Title: Classical Physics Lecture - 091123

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Collection: Classical Physics 2023/24

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Recap

- config space $Q \ni q$
- tangent space $TQ \ni (q, v)$ ← velocity
- phase space (cotangent sp) $P = T^*Q \ni (q, p)$
- histories $(\gamma \in C^\infty(\mathbb{R}, Q))$ $\quad \gamma : \mathbb{R} \longrightarrow Q$, $\quad t \longmapsto q = \gamma(t)$
- "phase space history" $z : \mathbb{R} \longrightarrow P$, $\quad t \longmapsto (q, p) = z(t)$

- Lagrangian function
$$L : TQ \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(q, v, t) \mapsto L(q, v, t)$$
- Action functional ← function on space of histori

$$S : C^\infty(\mathbb{R}, Q) \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(\gamma, t_0, t_1) \longmapsto S =$$

- Hamilton's action principle:

→ physically realized histories are
at fixed boundary condition

- On phase space $P = T^*Q$

- Hamilton's e.o.m

- Lagrangian function

$$L : TQ \times \mathbb{R} \to \mathbb{R}$$
$$(q, v, t) \mapsto L(q, v, t)$$

- Action functional
  - ↑ function on space of histories

$$S : C^\infty(\mathbb{R}, Q) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$
$$(\gamma, t_0, t_1) \longmapsto S = \int_{t_0}^{t_1} dt \; L(\gamma(t), \dot{\gamma}(t), t)$$

- Hamilton's action principle:

→ physically realized histories are extrema of the action, at fixed boundary conditions (in Q): $\delta S|_{\bar{\gamma}} = 0$, $\delta\gamma(t_0) = \delta\gamma(t_1) = 0$

↑ cf. tutorial!

- Hamilton action principle

$\Rightarrow$ Euler-Lagrange e.o.m.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \doteq 0$$

(equivalent to Newton if $L = \underbrace{T}_{\text{kinetic}} - \underbrace{V}_{\text{potential energy}}$)

- E.L. e.o.m are 2nd order
  ↪ can be written as a nice 1st order set of eqs. upon Legendre transform from v^i to $p_i := \dfrac{\partial L}{\partial v^i} \Leftrightarrow v^i = v^i(p)$

$$\rightsquigarrow H(q, p, t) = \sum_i p_i v$$

Hamilton's e.o.m.
$$\begin{cases} \dot{q}^i = \partial H / \partial p_i \\ \dot{p}_i = -\partial H / \partial q^i \end{cases}$$

- Hamilton's e.o.m iff

$, t)$

...on on
...of histories

$\mathbb{R} \to \mathbb{R}$

$\to S = \int_{t_0}^{t_1} dt \; L(\gamma(t), \dot\gamma(t), t)$

...inciple:

...es are extrema of the action, $\delta S|_{\bar\gamma} = 0$, $\delta\gamma(t_0) = \delta\gamma(t_1) = 0$

...nditions (in Q) :

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↳ can be written as a nice 1st order set of eqs. upon Legendre transform from v^i to $p_i := \frac{\partial L}{\partial v^i} \leftrightarrow v^i = v^i(p)$

$\leadsto H(q,p,t) = \sum_i p_i v^i - L(q^i, v^i(p), t)$

Hamilton's e.o.m.
$$\begin{cases} \dot q^i = \partial H / \partial p_i \\ \dot p_i = -\partial H / \partial q^i \end{cases}$$

- Hamilton's eom

iff

1st order action principle

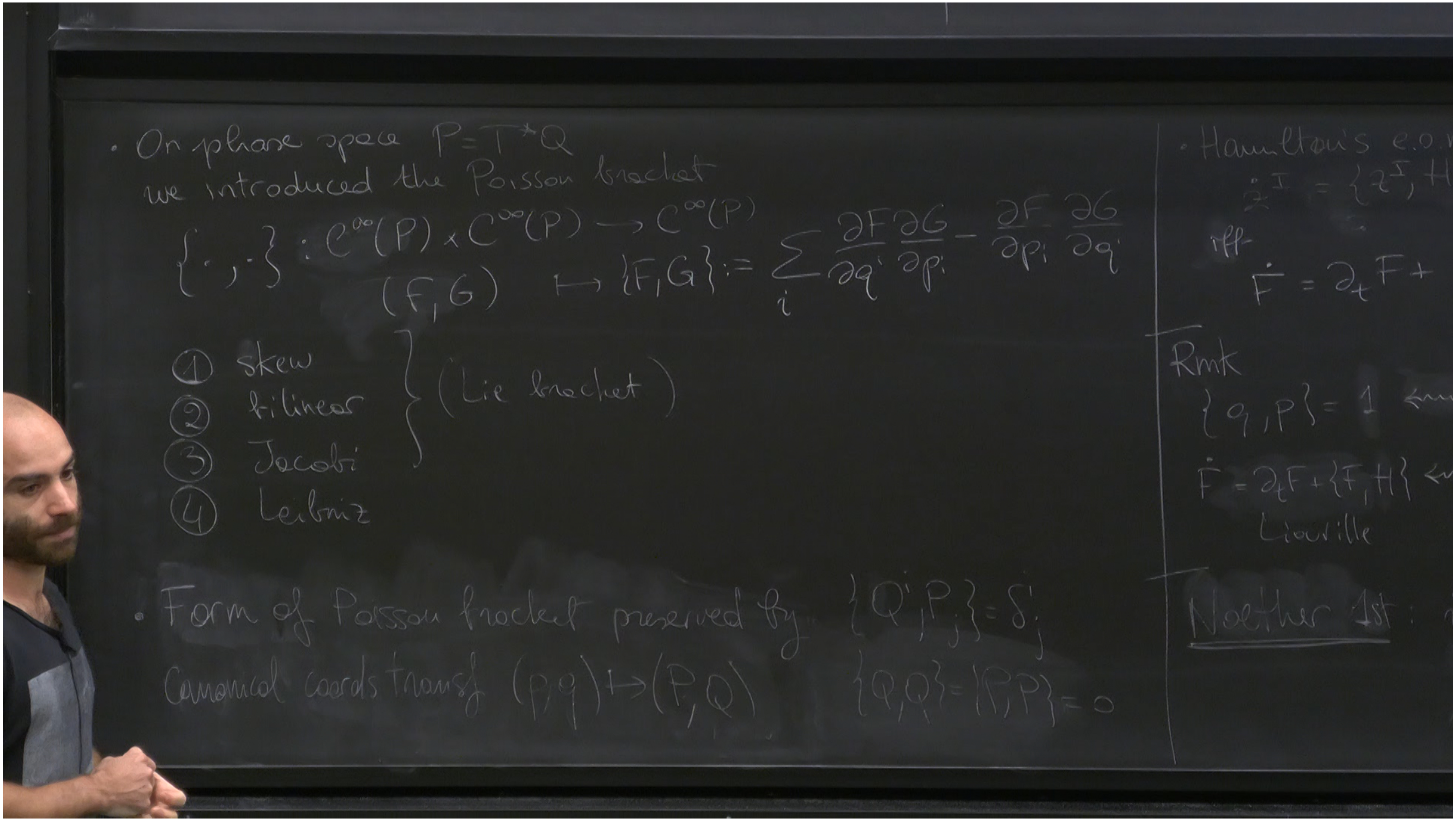
$$S_{1st}[z(t)] = \int_{t_0}^{t_1} \sum_i p_i \dot q^i - H(q,p,t)$$

$\delta S|_{\bar z} = 0$

with $\delta q(t_0) = \delta q(t_1) \doteq 0$

(not on p's, cf tutorial)

...s e.o.m iff
...δH...



• On phase space $P = T^*Q$
we introduced the Poisson bracket
$\{\cdot,\cdot\} : C^\infty(P) \times C^\infty(P) \longrightarrow C^\infty(P)$
$(F,G) \mapsto \{F,G\} := \sum_i \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i}$
① skew
② bilinear
③ Jacobi
④ Leibniz
(Lie bracket)
• Form of Poisson bracket preserved by
Canonical coords transf $(p,q) \mapsto (P,Q)$
$\{Q^i, P_j\} = \delta^i_j$
$\{Q,Q\} = \{P,P\} = 0$
• Hamilton's e.o.
iff
$\dot{F} = \partial_t F +$
Rmk
$\{q,p\} = 1$
$\dot{F} = \partial_t F + \{F,H\}$
Liouville
Noether 1st :

- Hamilton's e.o.m iff

$$\dot{z}^I = \{z^I, H\}$$

iff

$$\dot{F} = \partial_t F + \{F, H\} \quad \left(\begin{array}{c}\text{Liouville}\\ \text{eq.}\end{array}\right)$$

Rmk

$$\{q, p\} = 1 \quad \rightsquigarrow \quad [\hat{q}, \hat{p}] = i\hbar$$

$$\dot{F} = \partial_t F + \{F, H\} \quad \rightsquigarrow \quad \frac{d}{dt}\langle \hat{F} \rangle = \langle \partial_t \hat{F} \rangle + \frac{1}{i\hbar}\langle [\hat{F}, \hat{H}] \rangle$$

Liouville   Ehrenfest.

