Title: Classical Physics Lecture - 090823

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Proof (Noether 1st)

If $\tilde{\delta}_s q(t)$ is an infinitesimal sym of L, i.e. $\tilde{\delta}_s L = \frac{d}{dt} R_s$, then

$$Q_s(t) = \sum_i p_i \tilde{\delta}_s q^i - R_s$$ is conserved

on shell of the e.o.m.

$$\left[p_i(t) := \frac{\partial L}{\partial \dot{q}^i}(q(t), \dot{q}(t), t) \right]$$

ed

$$\tilde{\delta}_s L = \boxed{\frac{d}{dt} R_s}$$

$$= \sum_i \left(\frac{\partial L}{\partial q^i} \tilde{\delta} q^i + \frac{\partial L}{\partial \dot{q}^i} \tilde{\delta} \dot{q}^i \right) , \quad \tilde{\delta} \dot{q} = \frac{d}{dt} \tilde{\delta} q$$

$$= \boxed{\sum_i \underbrace{\left(\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right)}_{\text{E.L. eom}} \tilde{\delta} q^i + \frac{d}{dt} \left(\underbrace{\frac{\partial L}{\partial \dot{q}^i}}_{p_i} \tilde{\delta} q^i \right)}$$

Rearranging: $\frac{d}{dt} \left(\sum_i p_i \tilde{\delta} q^i - R_s \right) = - \sum_i (EL)_i \, \tilde{\delta} q^i \mathrel{\hat{=}} 0$

$, \quad \tilde{\delta}_s \dot{q} = \frac{d}{dt}\tilde{\delta}_s q$

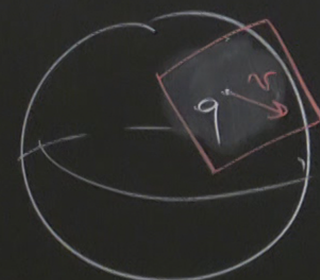
$$+ \frac{d}{dt}\Bigg(\underbrace{\frac{\partial L}{\partial \dot{q}^i}}_{p_i}\tilde{\delta}_s q^i\Bigg)$$

$$\tilde{\delta}_s q^i - R_s\Big) = -\sum_i (EL)_i\,\tilde{\delta}_s q^i \hat{=} 0$$

$TQ \ni (q^i, v^i)$

$T^*Q \ni (q^i, p_i)$

TQ



$$p_i = \frac{\partial L}{\partial v^i}$$

$$(v')^i = \sum_j R^i{}_j v^j$$

$$p_i' = \frac{\partial L(v')}{\partial v'^i} = \sum_j \frac{\partial v^j}{\partial v'^i} \frac{\partial L}{\partial v^j} = (R^{-1})^j{}_i \, p_j$$

inverse!

$$L(v') = L(v)$$

$$\sum_j p_j v^j = \sum_j p_j' v'^j$$

$T_q^* Q$ $\quad$ $T_q Q$

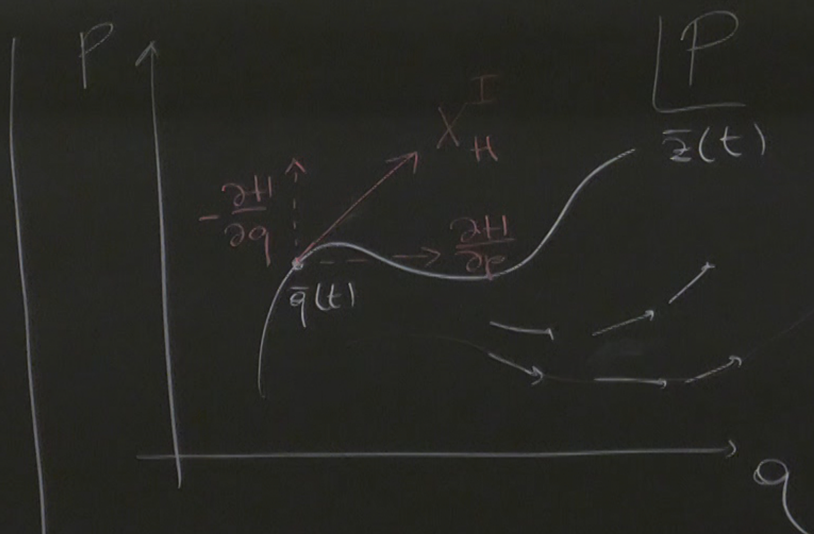
PHASE SPACE STRUCTURES

$\mathcal{P} := T^*Q$ phase space

$\cup$

$z^I = (q^i, p_i)$ $\quad I = 1, \dots, 2n$ $\quad i = 1, \dots, n$

$\dim(\mathcal{P})$ must be even.

