation -
+ isotropy & homogeneity
⟹ that all inertial frames are
related by either a Galilean or Lorentz transf.

ITY

t be the same

as for an

e uniform motion

y

rames are
Galileau or Lorentz. transf.

Let's work in 1 spatial dim.

If $K \ni (x,t)$ are inertial frames
$K' \ni (x',t')$

with ① $(x=0, t=0) \equiv (x'=0, t'=0)$ common origin O

② K' in moving at velocity v wrt K

$$\rightarrow \begin{cases} x' = \gamma(v)(x - vt) \\ t' = \gamma(v)\left(t - \frac{v}{c^2}x\right) \end{cases}$$

Lorentz transf.

$\gamma(v) =$

Lorentz f

l dim.

inertial frames

$\equiv (x'=0, t'=0)$ common origin O

ing at velocity v wrt k

$-vt)$

$t - \frac{v}{c^2} x)$

ost

$$\gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Lorentz factor

$v^2 < c^2$

$$\xrightarrow[\frac{v}{c^2} \to b]{c \to 0} \quad \begin{cases} x' = x \\ t' = t - b x \end{cases}$$

Carrollian transf.

$$\xrightarrow{c \to \infty} \quad \begin{cases} x' = x - vt \\ t' = t \end{cases}$$

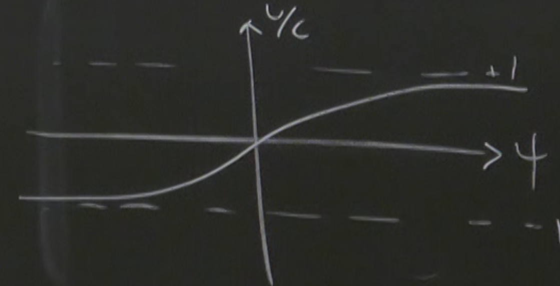
Galilean boost

related by either a Galilean or Lorentz transf.

Since $-1 < \frac{v}{c} < 1$

$\rightsquigarrow \quad \mathrm{th}\,\psi = \frac{v}{c}$, $\quad \psi = \mathrm{arcth}\left(\frac{v}{c}\right)$ "rapidity"

$$\mathrm{th}\,\psi := \frac{e^{\psi} - e^{-\psi}}{e^{\psi} + e^{-\psi}}$$



$$\mathrm{ch}\,\psi = \frac{e^{\psi} + e^{-\psi}}{2} > 0$$

$$\mathrm{sh}\,\psi = \frac{e^{\psi} - e^{-\psi}}{2}$$

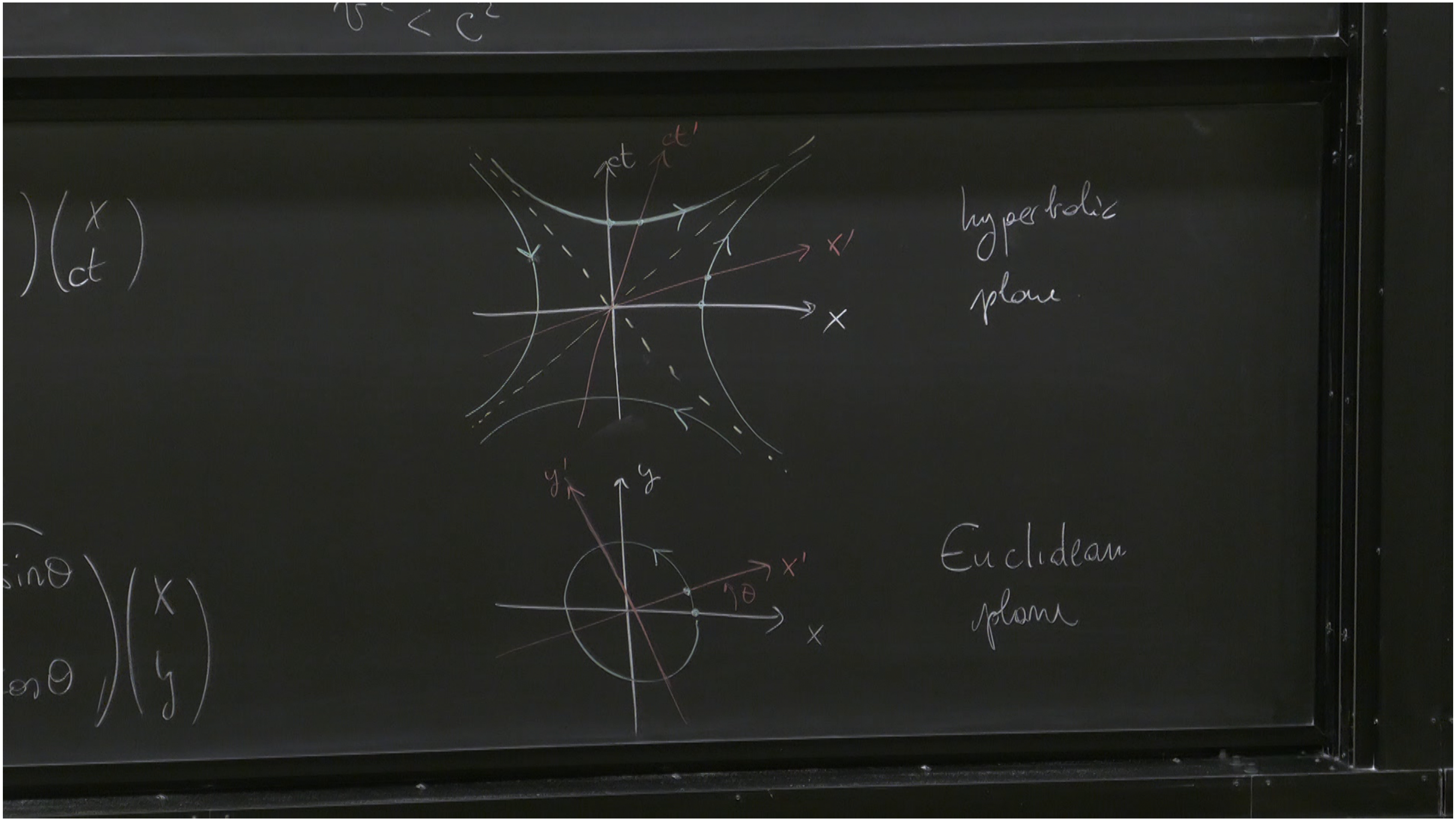
$$\mathrm{ch}^2\psi - \mathrm{sh}^2\psi = 1$$