<u>Noether 1st</u>: infinitesimal sym $\Rightarrow$ conserved quantity

$$\tilde{\delta}_s L = \frac{d}{dt} R_s \qquad Q_s = \sum_i p_i \tilde{\delta}_s q^i - R_s$$

Liouville thm

volume form on phase space

which in canonical coords reads

$$d\,vol_P = dq^1 dp_1 \cdots dq^n dp_n \quad \left(\begin{matrix}\text{Liouville}\\ \text{form}\end{matrix}\right)$$

$$= dQ^1 dP_1 \cdots dQ^n dP_n$$

$[\hat{F}, \hat{H}]$,

rved quantity

$\sum_i p_i \tilde{\delta}_s q^i - R_s$

$[\hat{F},\hat{H}]>$

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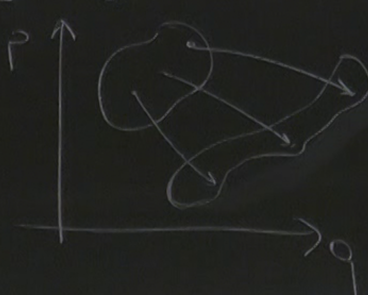
$$= dQ^1 dP_1 \cdots dQ^n dP_n$$

Rmk: time evolution is a canonical transf.

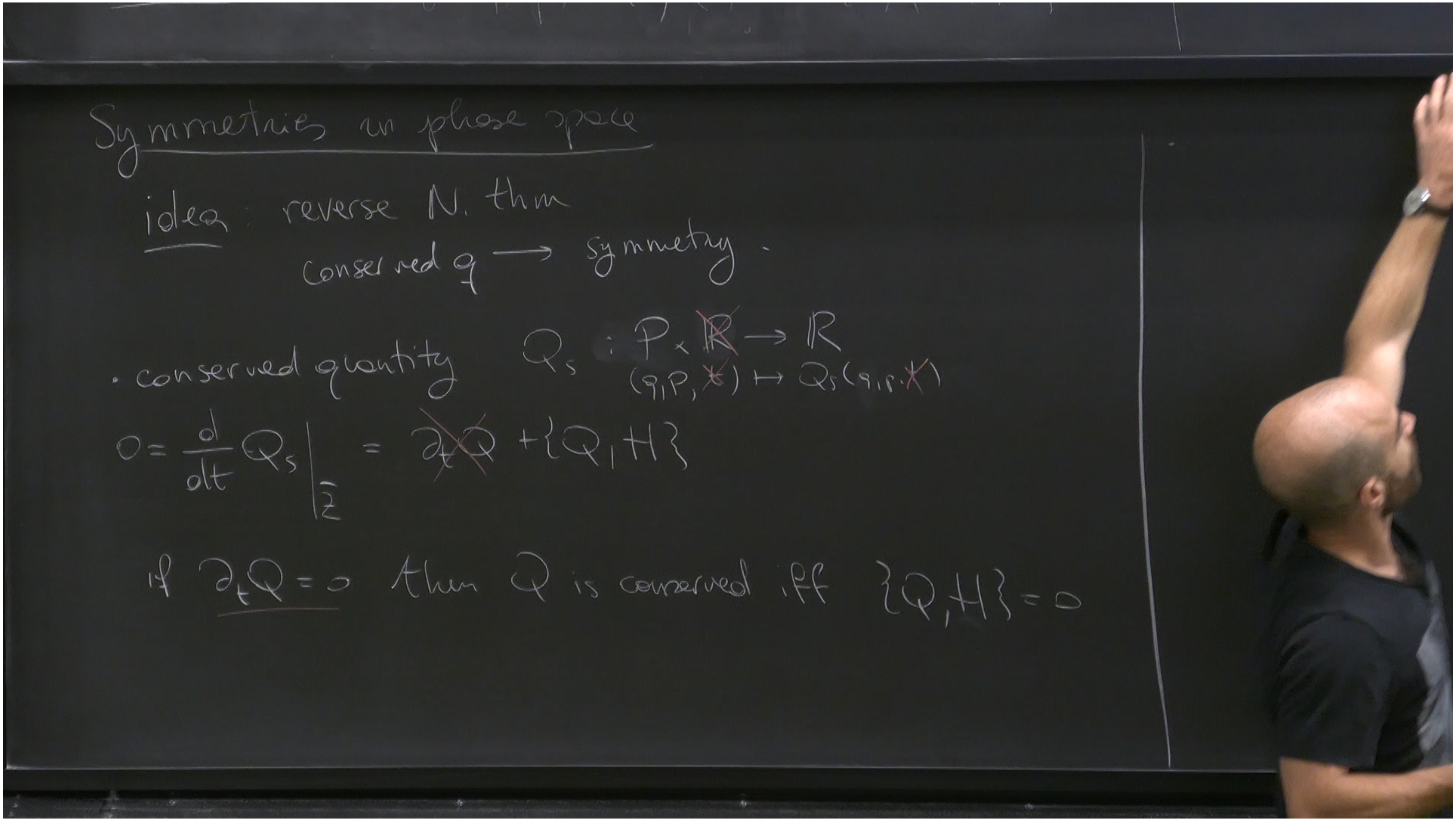
$\Longrightarrow$ 

time evol
from t_0 to t_1

$$Vol\big|_{t_0} = Vol\big|_{t_1}$$

i.e. time evol. preserve phase space vol.

$$\mathbb{P}(V) = \int_{V \subset P} \rho \, d\mathrm{vol}_P$$


Symmetries in phase space
idea: reverse N. thm
conserved q → symmetry.
· conserved quantity $Q_s : P \times \cancel{\mathbb{R}} \to \mathbb{R}$
$(q,p,\cancel{t}) \mapsto Q_s(q,p,\cancel{t})$
$0 = \frac{d}{dt} Q_s \Big|_{\bar{z}} = \cancel{\partial_t Q} + \{Q,H\}$
if $\partial_t Q = 0$ then Q is conserved iff $\{Q,H\} = 0$

- Given $Q : P \to \mathbb{R}$ define:

$$\tilde{\delta}_Q z^I := \{ z^I, Q \}$$

i.e. $$\begin{cases} \tilde{\delta}_Q q^i = \{q^i, Q\} = \dfrac{\partial Q}{\partial p_i} \\ \tilde{\delta}_Q p_i = \{p_i, Q\} = -\dfrac{\partial Q}{\partial q^i} \end{cases}$$

<u>Proposition</u>: if Q is conserved, then $\tilde{\delta}_Q$ is an infinitesimal sym of

$$L_{1st} = \Sigma p \dot{q} - H$$

$Q_S = \sum_i p_i \delta_S q^i - R_S$ $\longrightarrow q$ $Vol\big|_{t_0} = Vol\big|_{t_1}$ $Vol.$

then

Proof

$$\tilde{\delta}_Q L_{1st} := \lim_{\epsilon \to 0} \frac{L_{1st}(q + \epsilon \tilde{\delta} q, p + \epsilon \tilde{\delta} p, t) - L_{1st}(q, p, t)}{\epsilon}$$

$$= \sum_i \tilde{\delta} p_i \frac{d}{dt} q^i + p_i \frac{d}{dt} \tilde{\delta} q^i - \frac{\partial H}{\partial q^i} \tilde{\delta} q^i - \frac{\partial H}{\partial p_i} \tilde{\delta} p_i$$

$$= \sum_i \{p_i, Q\} \dot{q}^i - \{q^i, Q\} \dot{p}_i + \frac{d}{dt}\left(p_i \tilde{\delta} q^i\right) - \sum_I \frac{\partial H}{\partial z^I} \{z^I, Q\}$$

$Q_S = \sum_i p_i \delta_S q^i - R_S$ $\qquad \longrightarrow q \qquad Vol|_{t_0} = Vol|_{t_1} \qquad Vol.$

Proof

$$\tilde{\delta}_Q L_{1st} := \lim_{\epsilon \to 0} \frac{L_{1st}(q + \epsilon \tilde{\delta} q, p + \epsilon \tilde{\delta} p, t) - L_{1st}(q, p, t)}{\epsilon}$$

$$= \sum_i \tilde{\delta} p_i \frac{d}{dt} q^i + p_i \frac{d}{dt} \tilde{\delta} q^i - \underbrace{\frac{\partial H}{\partial q^i} \tilde{\delta} q^i - \frac{\partial H}{\partial p_i} \tilde{\delta} p_i} \equiv$$

$$= \sum_i \{p_i, Q\} \dot{q}^i - \{q^i, Q\} \dot{p}_i + \frac{d}{dt}\left(p_i \tilde{\delta} q^i\right) - \sum_I \underbrace{\frac{\partial H}{\partial z^I} \{z^I, Q\}}_{\text{Leibniz}} = \{H, Q\} = 0 \quad \boxed{\text{hyp}}$$