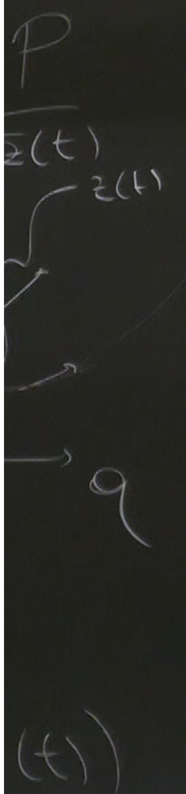


$$\dot{\bar z}^I = X_H^I(\bar z(t)) \qquad X^I = \begin{pmatrix} \partial H/\partial p_i \\ -\partial H/\partial q^i \end{pmatrix}$$

$$\begin{cases} \dot{\bar q}^i = \left.\dfrac{\partial H}{\partial p_i}\right|_{(\bar q, \bar p, t)} \\ \dot{\bar p}_i = -\left.\dfrac{\partial H}{\partial q^i}\right|_{(\bar q, \bar p, t)} \end{cases}$$

on shell history

$$\bar z(t) = (\bar q(t), \bar p(t))$$



$$\dot{z}^I = X_H^I(\bar{z}(t)) \qquad X_H^I = \begin{pmatrix} \partial H/\partial p_i \\ -\partial H/\partial q^i \end{pmatrix}$$ Hamiltonian vector field

$$\begin{cases} \dot{q}^i = \left.\dfrac{\partial H}{\partial p_i}\right|_{(\bar{q},\bar{p},t)} \\ \dot{p}_i = -\left.\dfrac{\partial H}{\partial q^i}\right|_{(\bar{q},\bar{p},t)} \end{cases}$$

Analogy w/ fluid dyn.

- history of system in P = trajectory of fluid particle
- Hamilt. v.f. X_H^I = velocity field of fluid

$$\dot{z}^I = X_H^I(\bar{z}(t)) \qquad X_H^I = \begin{pmatrix} \partial H/\partial p_i \\ -\partial H/\partial q^i \end{pmatrix}$$ Hamiltonian vector field

off shell

on shell

$$\begin{cases} \dot{q}^i = \left.\dfrac{\partial H}{\partial p_i}\right|_{(\bar{q},\bar{p},t)} \\ \dot{p}_i = -\left.\dfrac{\partial H}{\partial q^i}\right|_{(\bar{q},\bar{p},t)} \end{cases}$$

Analogy w/ fluid dyn.

- history of system in $\underline{P}$ = trajectory of fluid particle
- Hamilt. v.f. X_H^I = velocity field of fluid

$F : \mathcal{P} \times \mathbb{R} \longmapsto \mathbb{R}$

$(z, t) \longmapsto F(z, t)$

given an onshell history $\bar{z}(t)$:

$$f_{\bar{z}}(t) = F(\bar{z}(t), t)$$

$\frac{d}{dt} f_{\bar{z}}$ = change of F along the trajectory $= \frac{\partial F}{\partial t} + \sum_i \frac{\partial F}{\partial q^i} \dot{q}^i + \frac{\partial F}{\partial p_i} \dot{p}_i \Big|_{\bar{z}}$

$$= \frac{\partial F}{\partial t} + \sum_i \frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q^i} \Big|_{\bar{z}}$$

$$\frac{\partial F}{\partial p_i}\dot p_i \Big|_{\bar z}$$

$$\frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i}\Big)\Big|_{\bar z}$$

Poisson bracket

$$\{\cdot,\cdot\}: C^\infty(P)\times C^\infty(P) \longrightarrow C^\infty(P)$$

$$(F\ ,\ H) \longmapsto \{F,H\} = \sum_i \frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i}$$

$$\Rightarrow \dot f_z = \partial_t F + \{F,H\}\Big|_{\bar z}$$

or more conveniently

" $\dot F = \partial_t F + \{F,H\}$ "

$$\dot{z}^I = X_H^I(\bar{z}(t)) \qquad X_H^I = \begin{pmatrix} \partial H/\partial p_i \\ -\partial H/\partial q^i \end{pmatrix}$$ Hamiltonian vector field

(t) off shell

on shell

$$\begin{cases} \dot{q}^i = \dfrac{\partial H}{\partial p_i}\Big|_{(\bar{q},\bar{p},t)} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q^i}\Big|_{(\bar{q},\bar{p},t)} \end{cases}$$

vector field over P

vector $\downarrow$ one arrow at $p \in P$

field $\downarrow$ at all p's

Analogy w/ fluid dyn.