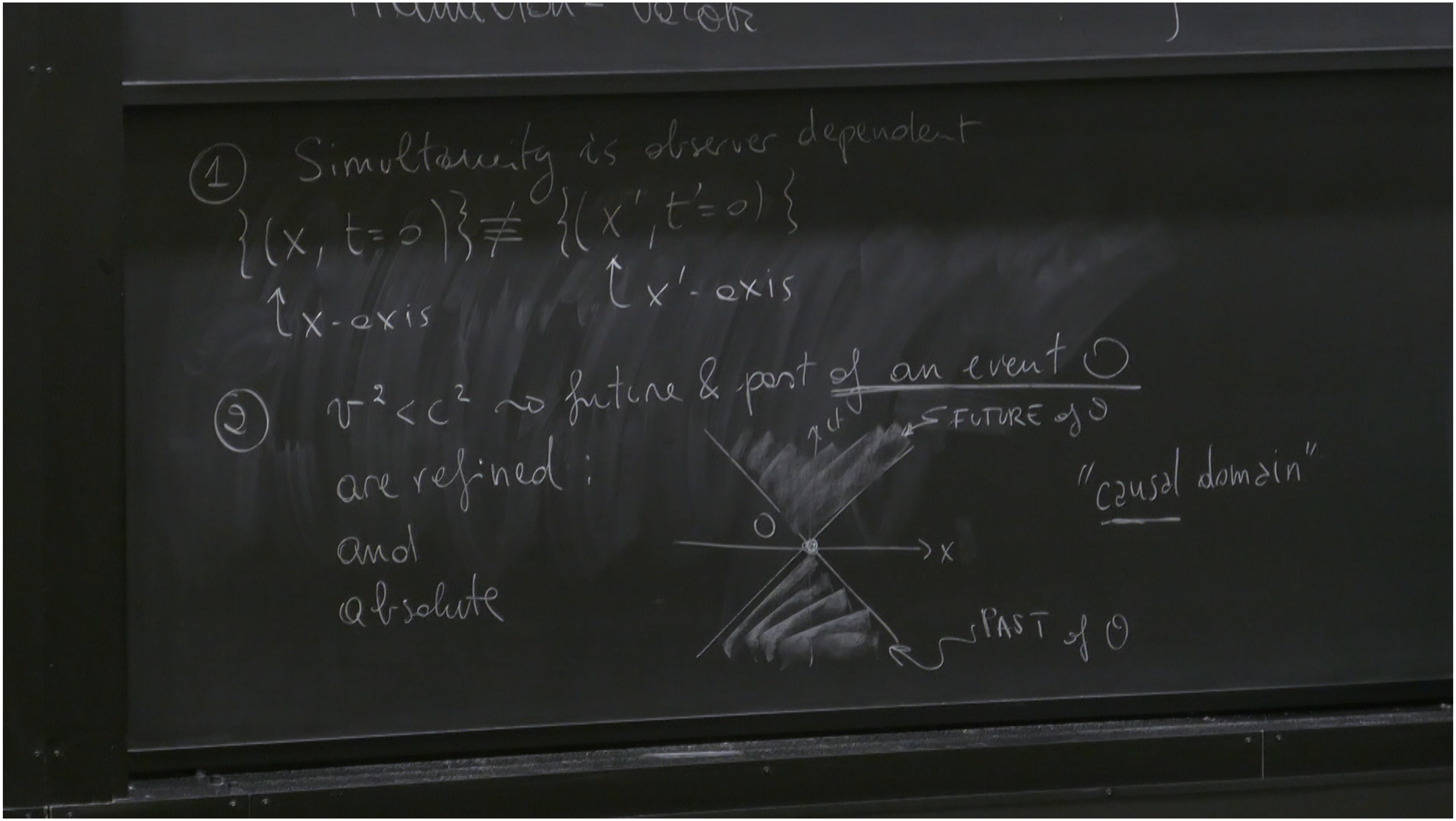
...nsf. | Lorentz boost

...entz factor
$v^2 < c^2$

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \underbrace{\begin{pmatrix} \operatorname{ch}\psi & -\operatorname{sh}\psi \\ -\operatorname{sh}\psi & \operatorname{ch}\psi \end{pmatrix}}_{\Lambda(\psi)} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \overbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}^{R(\theta)} \begin{pmatrix} x \\ y \end{pmatrix}$$


ct
ct'
x'
x
hyperbolic
plane
y'
y
θ
x'
x
Euclidean
plane


① Simultaneity is observer dependent
{(x, t=0)} ≢ {(x', t'=0)}
x-axis
x'-axis
② v² < c² ⤳ future & past of an event O
are refined :
and
absolute
ct
FUTURE of O
"causal domain"
O
x
PAST of O