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$$\tilde{\delta}_Q L_{1st} := \lim_{\epsilon \to 0} \frac{L_{1st}(q+\epsilon\tilde{\delta} q, p+\epsilon\tilde{\delta} p, t) - L_{1st}(q,p,t)}{\epsilon}$$

$$= \sum_i \tilde{\delta} p_i \frac{d}{dt} q^i + p_i \frac{d}{dt} \tilde{\delta} q^i - \underbrace{\frac{\partial H}{\partial q^i}\tilde{\delta} q^i - \frac{\partial H}{\partial p_i}\tilde{\delta} p_i}_{\equiv}$$

$$= \sum_i \{p_i, Q\}\dot{q}^i - \{q^i, Q\}\dot{p}_i + \frac{d}{dt}\left(p_i \tilde{\delta} q^i\right) - \sum_I \underbrace{\frac{\partial H}{\partial z^I}\{z^I, Q\}}_{\text{Leibniz}}$$

$$= \{H, Q\} \overset{\boxed{\text{hyp}}}{=} 0$$

$$(\partial_t Q \equiv 0) \quad = -\sum_i \frac{\partial Q}{\partial q^i}\dot{q}^i + \frac{\partial Q}{\partial p_i}\dot{p}_i + \frac{d}{dt}\left(p_i \tilde{\delta} q^i\right)$$

$$= \frac{d}{dt}\left(-Q + \sum_i p_i \tilde{\delta} q^i\right)$$

$$Q_S = \sum_i p_i \delta_S q^i - R_S \qquad \to q \qquad Vol|_{t_0} = Vol|_{t_1} \qquad Vol.$$

Proof

$$\tilde{\delta}_Q L_{1st} := \lim_{\epsilon \to 0} \frac{L_{1st}(q+\epsilon\tilde{\delta}q,\, p+\epsilon\tilde{\delta}p,\, t) - L_{1st}(q,p,t)}{\epsilon}$$

$$= \sum_i \tilde{\delta}p_i \frac{d}{dt} q^i + p_i \frac{d}{dt}\tilde{\delta}q^i - \underbrace{\frac{\partial H}{\partial q^i}\tilde{\delta}q^i - \frac{\partial H}{\partial p_i}\tilde{\delta}p_i}_{\equiv}$$

$$= \sum_i \{p_i, Q\}\dot{q}^i - \{q^i, Q\}\dot{p}_i + \frac{d}{dt}\left(p_i \tilde{\delta}q^i\right) - \sum_I \underbrace{\frac{\partial H}{\partial z^I}\{z^I, Q\}}_{\text{Leibniz}}$$

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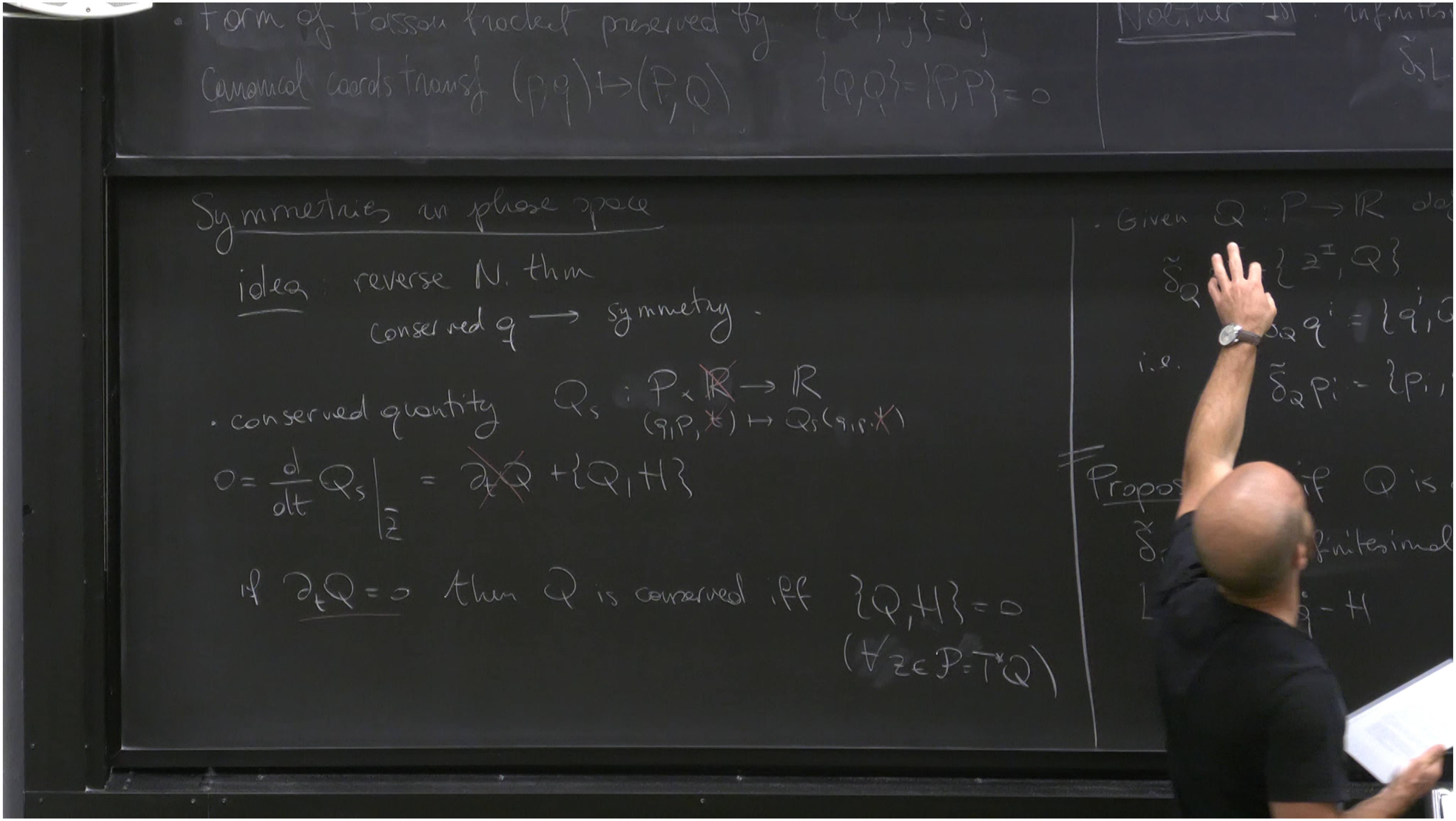
$$(\partial_t Q \equiv 0) \qquad = -\sum_i \frac{\partial Q}{\partial q^i}\dot{q}^i + \frac{\partial Q}{\partial p_i}\dot{p}_i + \frac{d}{dt}\left(p_i \tilde{\delta}q^i\right)$$

$$= \frac{d}{dt}\left(-Q + \sum_i p_i \tilde{\delta}q^i\right) \equiv \frac{d}{dt} R_S \to Q = \sum_i p_i \tilde{\delta}q^i - R_S$$

$\boxed{\checkmark}$


· Form of Poisson bracket preserved by {Q, P} = δ
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Noether thm : infinitesi
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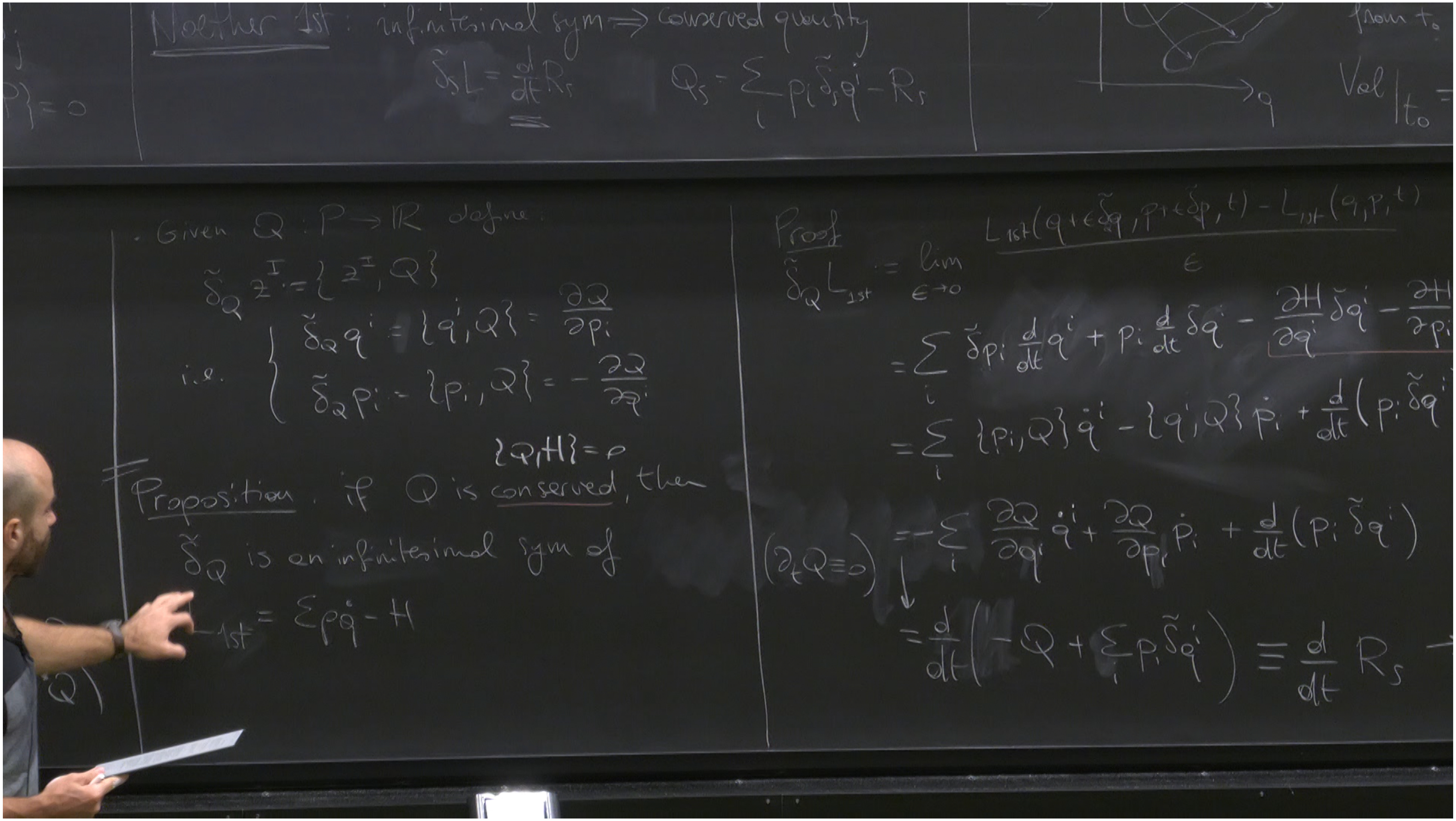
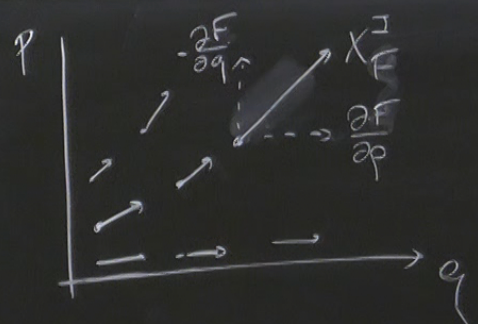
· conserved quantity Q_s : P × ℝ → ℝ
(q,p,t) ↦ Q_s(q,p,t)
0 = d/dt Q_s |_z = ∂_t Q + {Q, H}
if ∂_t Q = 0 then Q is conserved iff {Q,H} = 0
(∀ z ∈ P = T*Q)
· Given Q : P → ℝ
δ̃_Q z^I = {z^I, Q}
i.e.
δ̃_Q p_i = {p_i,
Propos
if Q is
finitesimal
q̇ − H


