Title: Classical Physics Lecture - 091123

Speakers: Aldo Riello

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Recap

- config space $Q \ni q$
- tangent space $TQ \ni (q, v)$ ← velocity
- phase space (cotangent sp.) $P = T^*Q \ni (q, p)$
- histories $\gamma : \mathbb{R} \longrightarrow Q$, $t \longmapsto q = \gamma(t)$ $(\gamma \in C^\infty(\mathbb{R}, Q))$
- "phase space history" $z : \mathbb{R} \longrightarrow P$, $t \longmapsto (q(t), p(t)) = z(t)$

- Lagrangian function
  $L : TQ \times \mathbb{R} \longrightarrow \mathbb{R}$
  $(q, v, t) \longmapsto L(q, v, t)$
- Action functional ← function on space of histories
  $S : C^\infty(\mathbb{R}, Q) \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$
  $(\gamma, t_0, t_1) \longmapsto S =$
- Hamilton's action princ'ple
- → physically realized histories are at fixed boundary conditions

- On phase space $P = T^*Q$
- Hamilton's e.o.m.

- Lagrangian function

$$L: TQ \times \mathbb{R} \to \mathbb{R}, \quad (q, v, t) \mapsto L(q, v, t)$$

- Action functional (function on space of histories)

$$S: C^\infty(\mathbb{R}, Q) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$
$$(\gamma, t_0, t_1) \longmapsto S = \int_{t_0}^{t_1} dt\ L(\gamma(t), \dot{\gamma}(t), t)$$

- Hamilton's action principle:

→ physically realized histories are extrema of the action, at fixed boundary conditions (in Q): $\delta S|_{\bar{\gamma}} = 0$, $\delta\gamma(t_0) = \delta\gamma(t_1) = 0$

↑ cf. tutorial!

- Hamilton action principle

⇒ Euler-Lagrange e.o.m.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \hat{=} 0$$

(kinetic, potential energy)

(equivalent to Newton if $L = T - V$)

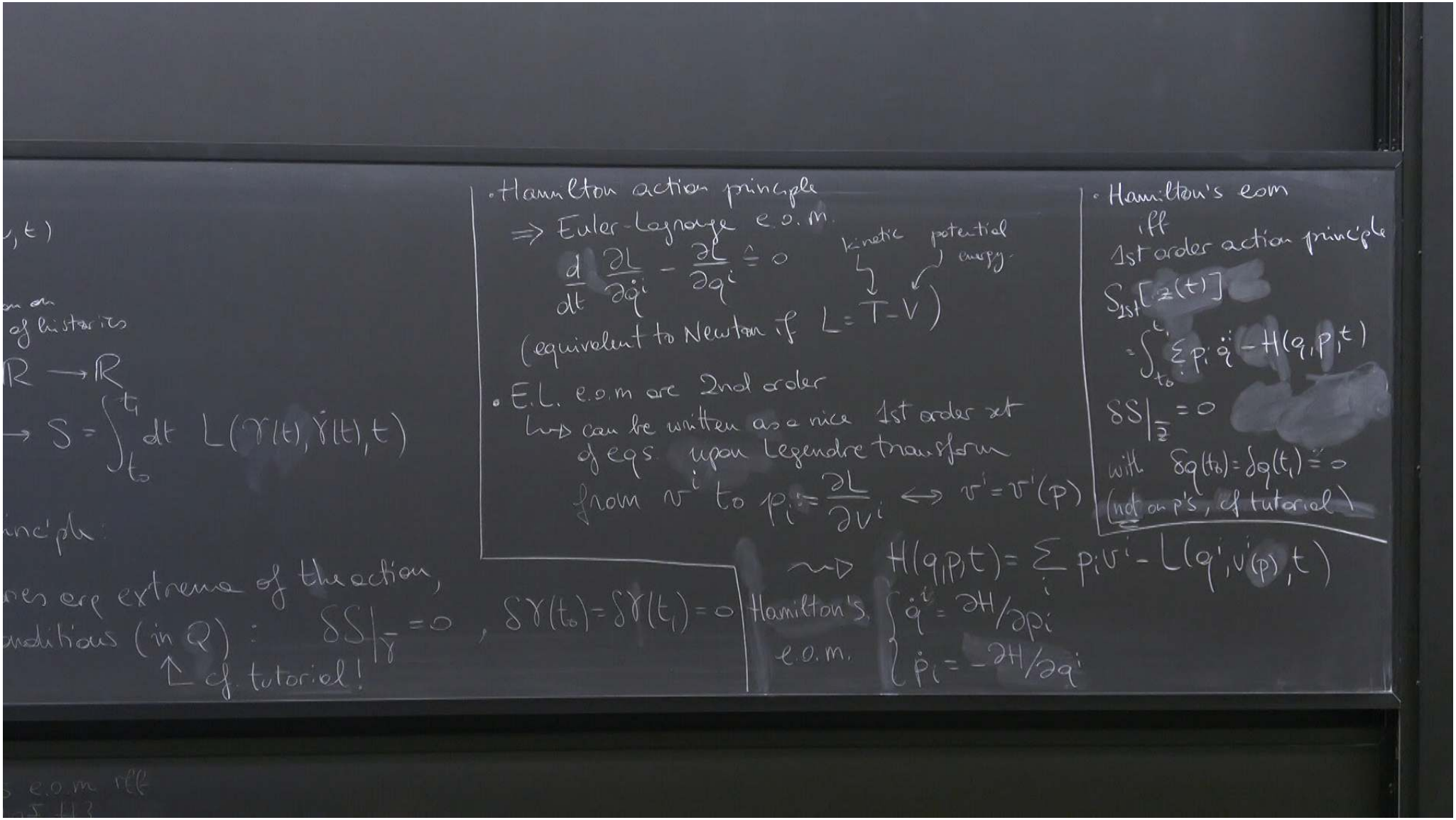
- E.L. e.o.m are 2nd order
  ↳ can be written as a nice 1st order set of eqs. upon Legendre transform from v^i to $p_i := \frac{\partial L}{\partial v^i} \leftrightarrow v^i = v^i(p)$

⇝ Hamilton's e.o.m.

$$H(q, p, t) = \sum_i p_i v^i$$

$$\begin{cases} \dot{q}^i = \partial H / \partial p_i \\ \dot{p}_i = -\partial H / \partial q^i \end{cases}$$

- Hamilton's e.o.m rel


, t)
of histories
$\mathbb{R} \longrightarrow \mathbb{R}$
$\rightarrow S = \int_{t_0}^{t_1} dt\ L(\gamma(t), \dot{\gamma}(t), t)$
inciple:
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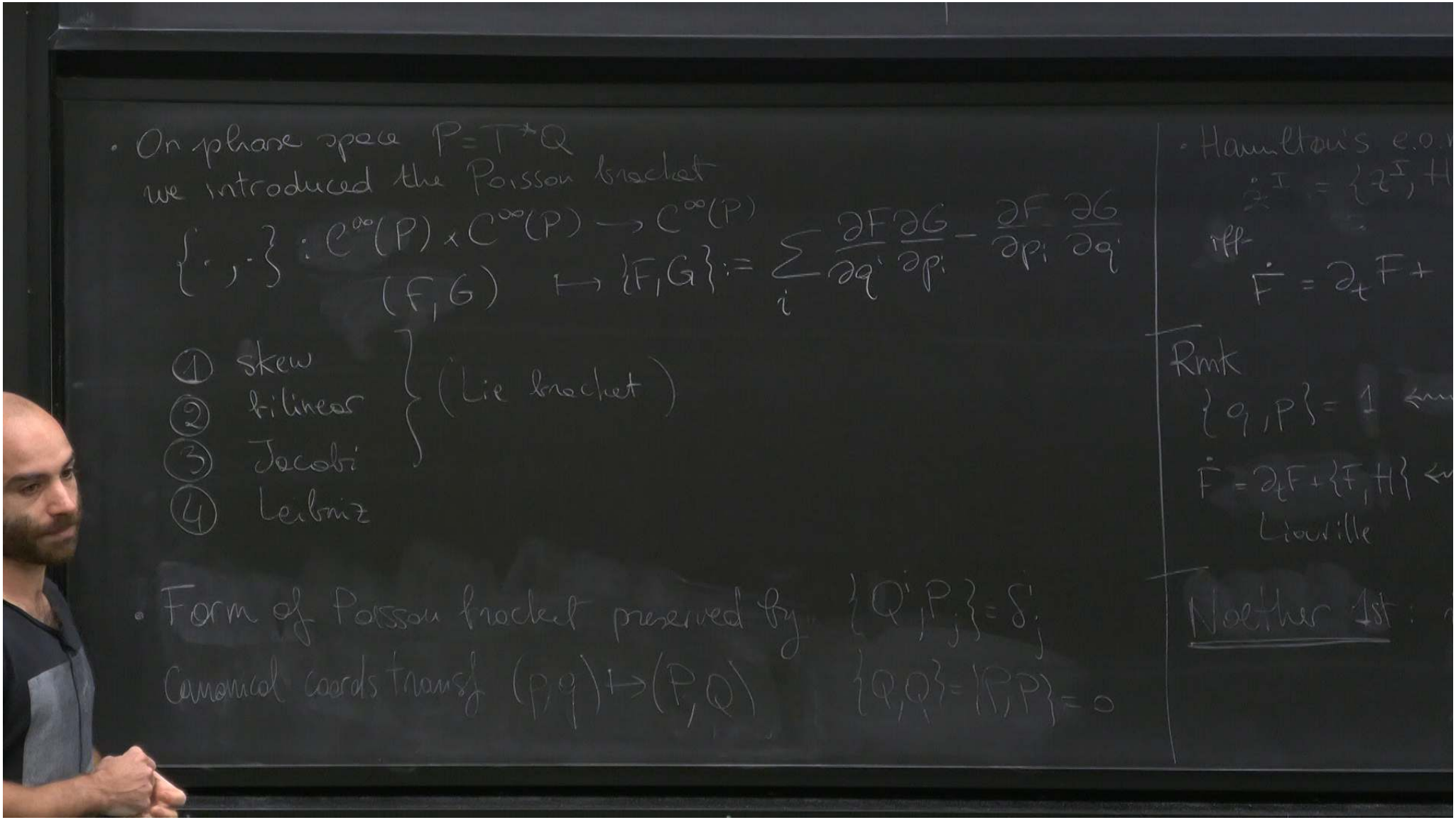
↑ cf. tutorial!
• Hamilton action principle
⇒ Euler-Lagrange e.o.m.
$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \doteq 0$
kinetic potential energy
(equivalent to Newton if $L = T - V$)
• E.L. e.o.m are 2nd order
↳ can be written as a nice 1st order set of eqs. upon Legendre transform from v^i to $p_i := \frac{\partial L}{\partial v^i} \leftrightarrow v^i = v^i(p)$
⇝ $H(q,p,t) = \sum_i p_i v^i - L(q^i, v^i(p), t)$
Hamilton's e.o.m. $\begin{cases} \dot{q}^i = \partial H/\partial p_i \\ \dot{p}_i = -\partial H/\partial q^i \end{cases}$
• Hamilton's eom iff 1st order action principle
$S_{1st}[z(t)] = \int_{t_0}^{t_1} \sum_i p_i \dot{q}^i - H(q,p,t)$
$\delta S|_{\bar{z}} = 0$
with $\delta q(t_0) = \delta q(t_1) \doteq 0$
(not on p's, cf tutorial)
e.o.m iff