- history of system in P = trajectory of fluid particle
- Hamlt. v.f. X_H^I = velocity field of fluid

$$\frac{\partial F}{\partial p_i}\dot{p}_i \Big|_{\bar{z}}$$

$$\frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i}\Big)\Big|_{\bar{z}}$$

Poisson bracket

$$\{\cdot,\cdot\} : C^\infty(P)\times C^\infty(P) \longrightarrow C^\infty(P)$$

$$(F, H) \longmapsto \{F,H\} = \sum_i \frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i}$$

$$\Rightarrow \dot{f}_{\bar{z}} = \partial_t F + \{F,H\}\Big|_{\bar{z}}$$

or more conveniently

$$\text{“}\ \dot{F} = \partial_t F + \{F,H\}\ \text{”}$$

$\leftrightarrow$ Hamilton's eom

properties of $\{\cdot,\cdot\}$

1) skew sym $\{F,G\}+\{G,F\}\equiv 0$

2) (bi) linear $\{F, aG+H\} = a\{F,G\}+\{F,H\}$, $a\in\mathbb{R}$

3) Jacobi $\{F,\{G,H\}\}+\{G,\{H,F\}\}+\{H,\{F,G\}\}\equiv 0 \leftarrow \left(\frac{\partial}{\partial q}\right.$

4) Leibniz $\{F,GH\}=\{F,G\}H+\{F,H\}G$

A vector space V $(=C^\infty(P))$ with a bracket satisfyin (1-3) is a Lie algebra, and the bracket is called a Lie bracket.

$\{\cdot,\cdot\}$

$\{F,G\}+\{G,F\}\equiv 0$

$\{F,aG+H\}=a\{F,G\}+\{F,H\}$

$a\in\mathbb{R}$

$\{F,\{G,H\}\}+\{G,\{H,F\}\}+\{H,\{F,G\}\}\equiv 0 \leftarrow \left(\frac{\partial}{\partial q}\frac{\partial}{\partial p}=\frac{\partial}{\partial p}\frac{\partial}{\partial q}\right)$

$\{F,GH\}=\{F,G\}H+\{F,H\}G$

$\left(=C^\infty(P)\right)$ with a bracket satisfying (1-3)

, and the bracket is called a Lie bracket.

Revisited time translation sym:

Recall H on a history is its energy.

$$\frac{d}{dt}E = \dot{H} = \partial_t H + \underbrace{\{H, H\}}_{\text{skew} \equiv 0} = \partial_t H \underset{\substack{\uparrow \\ \text{Legendre}}}{=} -\partial_t L$$

$\Rightarrow$ if $\partial_t H = 0$ then energy is conserved.

$$\Updownarrow$$

$$\partial_t L$$

$\partial_t L$

Rmk (Quantum theory)

$$\{q^i, q^j\} = 0 = \{p_i, p_j\}$$

$$\{q^i, p_j\} = \delta^i_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

Remind us of

$$[\hat{q}^i, \hat{q}^j] = 0 = [\hat{p}_i, \hat{p}_j]$$

$$[\hat{q}^i, \hat{p}_j] = i\hbar\, \delta^i_j$$

ory)

$\{P_i, P_j\}$

$\delta^i_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

$F \in C^\infty(P \times \mathbb{R})$

$, \quad \dot{F} = \partial_t F + \{F, H\}$

$\hat{F} : \underbrace{\mathcal{H}}_{\text{Hilbert space}} \times \underbrace{\mathbb{R}}_{\text{time}} \longrightarrow \mathcal{H}$

$[\hat{P}_i, \hat{P}_j]$

ith δ^i_j

Ehrenfest thm

$, \quad \frac{d}{dt}\langle \hat{F} \rangle = \langle \partial_t \hat{F} \rangle + \frac{1}{i\hbar}\langle [\hat{F}, \hat{H}] \rangle$

Canonical transf.

can we choose other q, p's that suit us better?

i.e. $(q,p) \mapsto (Q,P) = (Q(q,p), P(q,p))$

such that the formulation of Ham. mech does not change?