Noether 1st: infinitesimal sym ⟹ conserved quantity
$\tilde{\delta}_s L = \frac{d}{dt} R_s$
$Q_s = \sum_i p_i \tilde{\delta}_s q^i - R_s$
from t_0
$Vol|_{t_0} =$
· Given $Q : P \to \mathbb{R}$ define:
$\tilde{\delta}_Q z^I := \{z^I, Q\}$
i.e. $\tilde{\delta}_Q q^i = \{q^i, Q\} = \frac{\partial Q}{\partial p_i}$
$\tilde{\delta}_Q p_i = \{p_i, Q\} = -\frac{\partial Q}{\partial q^i}$
$\{Q, H\} = 0$
Proposition: if Q is conserved, then $\tilde{\delta}_Q$ is an infinitesimal sym of $L_{1st} = \sum p \dot{q} - H$
Proof
$\tilde{\delta}_Q L_{1st} := \lim_{\epsilon \to 0} \frac{L_{1st}(q + \epsilon \tilde{\delta} q, p + \epsilon \tilde{\delta} p, t) - L_{1st}(q, p, t)}{\epsilon}$
$= \sum_i \tilde{\delta} p_i \frac{d}{dt} q^i + p_i \frac{d}{dt} \tilde{\delta} q^i - \frac{\partial H}{\partial q^i} \tilde{\delta} q^i - \frac{\partial H}{\partial p_i}$
$= \sum_i \{p_i, Q\} \dot{q}^i - \{q^i, Q\} \dot{p}_i + \frac{d}{dt}(p_i \tilde{\delta} q^i$
$(\partial_t Q \equiv 0)$
$= -\sum_i \frac{\partial Q}{\partial q^i} \dot{q}^i + \frac{\partial Q}{\partial p_i} \dot{p}_i + \frac{d}{dt}(p_i \tilde{\delta} q^i)$
$= \frac{d}{dt}\left(-Q + \sum_i p_i \tilde{\delta} q^i\right) \equiv \frac{d}{dt} R_s$



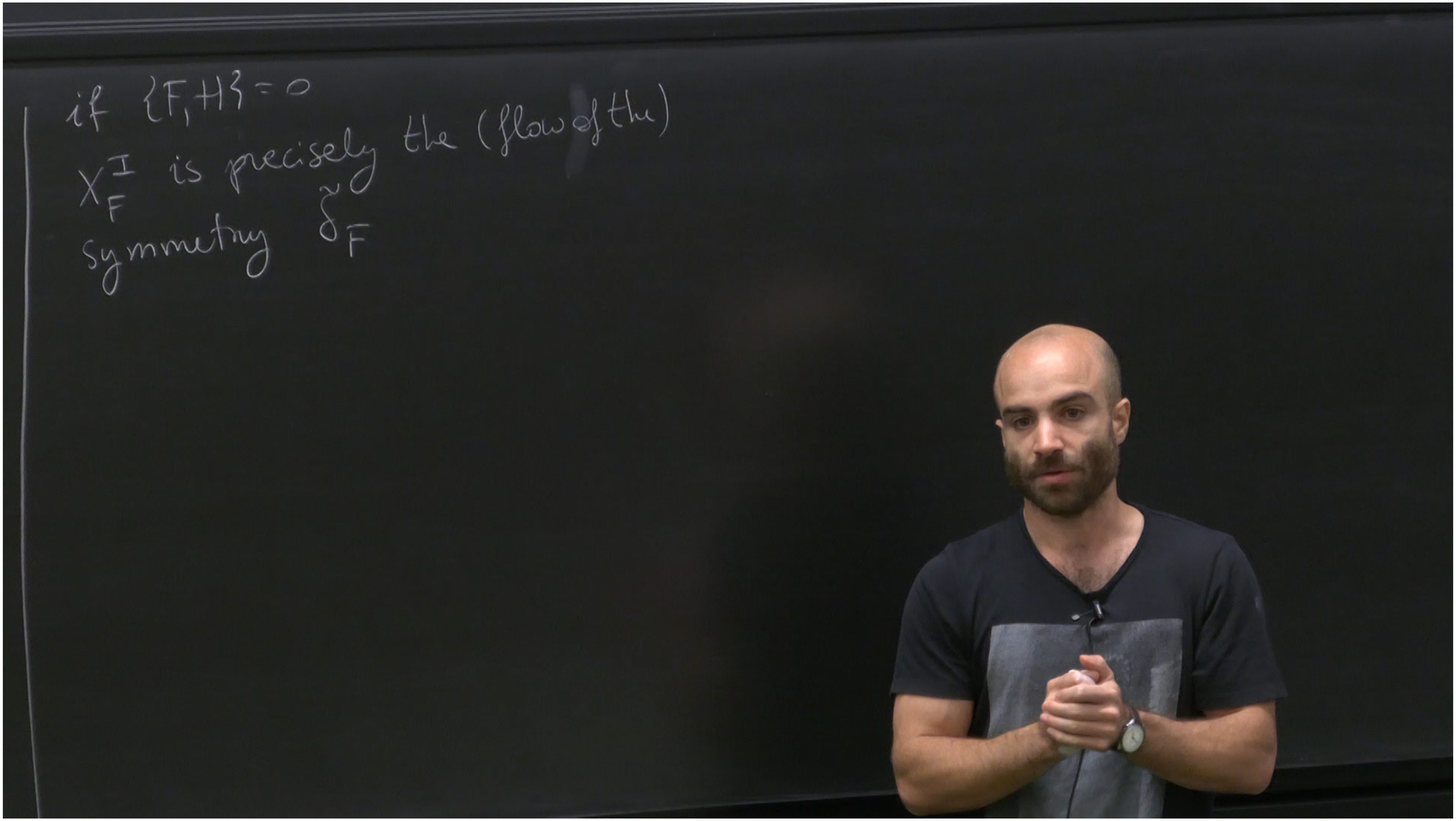
$$F \in C^\infty(P)$$

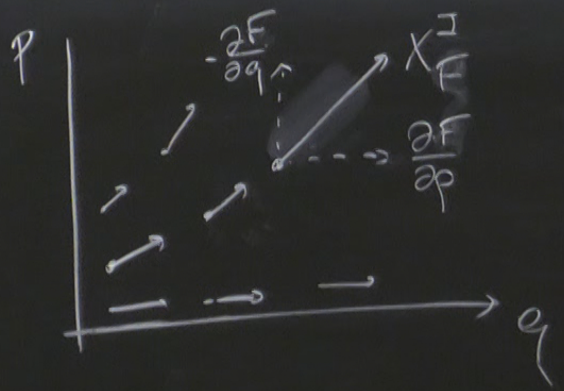
$$X^I_F(z) = \{z^I, F\}$$

Hamiltonian vector field associated to $F \in C^\infty(P)$

$$X_F = \{\cdot, F\} = -\{F, \cdot\}$$

$$X_F(f) = \{f, F\} = \sum_I X^I_F \frac{\partial f}{\partial z^I} = \nabla_X f$$


if $\{F, H\} = 0$
X_F^I is precisely the (flow of the)
symmetry $\tilde{\delta}_F$



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symmetry δ_F

• Do symmetries

(thin

if $\{F, H\} = 0$

X^I_F is precisely the (flow of the)

symmetry $\tilde{\delta}_F$

- Do symmetries commute?