• On phase space $P = T^*Q$ we introduced the Poisson bracket
$\{\cdot,\cdot\} : C^\infty(P) \times C^\infty(P) \to C^\infty(P)$
$(F,G) \mapsto \{F,G\} := \sum_i \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i}$
① skew
② bilinear
③ Jacobi
④ Leibniz
(Lie bracket)
• Form of Poisson bracket preserved by: $\{Q^i, P_j\} = \delta^i_j$
Canonical coords transf $(p,q) \mapsto (P,Q)$ $\{Q,Q\} = \{P,P\} = 0$
• Hamilton's e.o.
iff
$\dot{F} = \partial_t F +$
Rmk
$\{q,p\} = 1 \Leftrightarrow$
$\dot{F} = \partial_t F + \{F,H\} \Leftrightarrow$
Liouville
Noether 1st :

- Hamilton's e.o.m iff

$$\dot{z}^I = \{z^I, H\}$$

iff

$$\dot{F} = \partial_t F + \{F, H\} \quad \text{(Liouville eq.)}$$

Rmk

$$\{q, p\} = 1 \iff [\hat{q}, \hat{p}] = i\hbar$$

$$\dot{F} = \partial_t F + \{F, H\} \iff \frac{d}{dt}\langle \hat{F} \rangle = \langle \partial_t \hat{F} \rangle + \frac{1}{i\hbar}\langle [\hat{F}, \hat{H}] \rangle$$

Liouville — Ehrenfest.

Noether 1st: infinitesimal sym $\Rightarrow$ conserved quantity

$$\tilde{\delta}_s L = \frac{d}{dt} R_s \qquad Q_s = \sum_i p_i \tilde{\delta}_s q^i - R_s$$

Liouville thm

Volume form on phase space which in canonical coords reads

$$d\mathrm{vol}_P = dq^1 dp_1 \cdots dq^n dp_n \quad \text{(Liouville form)}$$

$$= dQ^1 dP_1 \cdots dQ^n dP_n$$

Liouville thm

Volume form on phase space which in canonical coords reads

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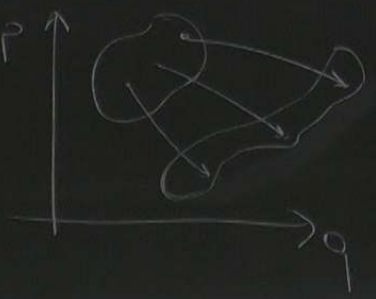
Rmk: time evolution is a canonical transf. i.e. time evol. preserve phase space Vol.

$\Rightarrow$



time evol from t_0 to t_1

$$\mathrm{Vol}|_{t_0} = \mathrm{Vol}|_{t_1}$$

$$P(V) = \int_{V \subset P} \rho \, d\mathrm{vol}_P$$

Symmetries in phase space

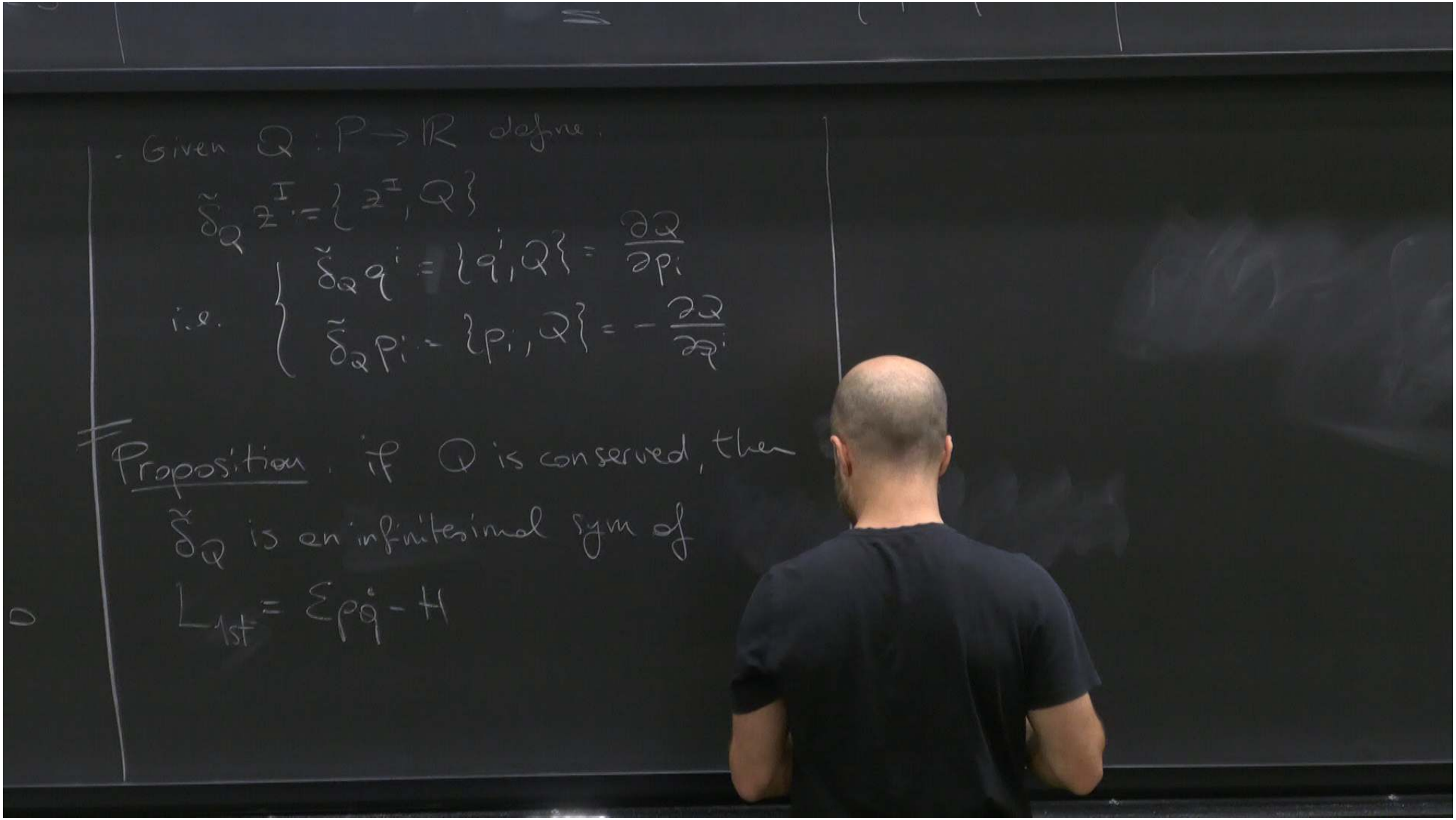
idea: reverse N. thm

conserved q $\longrightarrow$ symmetry.

- conserved quantity $Q_s : P \times \mathbb{R} \to \mathbb{R}$
  $(q,p,t) \mapsto Q_s(q,p,t)$

$$0 = \frac{d}{dt} Q_s \Big|_{\vec{z}} = \partial_t Q + \{Q, H\}$$

if $\partial_t Q = 0$ then Q is conserved iff $\{Q, H\} = 0$


Given $Q : P \to \mathbb{R}$ define:
$\check{\delta}_Q z^I := \{z^I, Q\}$
i.e. $\begin{cases} \check{\delta}_Q q^i = \{q^i, Q\} = \frac{\partial Q}{\partial p_i} \\ \check{\delta}_Q p_i = \{p_i, Q\} = -\frac{\partial Q}{\partial q^i} \end{cases}$
Proposition: if Q is conserved, then
$\check{\delta}_Q$ is an infinitesimal sym of
$L_{1st} = \Sigma p\dot{q} - H$

$Q_S = \sum_i p_i \delta q^i - R_S$ $\longrightarrow q$ $\mathrm{Vol}|_{t_0} = \mathrm{Vol}|_{t_1}$ Vol.

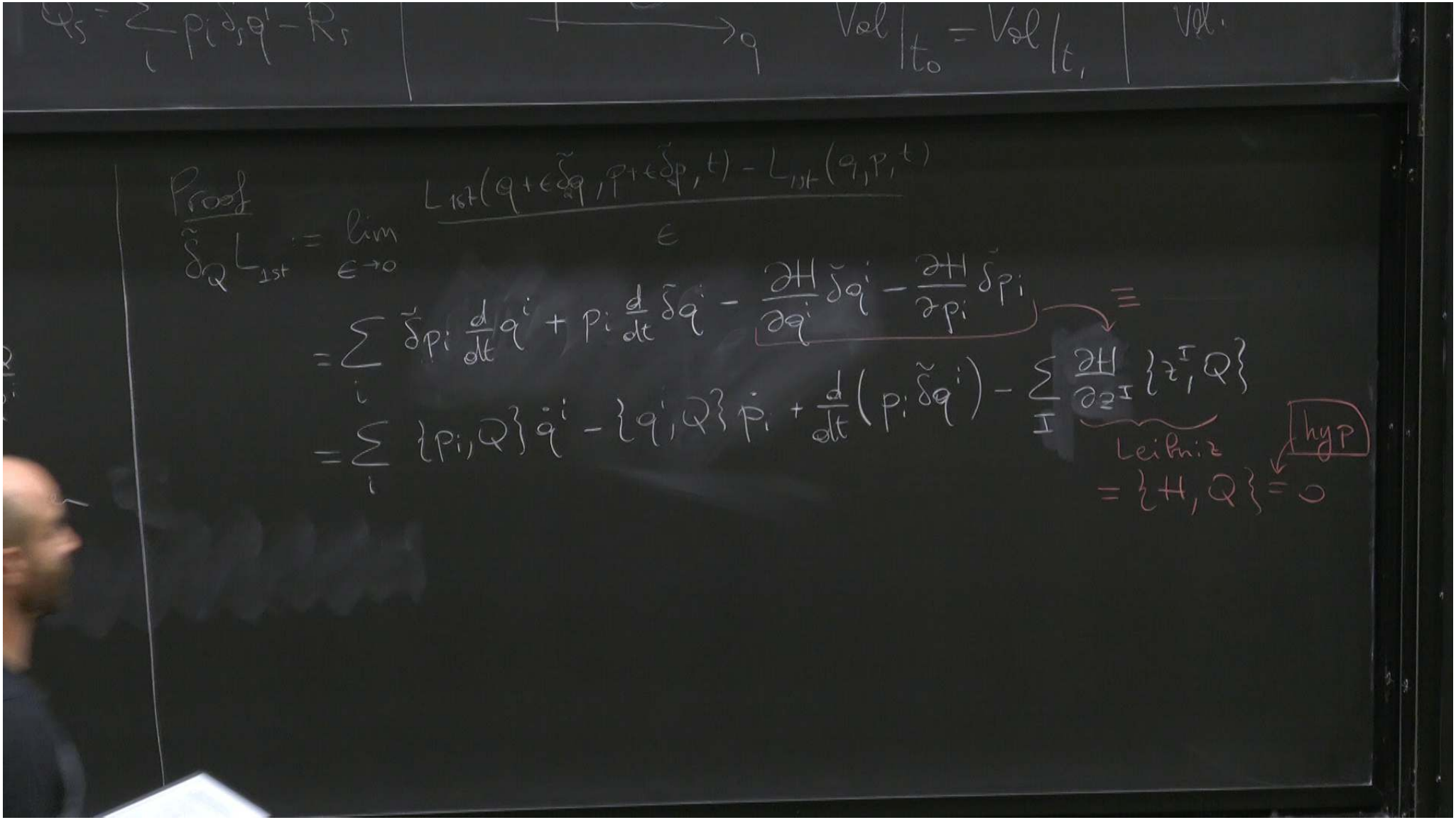
Proof

$$\tilde{\delta}_Q L_{1st} := \lim_{\epsilon \to 0} \frac{L_{1st}(q + \epsilon \tilde{\delta} q, p + \epsilon \tilde{\delta} p, t) - L_{1st}(q, p, t)}{\epsilon}$$