$$f(q,p) = F\big(Q(q,p), P(q,p)\big)$$ same for $g(q,p) = G(\;\dots\;)$

$$\{F,G\}_{(Q,P)} \overset{!}{=} \{f,g\}_{(q,p)}$$

$$\{f,g\}_{(q,p)} = \sum_i \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}$$

$$\frac{\partial f}{\partial q^i} = \sum_k \boxed{\frac{\partial F}{\partial Q^k}}\,\boxed{\frac{\partial Q^k}{\partial q^i}}$$

$$= \sum_{k,l} \{Q^k, Q^l\}_{(p,q)} \frac{\partial F}{\partial Q^k}\frac{\partial G}{\partial Q^l} + \text{same for } P$$

$$+ \{Q^k, P_l\}_{(q,p)} \left(\frac{\partial F}{\partial Q^k}\frac{\partial G}{\partial P_l} - \frac{\partial F}{\partial P_l}\frac{\partial G}{\partial Q^k} \right)$$

$$\overset{!}{=} \{F, G\}_{(P,Q)}$$

for P

$$\frac{\partial G}{\partial Q^k}$$

we need:

$$\{Q^k, Q^\ell\}_{(q,p)} = 0 = \{P_k, P_\ell\}_{(q,p)}$$

$$\{Q^k, P_\ell\}_{(q,p)} = \delta^k_\ell$$

we then call $(q,p) \mapsto (Q,P)$

a <u>canonical transformation</u>

Infinitesimal canonical transf.

$Q^i = q^i + \epsilon \delta q^i$, $\delta q^i = \delta q^i(q,p)$

$P_i = p_i + \epsilon \delta p_i$, $\delta p_i = \delta p_i(q,p)$

$\{Q^i, P_j\} \overset{!}{=} \delta^i_j$

$\qquad ||$

$\underbrace{\{q^i, p_j\}}_{\delta^i_j} + \epsilon \left(\underbrace{\{q^i, \delta p_j\} + \{\delta q^i, p_j\}}_{\overset{!}{=} 0} \right)$

iff. $\frac{\partial \delta p_j}{\delta p_i} + \frac{\partial \delta q^i}{\partial q^j} = 0$ $\quad (*)$

therefore if

$\begin{cases} \delta q^i = \{q^i, F\} = \frac{\partial F}{\partial p_i} \\ \delta p_i = \{p_i, F\} = -\frac{\partial F}{\partial q^i} \end{cases}$

for some $F(p,q)$, then $(*) = 0$

therefore if

$$\begin{cases} \delta q^i = \{q^i, F\} = \dfrac{\partial F}{\partial p_i} \\ \delta p_i = \{p_i, F\} = -\dfrac{\partial F}{\partial q^i} \end{cases}$$

for some $F(p,q)$, then $(*) = 0$

This means that

$\delta z^I = \{z^I, F\}$ is an

infinitesimal

canonical transf. !

Rmk

In particular

time evolution $(F = H)$

is a canonical transformation, ie

$$\begin{cases} q_0^i = \text{the value of } q^i \text{ at time } t_0 \\ p_0^i = \text{ — '' — } p_i \text{ — '' — } \end{cases}$$

$$\begin{cases} q_1^i = \text{ — '' — } t_1 \\ p_1^i \end{cases}$$

then $(q_0, p_0) \to (q_1, p_1)$

is a canonical

transf.

$\dot{z}^I = X_H^I(z(t))$ $\qquad X_H^I = \begin{pmatrix} \partial H/\partial p_i \\ -\partial H/\partial q^i \end{pmatrix}$ Hamiltonian vector field

$z(t)$ off shell

$\dot{z}^I = \{z^I, H\}|_z$

on shell

$X_H = \sum_I X_H^I \frac{\partial}{\partial z^I}$

$X_H^I(z) = \{z^I, H\}$

$\sum_I X_H^I \frac{\partial F}{\partial z^I} = \{F, H\} \longleftrightarrow \boxed{X_H = \{\cdot, H\}}$

a vector field is a "directional derivative" and in particular

$$X : C^\infty(P) \longrightarrow C^\infty(P)$$

$$X(af+g) = aX(f) + X(g)$$

$$X(fg) = f\,X(g) + g\,X(f)$$

therefore if

$$\begin{cases} \delta q^i = \{q^i, F\} = \frac{\partial F}{\partial p_i} \\ \delta p_i = \{p_i, F\} = -\frac{\partial F}{\partial q^i} \end{cases}$$

for some $F(p,q)$, then

This means that

$$\delta z^I = \{z^I, F\}$$

infinitesimal canonical transf.!