(think angular mom. & spin quantiz.)

$$X_G(X_F(f)) - X_F(X_G(f)) =$$

$$\equiv (\tilde{\delta}_G \tilde{\delta}_F - \tilde{\delta}_F \tilde{\delta}_G) f$$

if $\{F, H\} = 0$

X_F^I is precisely the (flow of the)

symmetry $\tilde{\delta}_F$

- Do symmetries commute?

(think angular mom. & spin quantiz.)

$$X_G(X_F(f)) - X_F(X_G(f)) = \sum_{I,J} \left(X_G^I \partial_I X_F^J - X_F^I \partial_I X_G^J\right) f$$

$$\equiv \left(\tilde{\delta}_G \tilde{\delta}_F - \tilde{\delta}_F \tilde{\delta}_G\right) f$$

$$\partial_I \equiv \frac{\partial}{\partial z^I}$$

if $\{F, H\} = 0$

X_F^I is precisely the (flow of the)

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$$X_G(X_F(f)) - X_F(X_G(f)) = \sum_{I,J} \left(X_G^I \partial_I X_F^J - X_F^I \partial_I X_G^J \right) \partial_J f$$

$$\equiv (\tilde{\delta}_G \tilde{\delta}_F - \tilde{\delta}_F \tilde{\delta}_G) f$$

$$\partial_I \equiv \frac{\partial}{\partial z^I}$$

if $\{F,H\}=0$

X_F^I is precisely the (flow of the) symmetry $\tilde{\delta}_F$

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$$X_G(X_F(f)) - X_F(X_G(f)) = \sum_{I,J}\left(X_G^{I}\,\partial_{I} X_F^{J} - X_F^{I}\,\partial_{I} X_G^{J}\right)\partial_{J} f$$

$$\equiv \left(\tilde{\delta}_G\,\tilde{\delta}_F - \tilde{\delta}_F\,\tilde{\delta}_G\right) f$$

$$\partial_I \equiv \frac{\partial}{\partial z^I}$$

<u>Def</u>: X, Y are vector fields

$Z := [X, Y]$ is a third v.f.

$$Z^J := X^I \partial_I Y^J - Y^I \partial_I X^J$$

and it is called the "Lie bracket" of X & Y

$$[\cdot,\cdot] : \mathfrak{X}^1(P) \times \mathfrak{X}^1(P) \longrightarrow \mathfrak{X}^1(P)$$

satisfies 1 - 3

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\mathfrak(X)(P) space of v.f. over P

$$[\,,\cdot\,] : \mathfrak{X}^1(P) \times \mathfrak{X}^1(P) \longrightarrow \mathfrak{X}^1(P)$$

satisfies 1 - 3 and turns $\mathfrak{X}^1(P)$ into a Lie algebra.

Is the Lie bracket of two Ham. v.f. a H.v.f. itself?

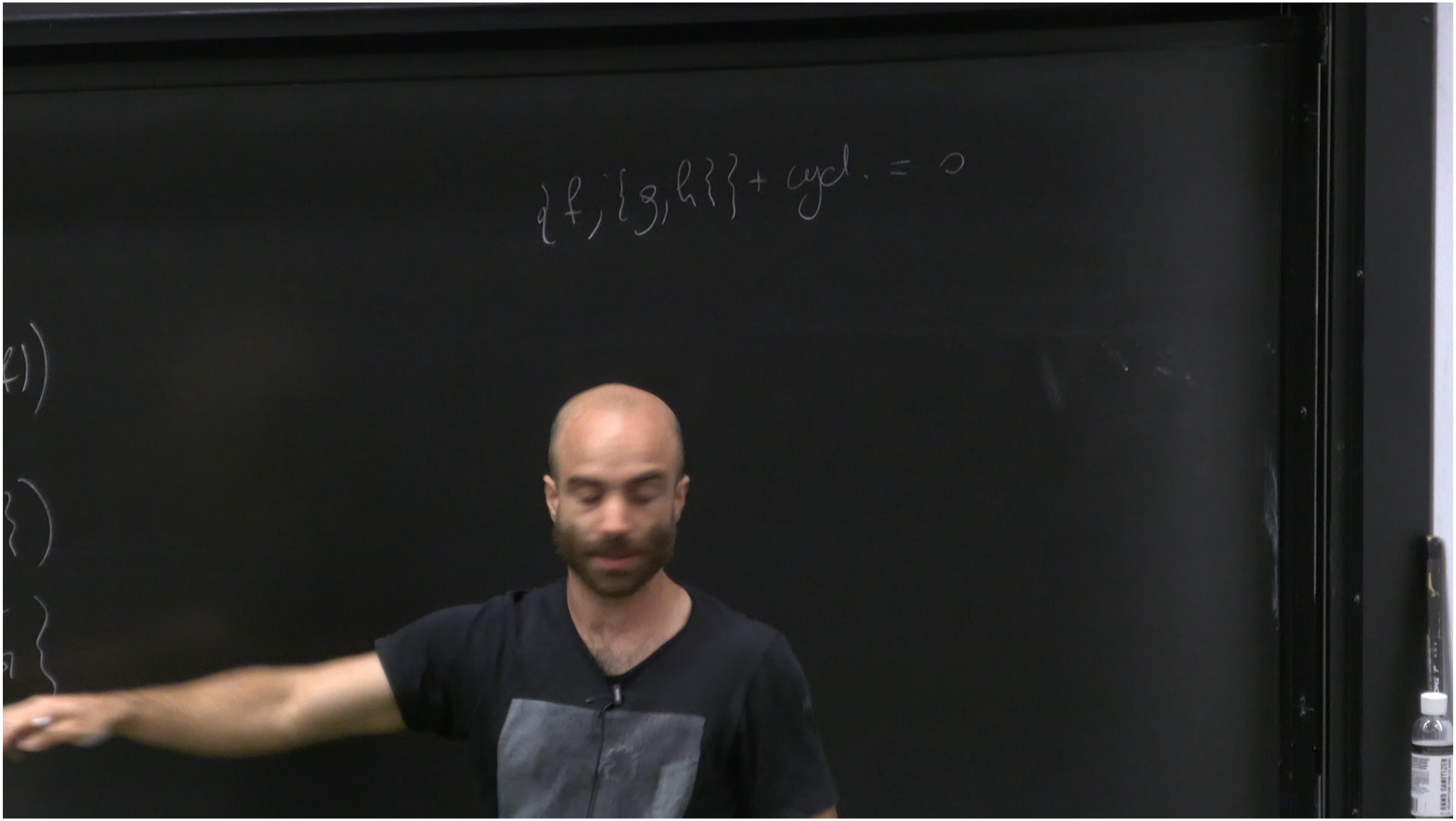
Yes!

Proof: $[X_F, X_G](f) = X_F(X_G(f)) - X_G(X_F(f))$

$= X_F(\{f, G\}) - X_G(\{f, F\})$

$= \{\{f, G\}, F\} - \{\{f, F\}, G\}$

$=$


{f,{g,h}} + cycl. = 0

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Proof: $[X_F, X_G](f) = X_F(X_G(f)) - X_G(X_F(f))$