$$= \sum_i \tilde{\delta} p_i \frac{d}{dt} q^i + p_i \frac{d}{dt} \tilde{\delta} q^i - \frac{\partial H}{\partial q^i} \tilde{\delta} q^i - \frac{\partial H}{\partial p_i} \tilde{\delta} p_i$$

$$= \sum_i \{p_i, Q\} \dot{q}^i - \{q^i, Q\} \dot{p}_i + \frac{d}{dt}\left(p_i \tilde{\delta} q^i\right) - \sum_I \frac{\partial H}{\partial z^I} \{z^I, Q\}$$

then


Proof
$\tilde{\delta}_Q L_{1st} := \lim_{\epsilon \to 0} \frac{L_{1st}(q+\epsilon\tilde{\delta}q, p+\epsilon\tilde{\delta}p, t) - L_{1st}(q,p,t)}{\epsilon}$
$= \sum_i \tilde{\delta}p_i \frac{d}{dt} q^i + p_i \frac{d}{dt}\tilde{\delta}q^i - \frac{\partial H}{\partial q^i}\tilde{\delta}q^i - \frac{\partial H}{\partial p_i}\tilde{\delta}p_i \equiv$
$= \sum_i \{p_i, Q\}\dot{q}^i - \{q^i, Q\}\dot{p}_i + \frac{d}{dt}(p_i \tilde{\delta}q^i) - \sum_I \frac{\partial H}{\partial z^I}\{z^I, Q\}$
Leibniz
hyp
$= \{H, Q\} = 0$

$Q_S = \sum_i p_i \delta q^i - R_S$ | $\longrightarrow q$ | $\mathrm{Vol}|_{t_0} = \mathrm{Vol}|_{t_1}$ | Vol.

Proof

$$\tilde{\delta}_Q L_{1st} := \lim_{\epsilon \to 0} \frac{L_{1st}(q+\epsilon\tilde{\delta}q, p+\epsilon\tilde{\delta}p, t) - L_{1st}(q,p,t)}{\epsilon}$$

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$$= \sum_i \{p_i, Q\}\dot{q}^i - \{q^i, Q\}\dot{p}_i + \frac{d}{dt}(p_i \tilde{\delta}q^i) - \sum_I \frac{\partial H}{\partial z^I}\{z^I, Q\}$$

Leibniz $= \{H, Q\} = 0$ (hyp)

$$(\partial_t Q \equiv 0) \quad = -\sum_i \frac{\partial Q}{\partial q^i}\dot{q}^i + \frac{\partial Q}{\partial p_i}\dot{p}_i + \frac{d}{dt}(p_i \tilde{\delta}q^i)$$

$$= \frac{d}{dt}\left(-Q + \sum_i p_i \tilde{\delta}q^i\right)$$

then

$Q_S = \sum_i p_i \delta q^i - R_S$ | $\longrightarrow q$ | $Vol|_{t_0} = Vol|_{t_1}$ | Vol.

Proof

$$\tilde{\delta}_Q L_{1st} := \lim_{\epsilon \to 0} \frac{L_{1st}(q+\epsilon\tilde{\delta}q, p+\epsilon\tilde{\delta}p, t) - L_{1st}(q,p,t)}{\epsilon}$$

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$$= \sum_i \{p_i, Q\}\dot{q}^i - \{q^i, Q\}\dot{p}_i + \frac{d}{dt}(p_i \tilde{\delta}q^i) - \sum_I \frac{\partial H}{\partial z^I}\{z^I, Q\}$$

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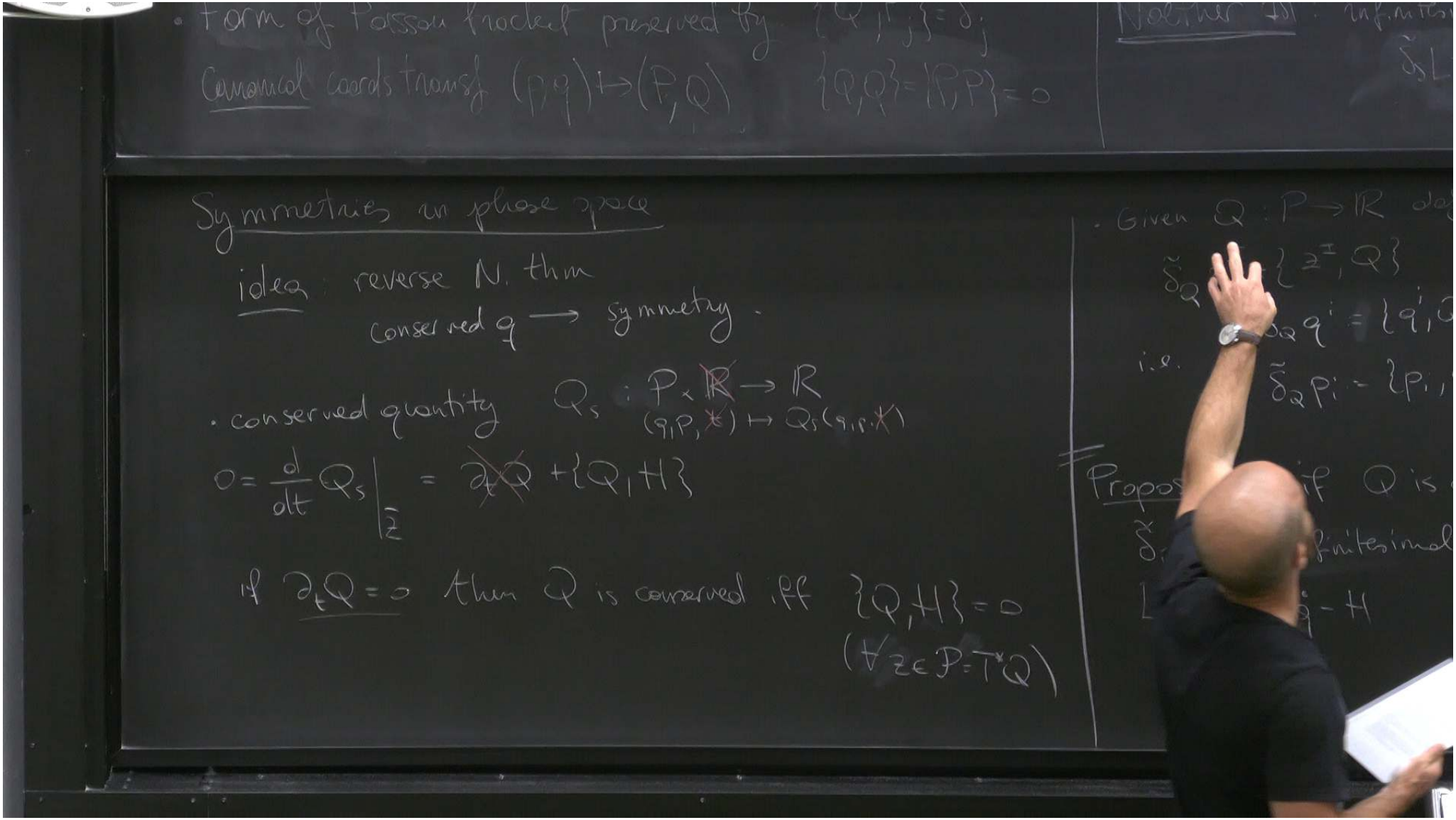
$$= \frac{d}{dt}\left(-Q + \sum_i p_i \tilde{\delta}q^i\right) \equiv \frac{d}{dt} R_S \rightarrow Q = \sum_i p_i \tilde{\delta}q^i - R_S$$

☑


form of Poisson bracket preserved by
Canonical coords transf $(p,q) \mapsto (P,Q)$
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Noether
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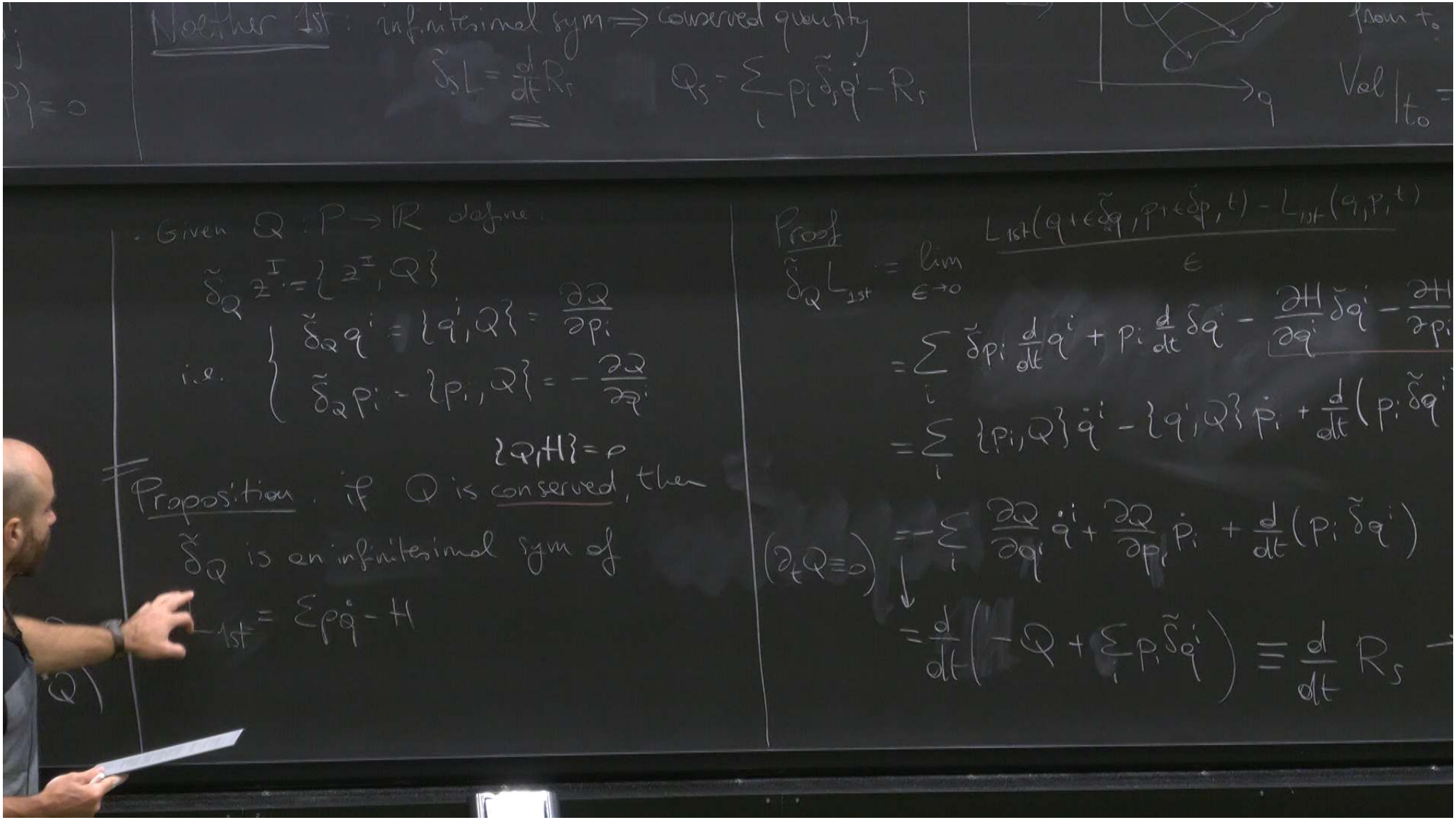
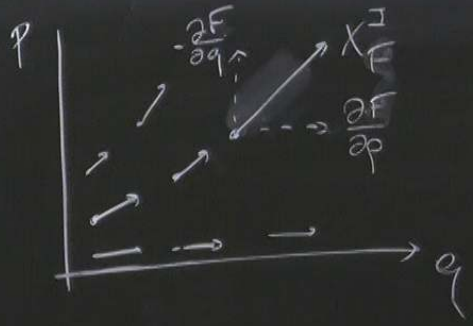
$(q,p,t) \mapsto Q_s(q,p,t)$
$0 = \frac{d}{dt} Q_s \Big|_{\bar{z}} = \partial_t Q + \{Q, H\}$
if $\partial_t Q = 0$ then Q is conserved iff $\{Q,H\} = 0$
$(\forall z \in P = T^*Q)$
• Given $Q : P \to \mathbb{R}$
$\delta_Q z^I = \{z^I, Q\}$
i.e.
$\delta_Q q^i = \{q^i,$
$\delta_Q p_i = \{p_i,$
Propos
if Q is
finitesimal
$q - H$


Noether 1st: infinitesimal sym ⇒ conserved quantity
$\tilde{\delta}_S L = \frac{d}{dt} R_S$
$Q_S = \sum_i p_i \tilde{\delta}_S q^i - R_S$
from t_0
q
$\mathrm{Vol}|_{t_0}$
Given $Q : P \to \mathbb{R}$ define:
$\tilde{\delta}_Q z^I := \{z^I, Q\}$
i.e. $\begin{cases} \tilde{\delta}_Q q^i = \{q^i, Q\} = \frac{\partial Q}{\partial p_i} \\ \tilde{\delta}_Q p_i = \{p_i, Q\} = -\frac{\partial Q}{\partial q^i} \end{cases}$
$\{Q, H\} = 0$
Proposition: if Q is conserved, then $\tilde{\delta}_Q$ is an infinitesimal sym of $L_{1st} = \Sigma p \dot{q} - H$
Proof
$\tilde{\delta}_Q L_{1st} := \lim_{\epsilon \to 0} \frac{L_{1st}(q + \epsilon \tilde{\delta}_Q q, p + \epsilon \tilde{\delta}_Q p, t) - L_{1st}(q, p, t)}{\epsilon}$
$= \sum_i \tilde{\delta} p_i \frac{d}{dt} q^i + p_i \frac{d}{dt} \tilde{\delta} q^i - \frac{\partial H}{\partial q^i} \tilde{\delta} q^i - \frac{\partial H}{\partial p_i}$
$= \sum_i \{p_i, Q\} \dot{q}^i - \{q^i, Q\} \dot{p}_i + \frac{d}{dt}(p_i \tilde{\delta} q^i$
$(\partial_t Q \equiv 0) = -\sum_i \frac{\partial Q}{\partial q^i} \dot{q}^i + \frac{\partial Q}{\partial p_i} \dot{p}_i + \frac{d}{dt}(p_i \tilde{\delta} q^i)$
$= \frac{d}{dt}\left(-Q + \sum_i p_i \tilde{\delta} q^i\right) \equiv \frac{d}{dt} R_S$

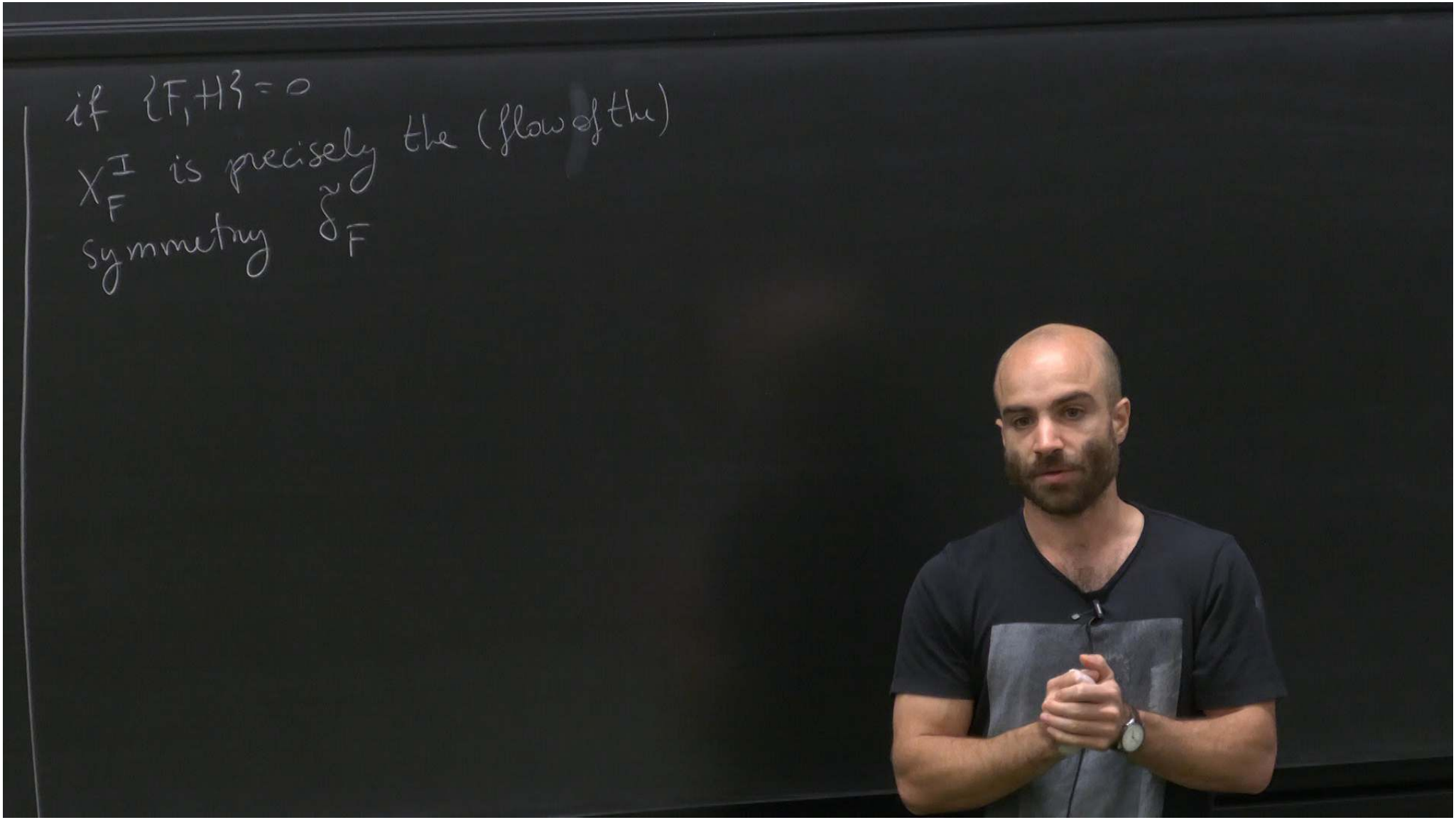


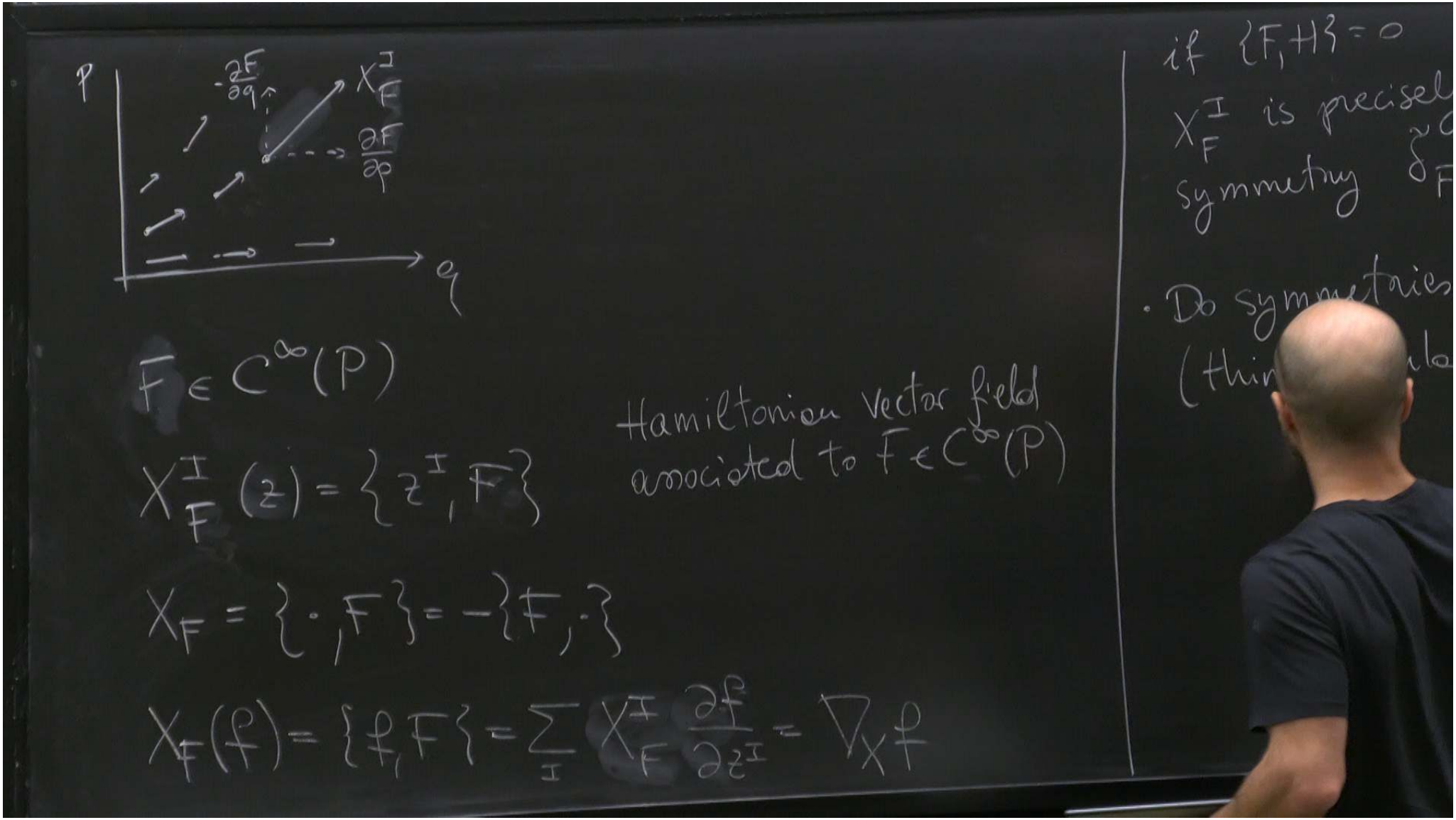
$F \in C^\infty(P)$

$X_F^I(z) = \{z^I, F\}$ Hamiltonian vector field associated to $F \in C^\infty(P)$