$= X_F(\{f, G\}) - X_G(\{f, F\})$

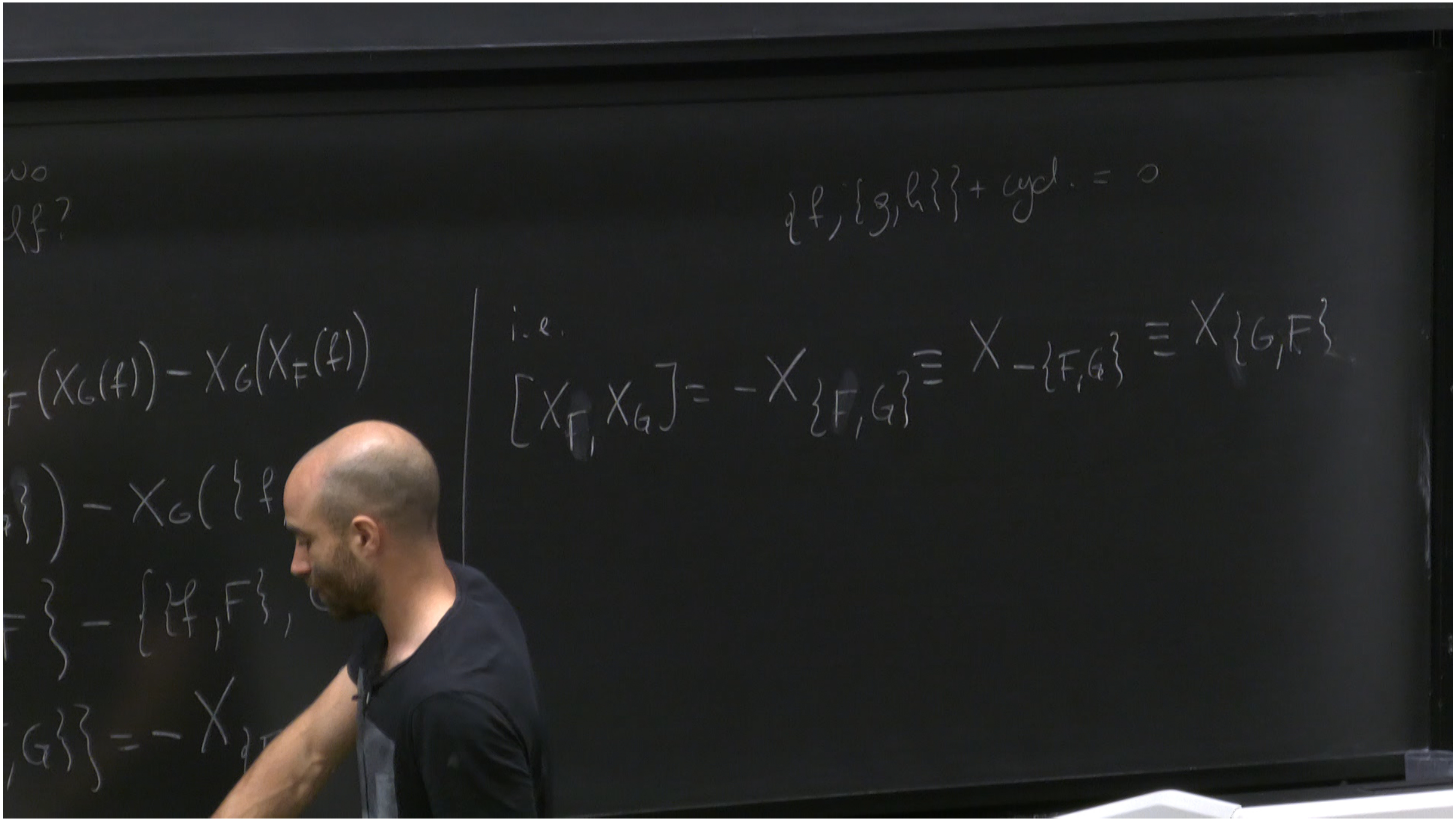
$= \{\{f, G\}, F\} - \{\{f, F\}, G\}$

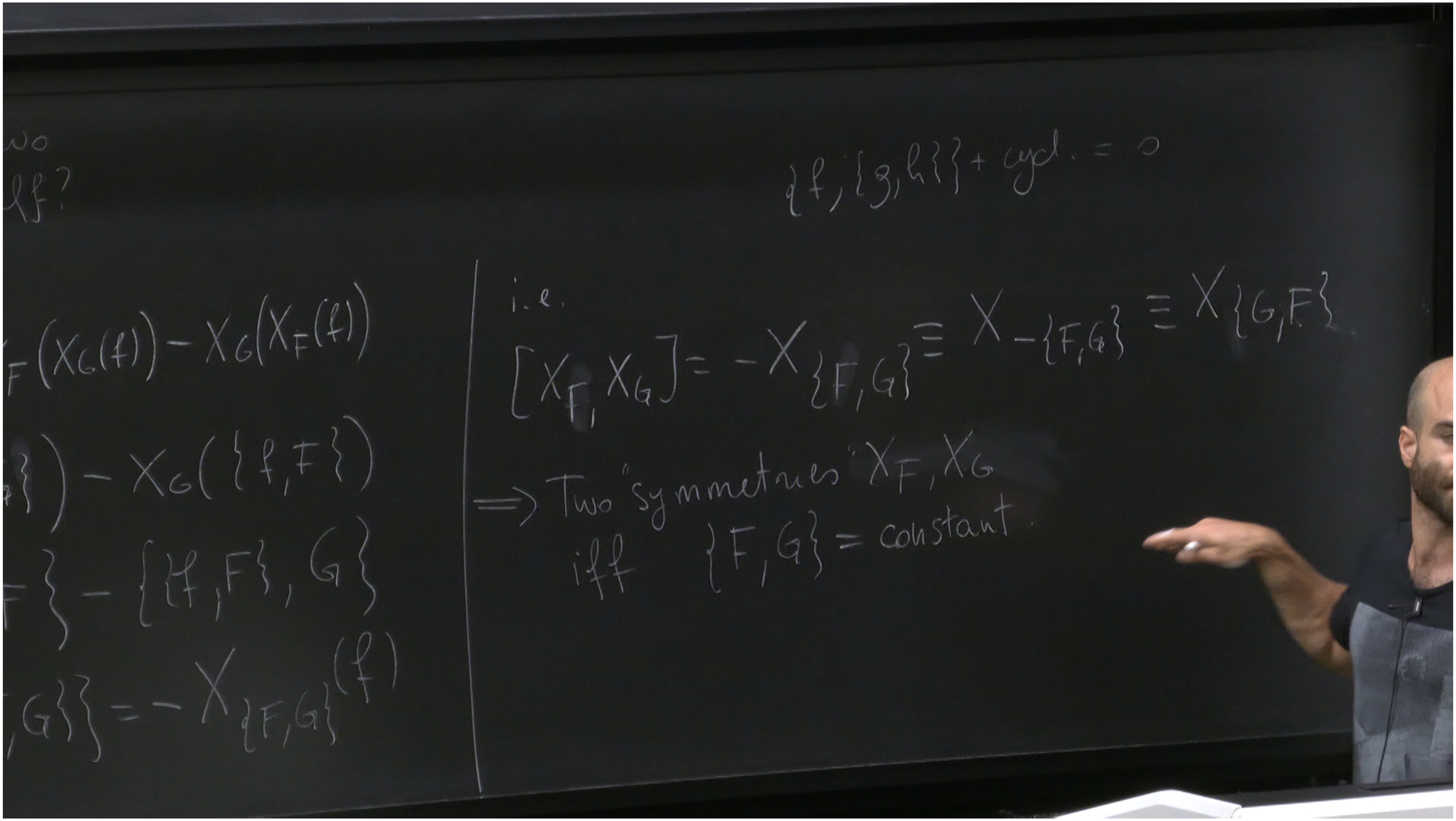
Jacobi

$= -\{f, \{F, G\}\} = -X_{\{F,G\}}(f)$

i.e. $[X_F, X_G] = -X_{\{F,G\}} \equiv$

$\{f, \{g, h\}\}$


{f,{g,h}} + cycl. = 0
i.e.
[X_F, X_G] = −X_{F,G} ≡ X_−{F,G} ≡ X_{G,F}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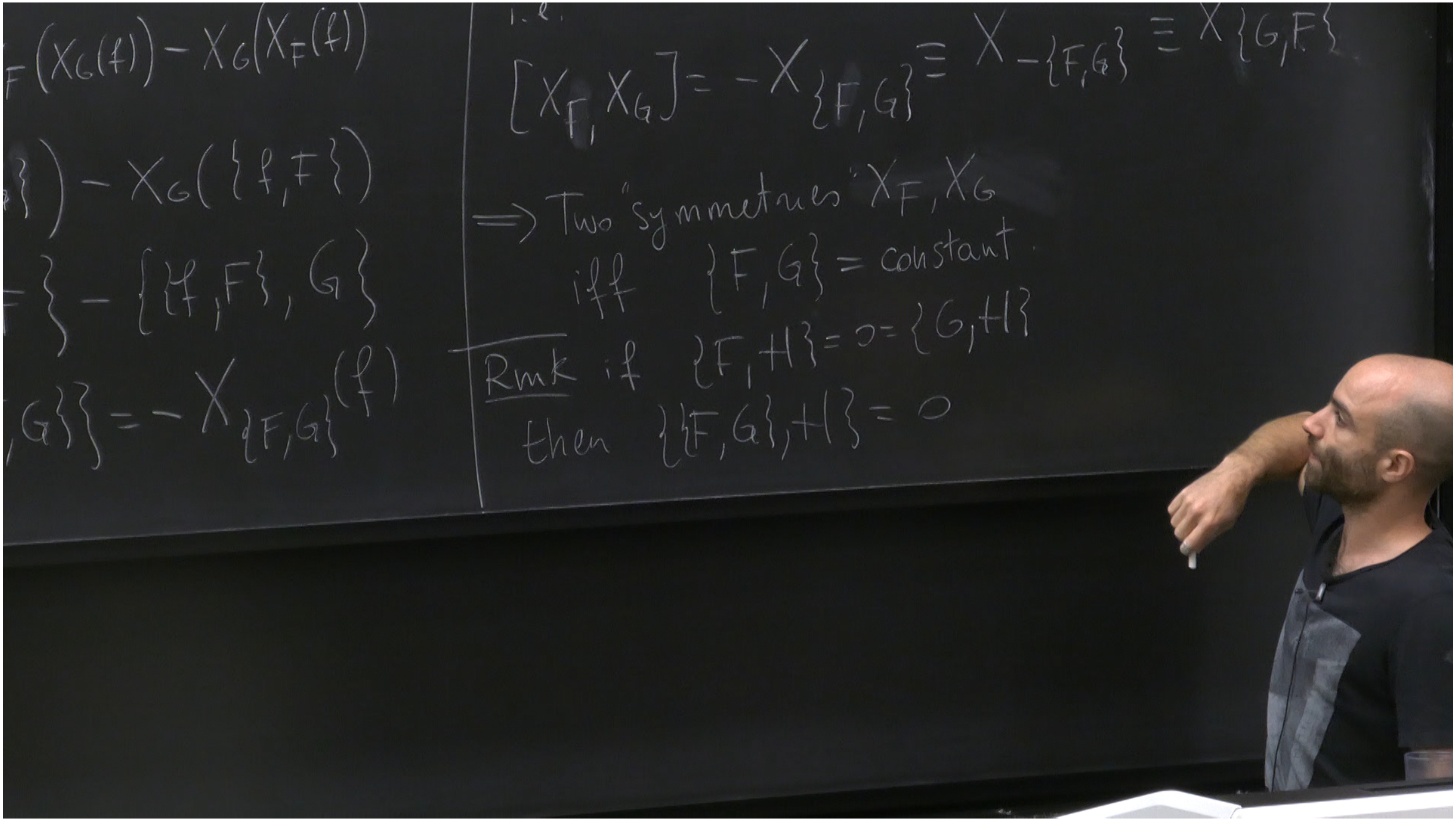
⟹ Two "symmetries" X_F, X_G
iff {F,G} = constant.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⟹ Two "symmetries" X_F, X_G
iff {F, G} = constant.
Rmk if {F, H} = 0 = {G, H}
then {{F, G}, H} = 0

$$\ldots_F(X_G(f)) - X_G(X_F(f))$$

$$\ldots\}) - X_G(\{f,F\})$$

$$\ldots F\} - \{\{f,F\},G\}$$

$$\ldots,G\}\} = -X_{\{F,G\}}(f)$$

i.e.

$$[X_F, X_G] = -X_{\{F,G\}} \equiv X_{-\{F,G\}} \equiv X_{\{G,F\}}$$

$\Longrightarrow$ Two "symmetries" X_F, X_G <u>commute</u>
iff $\{F,G\}$ = constant.