$X_F = \{\cdot, F\} = -\{F, \cdot\}$

$X_F(f) = \{f, F\} = \sum_I X_F^I \frac{\partial f}{\partial z^I} = \nabla_X f$


if $\{F, H\} = 0$
X_F^I is precisely the (flow of the)
Symmetry $\tilde{\delta}_F$


p
q
$-\frac{\partial F}{\partial q}$
X_F^I
$\frac{\partial F}{\partial p}$
$F \in C^\infty(P)$
$X_F^I(z) = \{z^I, F\}$
Hamiltonian vector field associated to $F \in C^\infty(P)$
$X_F = \{\cdot, F\} = -\{F, \cdot\}$
$X_f(f) = \{f, F\} = \sum_I X_F^I \frac{\partial f}{\partial z^I} = \nabla_X f$
if $\{F, H\} = 0$
X_F^I is precisel
symmetry δ_F
• Do symmetries
(thin

if $\{F, H\} = 0$

X_F^I is precisely the (flow of the) symmetry $\tilde{\delta}_F$

- Do symmetries commute?
  (think angular mom. & spin quantiz.)

$$X_G(X_F(f)) - X_F(X_G(f)) =$$

$$\equiv (\tilde{\delta}_G \tilde{\delta}_F - \tilde{\delta}_F \tilde{\delta}_G) f$$

if $\{F, H\} = 0$
X_F^I is precisely the (flow of the)
symmetry $\tilde{\delta}_F$

- Do symmetries commute?
(think angular mom. & spin quantiz.)

$$X_G(X_F(f)) - X_F(X_G(f)) = \sum_{I,J} \left(X_G^I \partial_I X_F^J - X_F^I \partial_I X_G^J \right) f$$

$$\equiv \left(\tilde{\delta}_G \tilde{\delta}_F - \tilde{\delta}_F \tilde{\delta}_G \right) f$$

$$\partial_I \equiv \frac{\partial}{\partial z^I}$$

if $\{F, H\} = 0$
X_F^I is precisely the (flow of the)
symmetry $\tilde{\delta}_F$

- Do symmetries commute?
(think angular mom. & spin quantiz.)

$$X_G(X_F(f)) - X_F(X_G(f)) = \sum_{I,J} \left(X_G^I \partial_I X_F^J - X_F^I \partial_I X_G^J \right) \partial_J f$$

$$\equiv \left(\tilde{\delta}_G \tilde{\delta}_F - \tilde{\delta}_F \tilde{\delta}_G \right) f$$

$$\partial_I \equiv \frac{\partial}{\partial z^I}$$

if $\{F, H\} = 0$

X_F^I is precisely the (flow of the) symmetry $\tilde{\delta}_F$

- Do symmetries commute?

(think angular mom. & spin quantiz.)

$$X_G(X_F(f)) - X_F(X_G(f)) = \sum_{I,J} \left(X_G^{\textcircled{I}} \partial_{\textcircled{I}} X_F^{\textcircled{J}} - X_F^I \partial_I X_G^{\textcircled{J}} \right) \partial_{\textcircled{J}} f$$

$$\equiv \left(\tilde{\delta}_G \tilde{\delta}_F - \tilde{\delta}_F \tilde{\delta}_G \right) f$$

$$\partial_I \equiv \frac{\partial}{\partial z^I}$$

Def: X, Y are vector fields

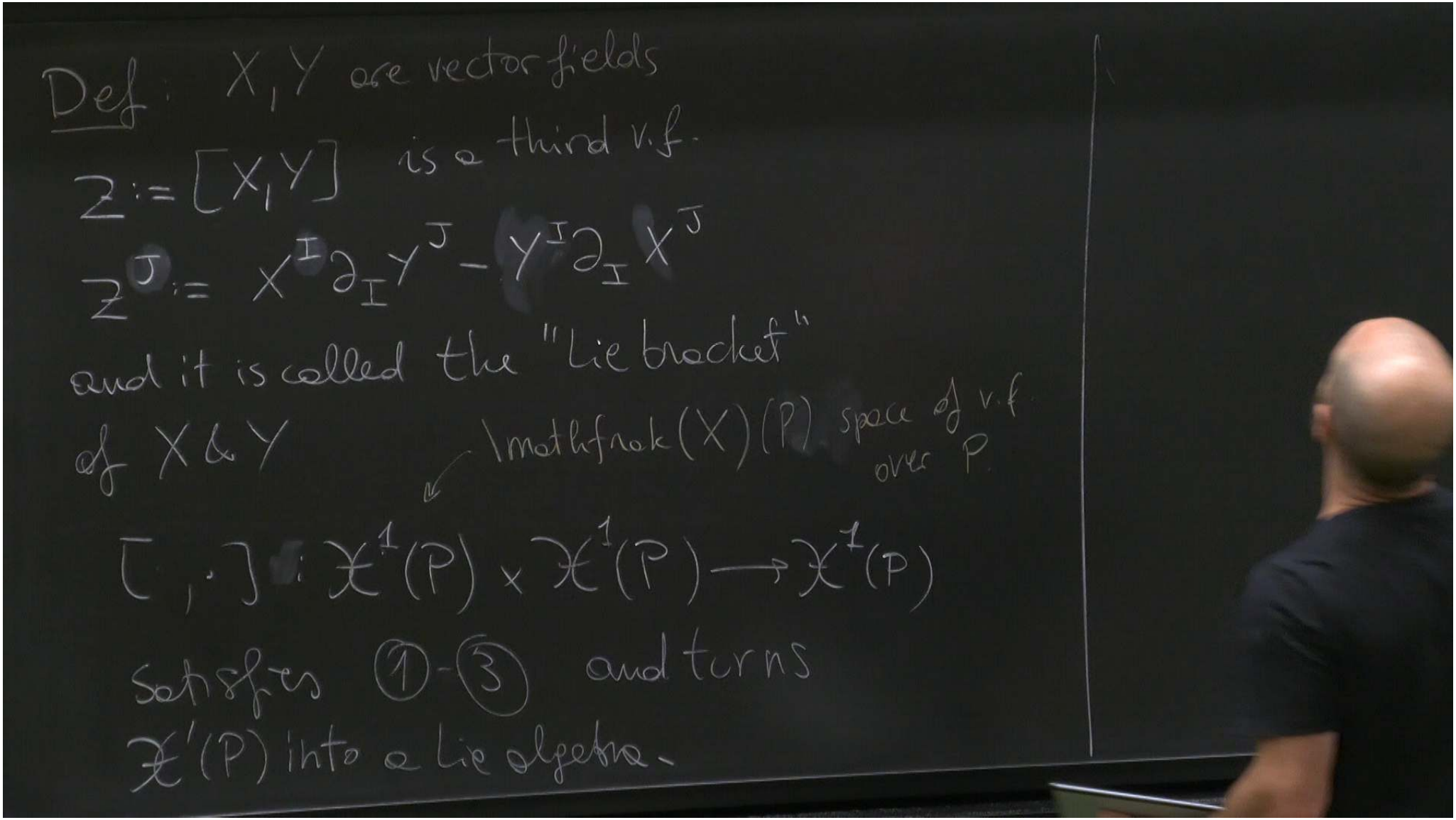
$Z := [X, Y]$ is a third v.f.

$$Z^J := X^I \partial_I Y^J - Y^I \partial_I X^J$$

and it is called the "Lie bracket" of X & Y

$$[\cdot,\cdot] : \mathfrak{X}(P) \times \mathfrak{X}(P) \longrightarrow \mathfrak{X}(P)$$

satisfies ① - ③


Def: X, Y are vector fields
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of X & Y
\mathfrak{X}^1(P) space of v.f. over P
$[\cdot,\cdot] : \mathfrak{X}^1(P) \times \mathfrak{X}^1(P) \longrightarrow \mathfrak{X}^1(P)$
Satisfies ① - ③ and turns
$\mathfrak{X}^1(P)$ into a Lie algebra.

Is the Lie bracket of two Ham. v.f. a H.v.f. itself?

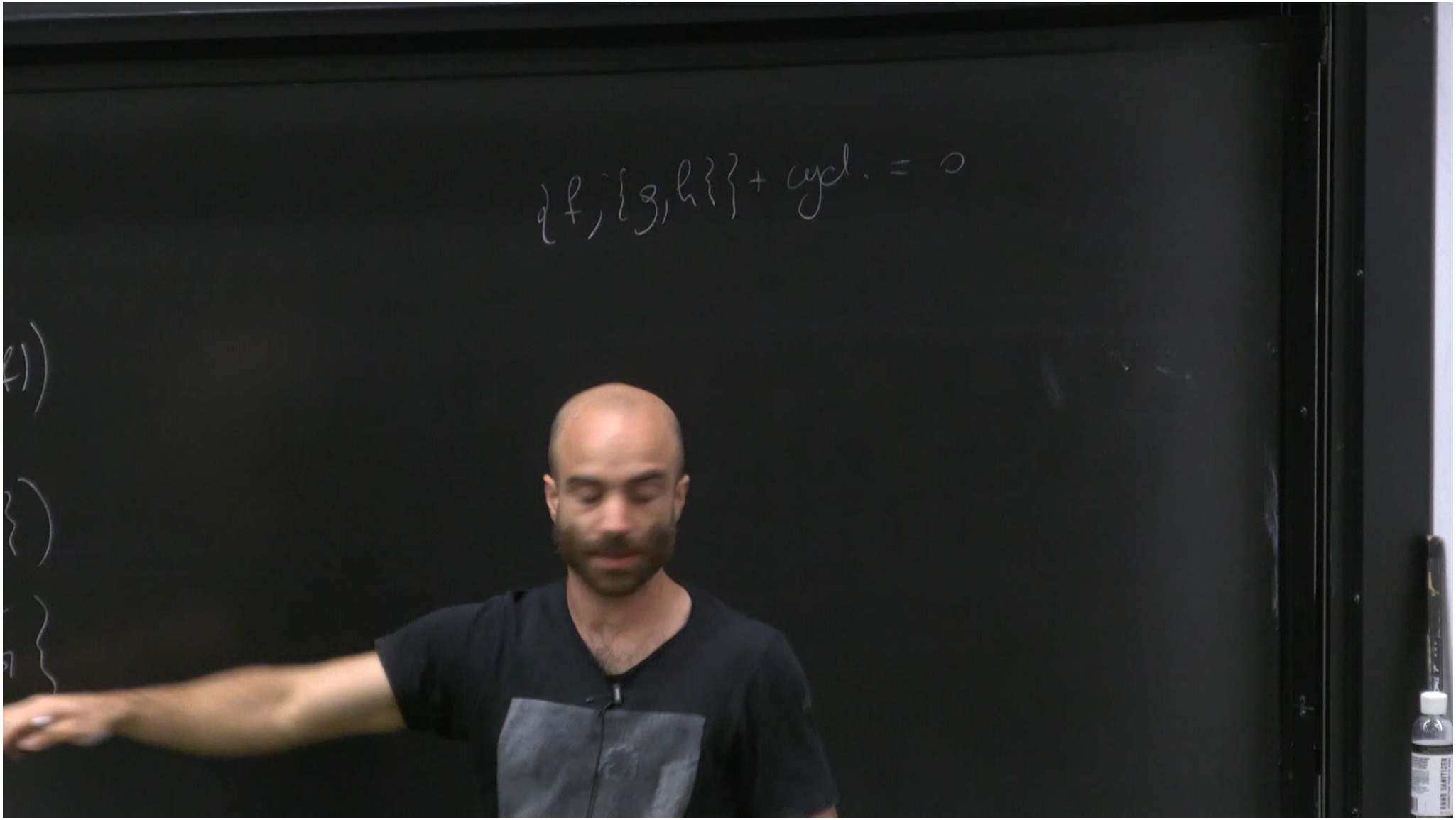
Yes!

Proof: $[X_F, X_G](f) = X_F(X_G(f)) - X_G(X_F(f))$

$= X_F(\{f, G\}) - X_G(\{f, F\})$

$= \{\{f, G\}, F\} - \{\{f, F\}, G\}$

$=$


{f,{g,h}} + cycl. = 0

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Proof: $[X_F, X_G](f) = X_F(X_G(f)) - X_G(X_F(f))$