<u>Rmk</u> if $\{F,H\} = 0 = \{G,H\}$
then $\{\{F,G\},H\} = 0$

$[\cdot,\cdot] : \mathfrak{X}^1(P)\times\mathfrak{X}^1(P)\longrightarrow\mathfrak{X}^1(P)$

satisfies ①-③ and turns $\mathfrak{X}^1(P)$ into a Lie algebra.

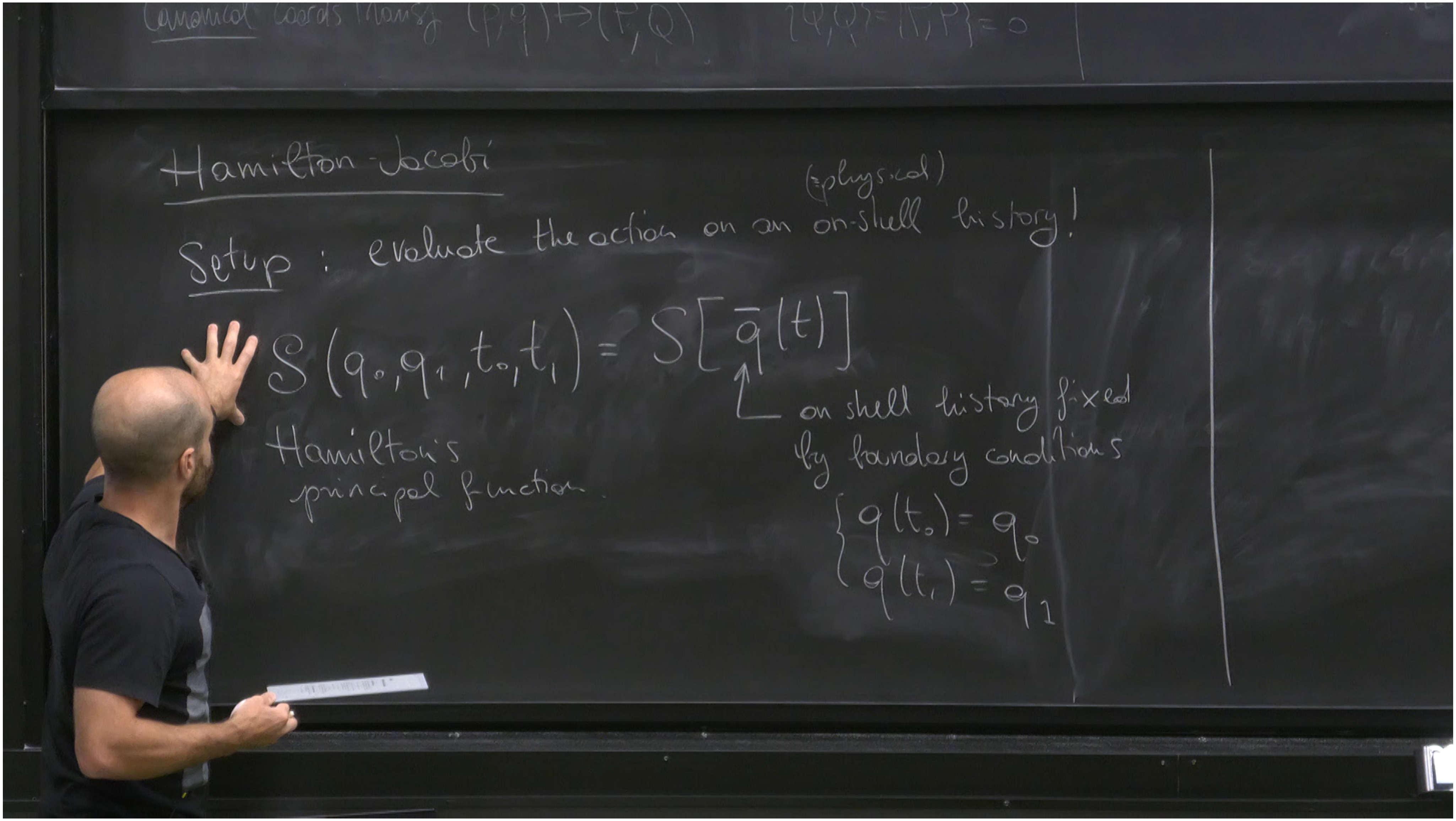
$$= \{\{f,G\},F\} - \{\{f,F\},G\}$$

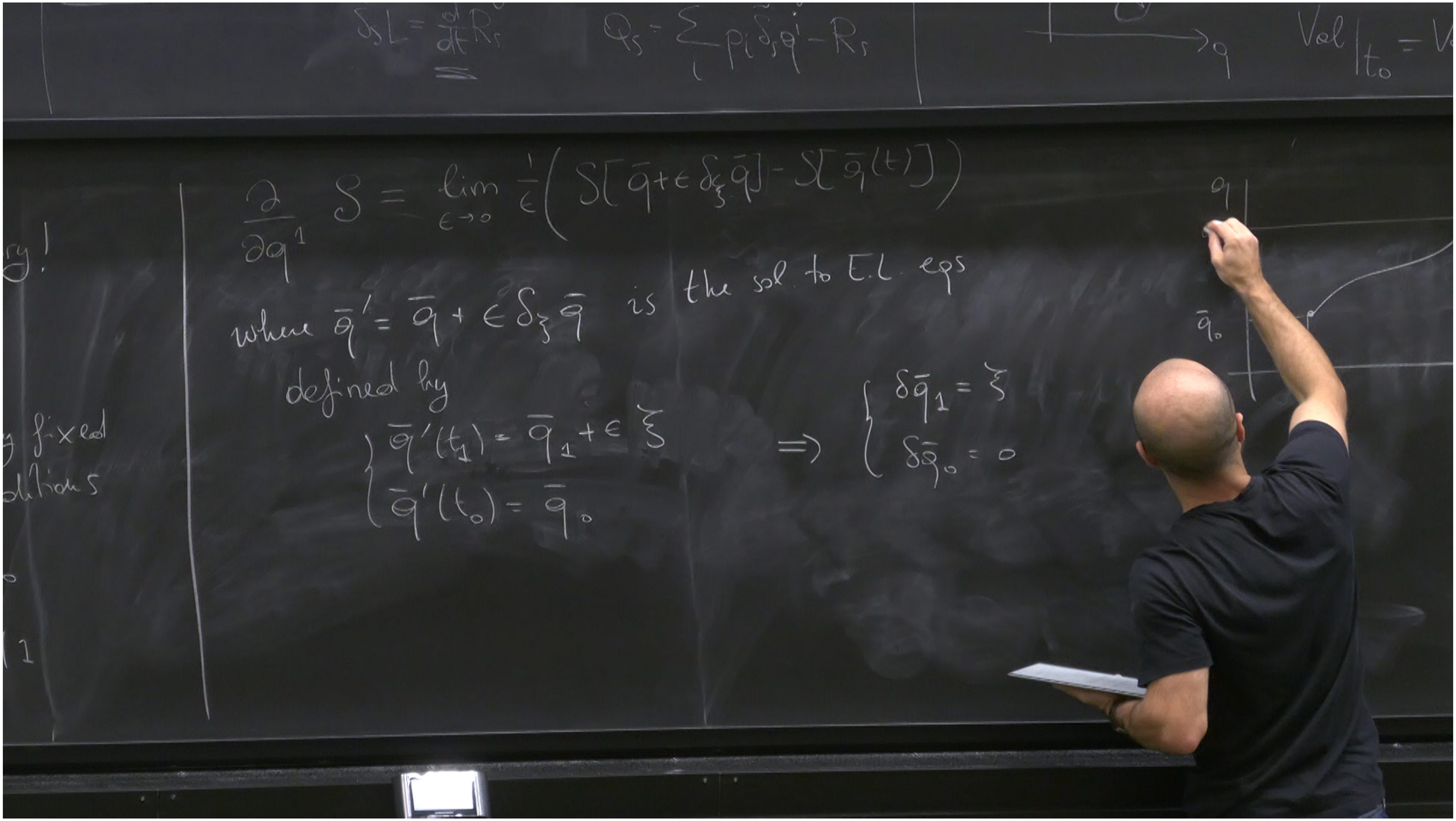
Jacobi

$$= -\{f,\{F,G\}\} = -X_{\{F,G\}}(f)$$

"Quantization"

P phase sp	$\longleftrightarrow$	$\mathcal{H}$ Hilbert sp
$z=(q,p)\in P$	$\longleftrightarrow$	$\psi\in\mathcal{H}$ state
$F\in C^\infty(P)$ real	$\longleftrightarrow$	$\hat{F}:\mathcal{H}\to\mathcal{H}$ Hermitian
$F(z)$		$\langle\psi\vert F\vert\psi\rangle$
Liouville	$\longleftrightarrow$	Ehrenfest
$\{F,G\}$ Poisson	$\longleftrightarrow$	$[\hat{F},\hat{G}]$
$X_F=\{\cdot,F\}$	$\longleftrightarrow$	$[\ ,\hat{F}]$ infinitesimal unitary transf $U=e^{-\frac{i\epsilon\hat{F}}{\hbar}}$, $U^\dagger\hat{G}U=\frac{\epsilon}{i\hbar}[\hat{G},\hat{F}]+O(\epsilon^2)$

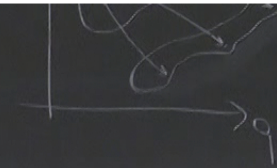

Canonical coords transf (p,q) ↦ (P,Q) {Q,Q}={P,P}=0
Hamilton-Jacobi
Setup : evaluate the action on an on-shell (=physical) history!
S(q₀,q₁,t₀,t₁) = S[q̄(t)]
Hamilton's principal function
on shell history fixed by boundary conditions
q(t₀) = q₀
q(t₁) = q₁



$\delta_\xi L = \frac{d}{dt} R_\xi$
$Q_\xi = \sum_i p_i \delta_\xi q^i - R_\xi$
$Vol|_{t_0} =$
$\frac{\partial}{\partial q^1} S = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(S[\bar{q} + \epsilon \delta_\xi \bar{q}] - S[\bar{q}(t)] \right)$
where $\bar{q}' = \bar{q} + \epsilon \delta_\xi \bar{q}$ is the sol to E.L. eqs
defined by
$\bar{q}'(t_1) = \bar{q}_1 + \epsilon \xi$
$\bar{q}'(t_0) = \bar{q}_0$
$\Longrightarrow \delta \bar{q}_1 = \xi$
$\delta \bar{q}_0 = 0$
fixed

oether 1st : infinitesimal sym $\Rightarrow$ conserved quantity

$$\tilde{\delta}_\xi L = \frac{d}{dt} R_\xi \qquad Q_\xi = \sum_i p_i \tilde{\delta}_\xi q^i - R_\xi$$



from t_0 to t_1

$$Vol\big|_{t_0} = Vol\big|_{t_1}$$

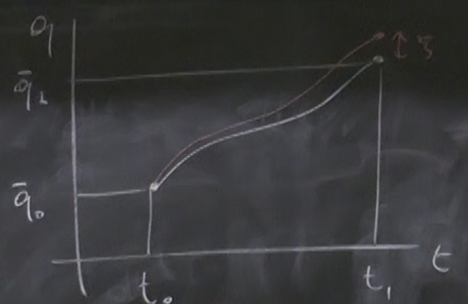
phase space Vol.

$$\sum_i \xi^i \frac{\partial}{\partial (q^1)^i} S = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(S[\bar{q} + \epsilon \delta_\xi \bar{q}] - S[\bar{q}(t)] \right) = \delta_\xi S$$

where $\bar{q}' = \bar{q} + \epsilon \delta_\xi \bar{q}$ is the sol. to E.L. eqs

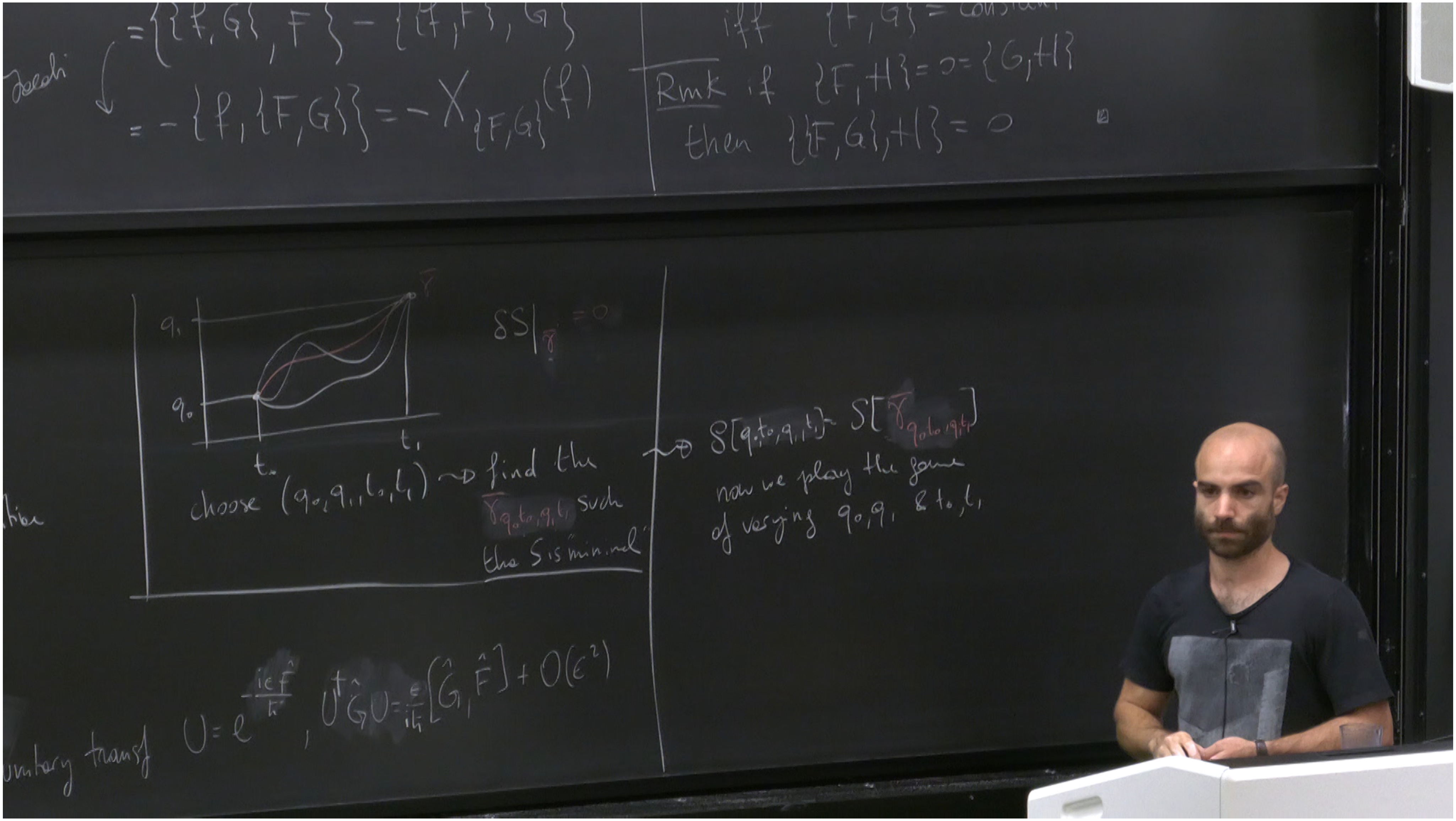
defined by

$$\begin{cases} \bar{q}'(t_1) = \bar{q}_1 + \epsilon \xi \\ \bar{q}'(t_0) = \bar{q}_0 \end{cases} \Rightarrow \begin{cases} \delta \bar{q}_1 = \xi \\ \delta \bar{q}_0 = 0 \end{cases}$$



$$\frac{\partial S}{\partial (q^1)^i} = \bar{p}_i(t_1)$$

$$\sum_i \xi^i \frac{\partial}{\partial (q^1)^i} S = \delta_\xi S = \int_{t_0}^{t_1} dt\, \delta_\xi L = \int_{t_0}^{t_1} dt \sum_i \underbrace{(\text{Euler-Lagr})_i}_{\equiv 0 \ (\bar{q} \text{ on shell!})} \delta_\xi \bar{q}^i + \left[\sum_i \bar{p}_i \delta_\xi \bar{q}^i \right]_{t_0}^{t_1} = \sum_i \bar{p}_i(t_1) \xi^i$$


= {{f, G}, F} − {{f, F}, G}
= −{f, {F, G}} = −X_{F,G}(f)
iff {F, G} = constant
Rmk if {F, H} = 0 = {G, H}
then {{F, G}, H} = 0
δS|_γ̄ = 0
q1
q0
t0
t1
γ̄
choose (q0, q1, t0, t1) ⇝ find the γ̄_{q0,t0,q1,t1} such that S is "minimal"
⇝ S[q0,t0,q1,t1] = S[γ̄_{q0,t0,q1,t1}]
now we play the game of varying q0, q1 & t0, t1
unitary transf U = e^{−iεF̂/ħ}, U†ĜU = ε/iħ [Ĝ, F̂] + O(ε²)