$= X_F(\{f, G\}) - X_G(\{f, F\})$

$= \{\{f, G\}, F\} - \{\{f, F\}, G\}$

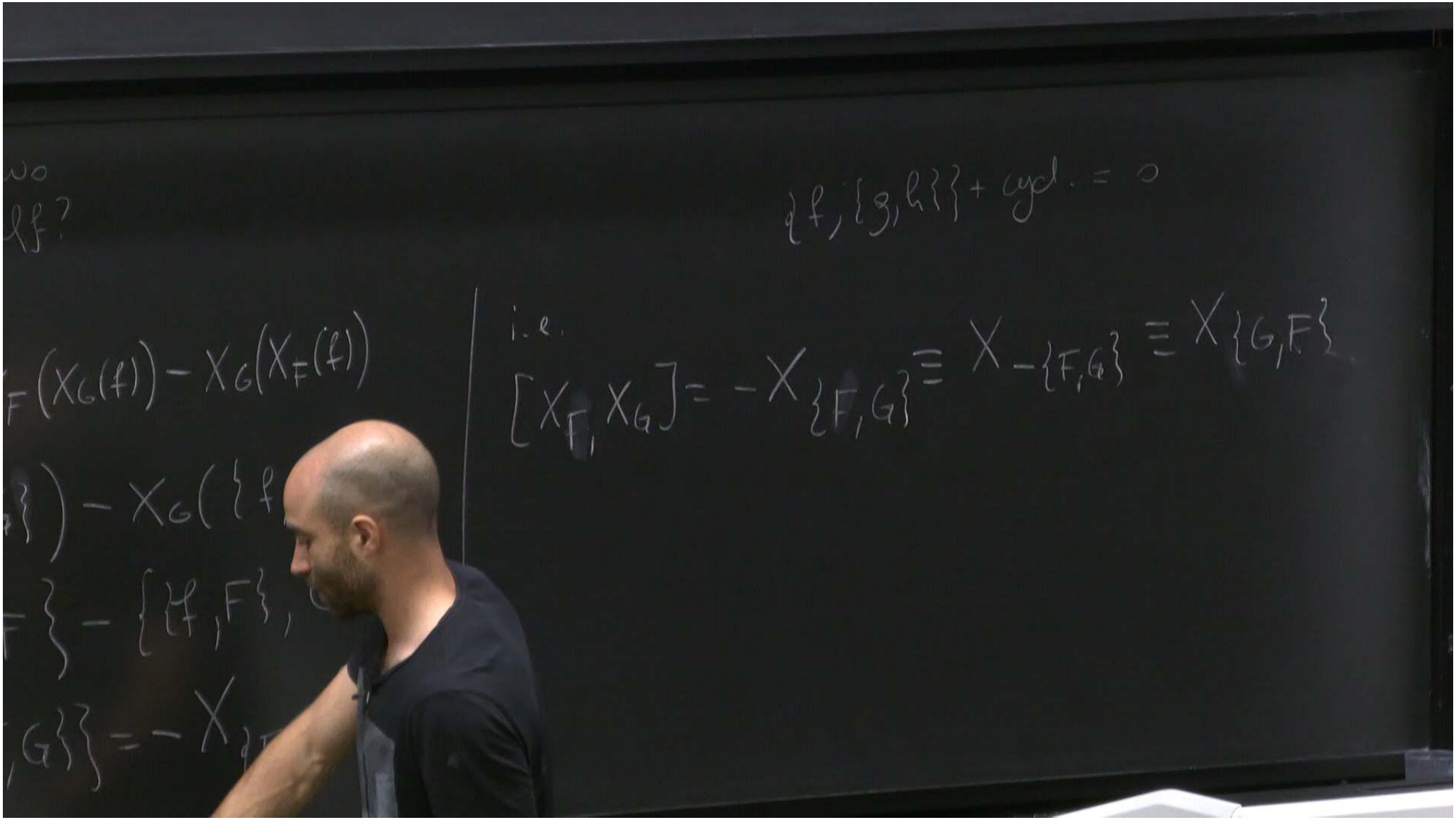
Jacobi $\downarrow$

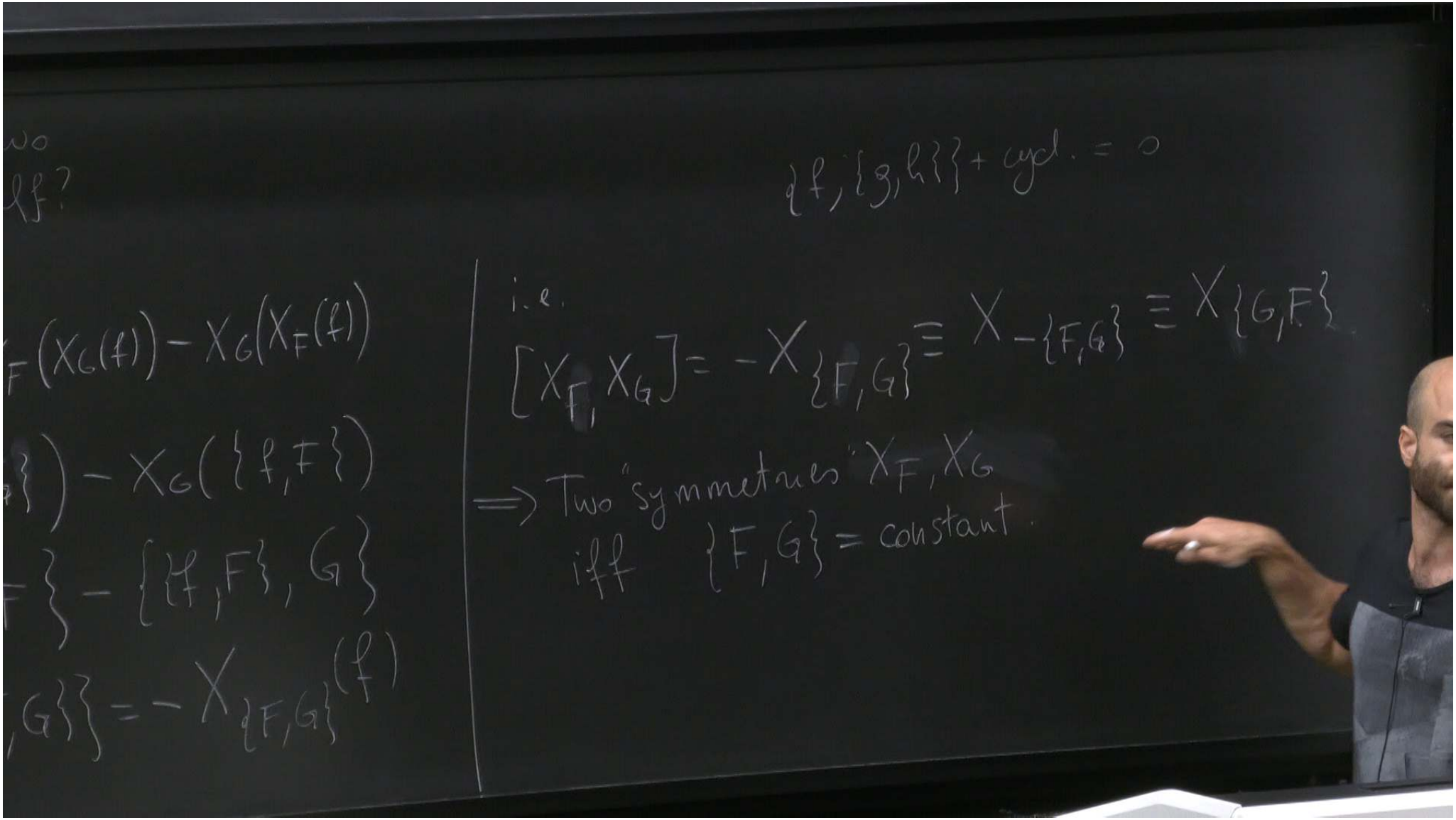
$= -\{f, \{F, G\}\} = -X_{\{F,G\}}(f)$

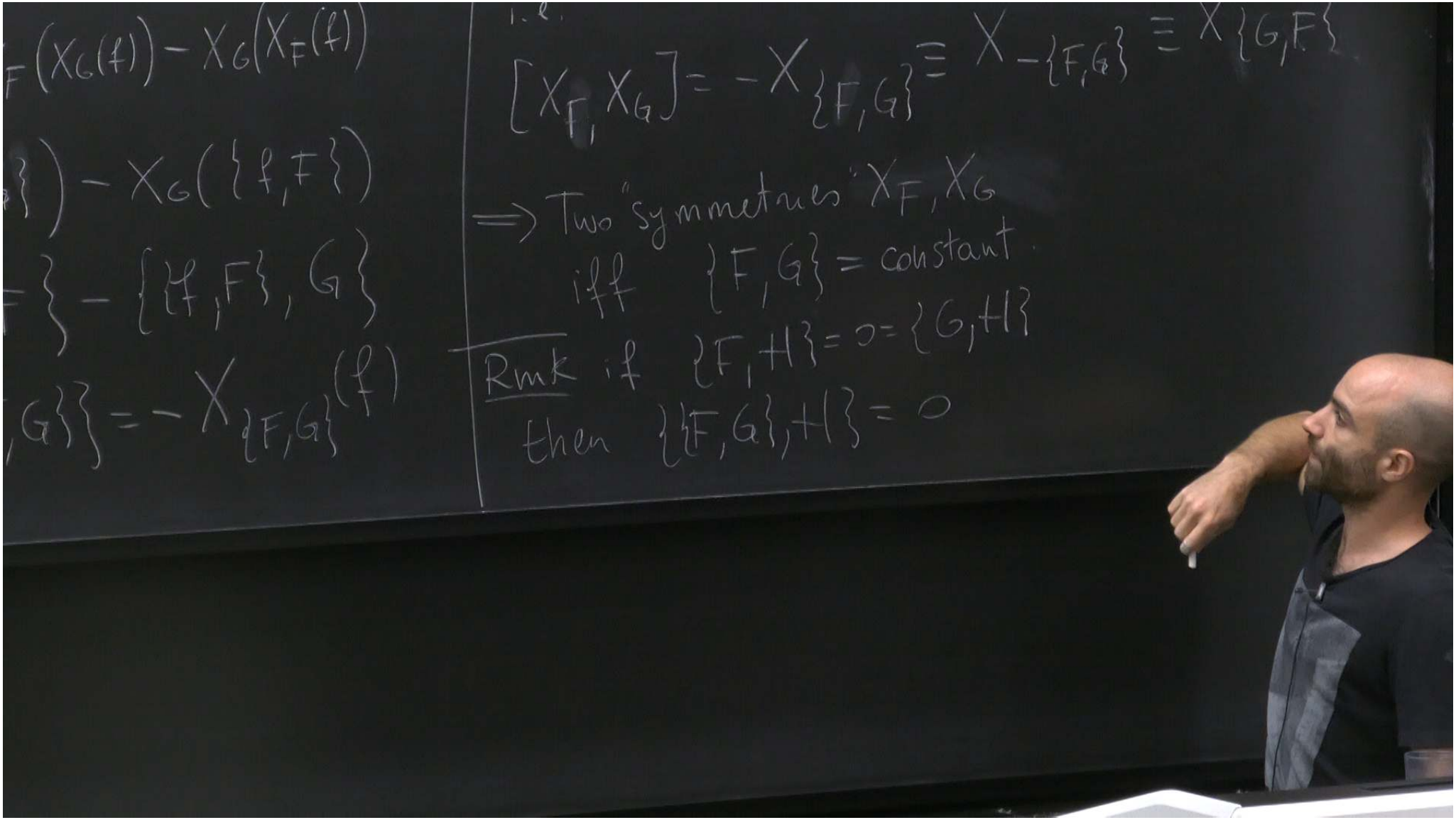
i.e.

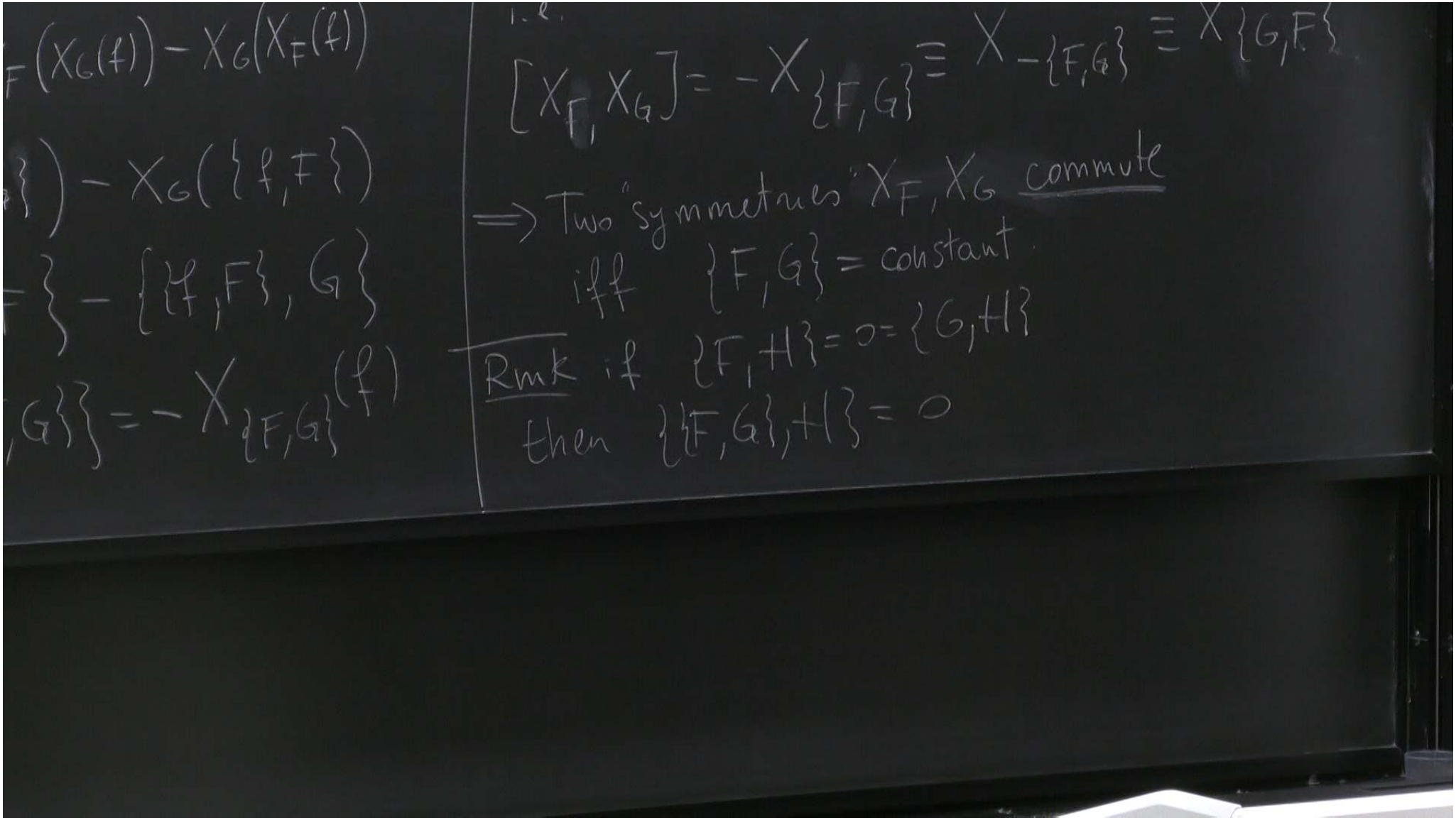
$[X_F, X_G] = -X_{\{F,G\}}$

$\{f, \{g, h\}$


$\{f,\{g,h\}\}+\text{cycl.}=0$
i.e.
$[X_F, X_G] = -X_{\{F,G\}} \equiv X_{-\{F,G\}} \equiv X_{\{G,F\}}$


$\{f,\{g,h\}\} + \text{cycl.} = 0$
$X_F(X_G(f)) - X_G(X_F(f))$
$\}) - X_G(\{f,F\})$
$\} - \{\{f,F\},G\}$
$,G\}\} = -X_{\{F,G\}}(f)$
i.e.
$[X_F, X_G] = -X_{\{F,G\}} \equiv X_{-\{F,G\}} \equiv X_{\{G,F\}}$
$\Longrightarrow$ Two "symmetries" X_F, X_G
iff $\{F,G\} = \text{constant}$.


$[X_F, X_G] = -X_{\{F,G\}} \equiv X_{-\{F,G\}} \equiv X_{\{G,F\}}$
$\Rightarrow$ Two "symmetries" X_F, X_G
iff $\{F,G\} =$ constant.
Rmk if $\{F,H\} = 0 = \{G,H\}$
then $\{\{F,G\},H\} = 0$


$X_F(X_G(f)) - X_G(X_F(f))$
$\}) - X_G(\{f,F\})$
$\} - \{\{f,F\},G\}$
$,G\}\} = -X_{\{F,G\}}(f)$
$[X_F, X_G] = -X_{\{F,G\}} \equiv X_{-\{F,G\}} \equiv X_{\{G,F\}}$
$\Longrightarrow$ Two "symmetries" X_F, X_G commute
iff $\{F,G\} =$ constant
Rmk if $\{F,+1\} = 0 = \{G,+1\}$
then $\{\{F,G\},+1\} = 0$

$[\,,\,] : \mathfrak{X}^1(P) \times \mathfrak{X}^1(P) \longrightarrow \mathfrak{X}^1(P)$

satisfies ① - ③ and turns $\mathfrak{X}^1(P)$ into a Lie algebra.

Jacobi $= \{\{F,G\},F\} - \{\{f,F\},G\}$

$= -\{f,\{F,G\}\} = -X_{\{F,G\}}(f)$

"Quantization"

$\mathcal{P}$ phase space $\longleftrightarrow$ $\mathcal{H}$ Hilbert sp

$z=(q,p) \in \mathcal{P}$ $\longleftrightarrow$ $\psi \in \mathcal{H}$ state

$F \in C^\infty(\mathcal{P})$ real $\longleftrightarrow$ $\hat{F} : \mathcal{H} \to \mathcal{H}$ Hermitian

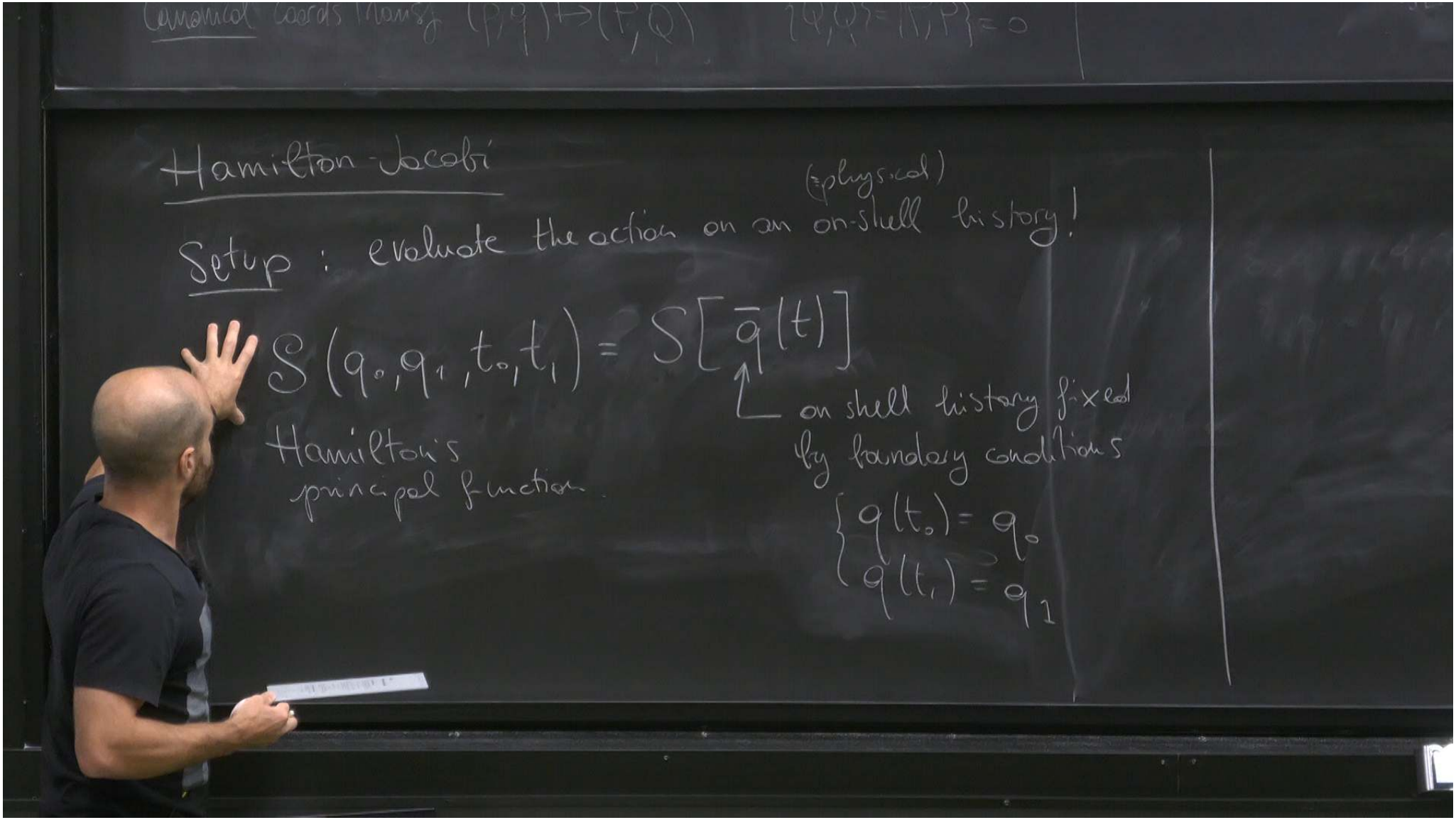
$\langle \psi | F | \psi \rangle$

$F(z)$

Liouville $\longleftrightarrow$ Ehrenfest

$\{F,G\}$ Poisson $\longleftrightarrow$ $[\hat{F},\hat{G}]$

$X_F = \{\cdot, F\}$ $\longleftrightarrow$ $[\,,\hat{F}]$ infinitesimal unitary transf $U = e^{-\frac{i\epsilon\hat{F}}{\hbar}}$, $U^\dagger \hat{G} U = \frac{\epsilon}{i\hbar}[\hat{G},\hat{F}] + O(\epsilon^2)$


Canonical Coords Transf. $(p,q) \mapsto (P,Q)$ $\{Q,P\} = 0$
Hamilton-Jacobi
(=phys. sol)
Setup: evaluate the action on an on-shell history!
$S(q_0, q_1, t_0, t_1) = S[\bar{q}(t)]$
on shell history fixed by boundary conditions
$\begin{cases} q(t_0) = q_0 \\ q(t_1) = q_1 \end{cases}$
Hamilton's principal function

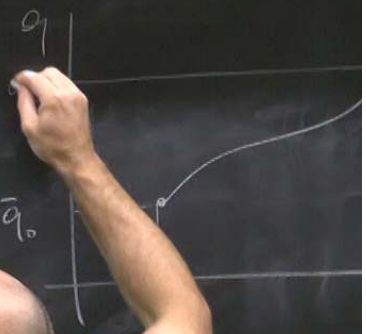
$\delta_\xi L = \frac{d}{dt} R_\xi$ $\quad Q_\xi = \sum_i p_i \delta_\xi q^i - R_\xi$ $\quad \text{Vol}|_{t_0} = V$

$$\frac{\partial}{\partial q^1} S = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(S[\bar{q} + \epsilon \delta_\xi \bar{q}] - S[\bar{q}(t)] \right)$$

where $\bar{q}' = \bar{q} + \epsilon \delta_\xi \bar{q}$ is the sol to E.L. eqs

defined by

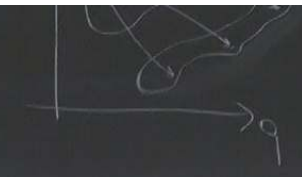
$$\begin{cases} \bar{q}'(t_1) = \bar{q}_1 + \epsilon \xi \\ \bar{q}'(t_0) = \bar{q}_0 \end{cases} \Rightarrow \begin{cases} \delta \bar{q}_1 = \xi \\ \delta \bar{q}_0 = 0 \end{cases}$$



Noether 1st: inf. variational sym $\Rightarrow$ conserved quantity

$\delta_\xi L = \frac{d}{dt} R_\xi$ $\qquad Q_\xi = \sum_i p_i \delta_\xi q^i - R_\xi$

$\Rightarrow$



flow $t_0 \to t_1$

$\mathrm{Vol}|_{t_0} = \mathrm{Vol}|_{t_1}$

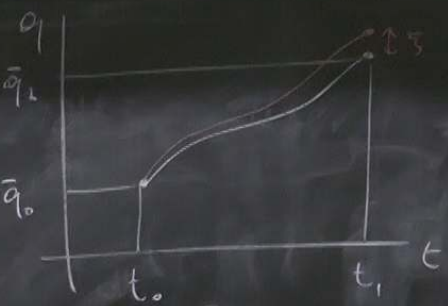
phase space Vol.

$$\sum_i \xi^i \frac{\partial}{\partial q^{1i}} S = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(S[\bar{q} + \epsilon \delta_\xi \bar{q}] - S[\bar{q}(t)] \right) = \delta_\xi S$$

where $\bar{q}' = \bar{q} + \epsilon \delta_\xi \bar{q}$ is the sol to E.L. eqs

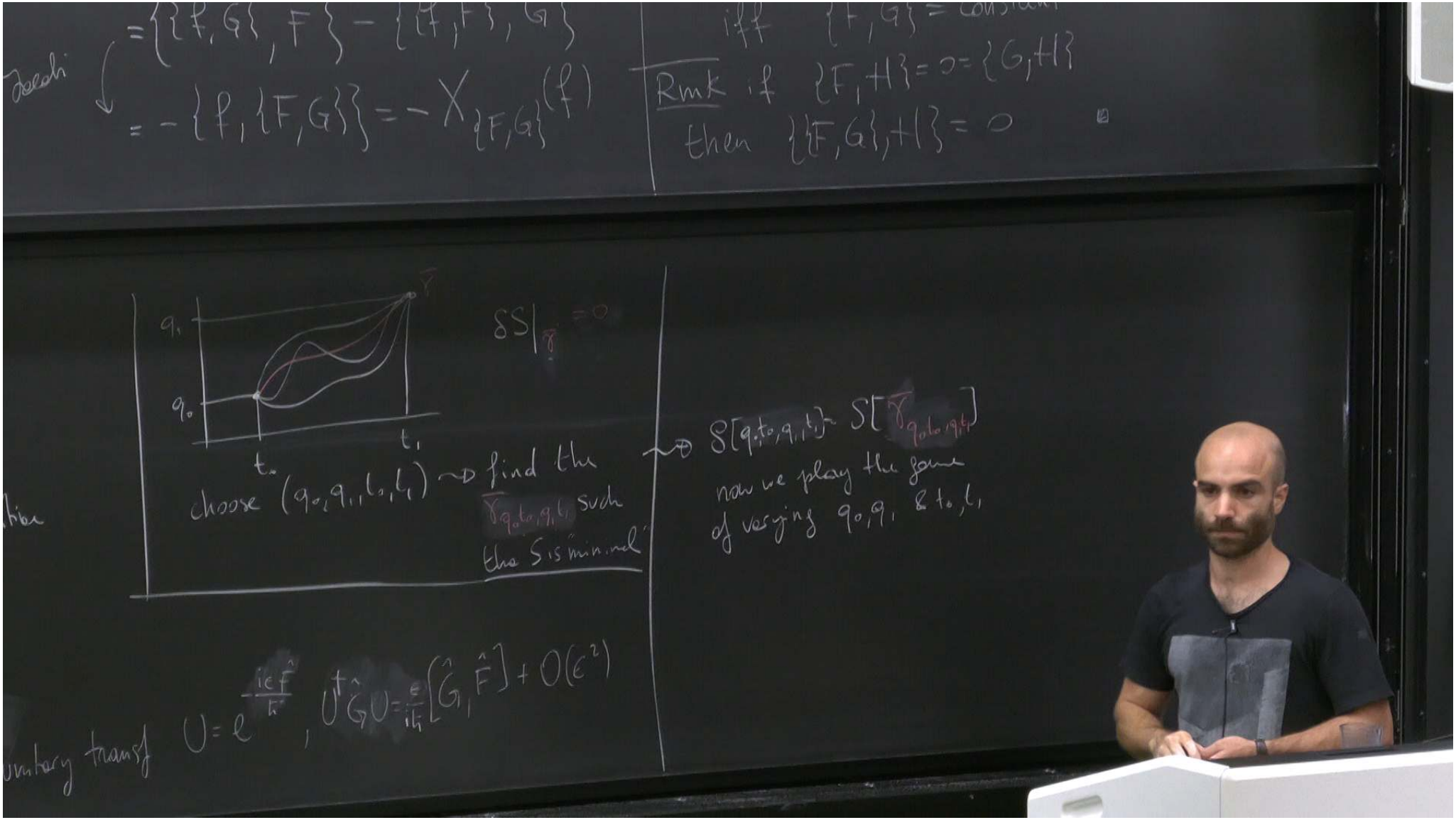
defined by

$$\begin{cases} \bar{q}'(t_1) = \bar{q}_1 + \epsilon \xi \\ \bar{q}'(t_0) = \bar{q}_0 \end{cases} \Rightarrow \begin{cases} \delta \bar{q}_1 = \xi \\ \delta \bar{q}_0 = 0 \end{cases}$$



$$\frac{\partial S}{\partial q^{1i}} = \bar{p}_i(t_1)$$

$$\sum_i \xi^i \frac{\partial}{\partial q^{1i}} S = \delta_\xi S = \int_{t_0}^{t_1} dt\, \delta_\xi L = \int_{t_0}^{t_1} dt \underbrace{\sum_i (\text{Euler-Lagr})_i \delta_\xi \bar{q}^i}_{=0 \ (\bar{q} \text{ on shell!})} + \left[\sum_i \bar{p}_i \delta_\xi \bar{q}^i \right]_{t_0}^{t_1} = \sum_i \bar{p}_i(t_1) \xi^i$$


Rmk
Jacobi
choose
find the
such
the S is minimal
now we play the game
of varying