Title: Classical Physics Lecture - 090623

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ACTION PRINCIPLE
& LAGRANGIAN MECHANICS

Setup

- generalized coordinates q^i, $i=1,\dots,N$
  they parametrize configuration space $Q \ni q$
- generalized velocities v^i
- tangent space to configuration space $(q,v) \in TQ$

Ex 1 (Particle in $\mathbb{R}^3$)

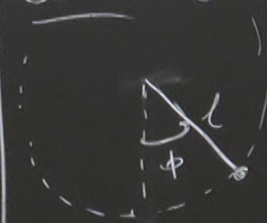
$Q = \mathbb{R}^3$

gen. coords #1 : $\vec{x}$ Cartesian

#2 : (r, θ, φ) spherical coords

$$TQ = \underset{\substack{\Downarrow \\ q}}{\mathbb{R}^3} \times \underset{\substack{\Downarrow \\ v}}{\mathbb{R}^3}$$

Ex 2 (Pendulum)



$Q = S^1$

$q = \theta \in (0, 2\pi)$

$$TQ = \underset{\substack{\Downarrow \\ q}}{S^1} \times \underset{\substack{\Downarrow \\ v}}{\mathbb{R}}$$

Ex 3 (Rigid body)

$$Q = \underbrace{\mathbb{R}^3}_{\text{c.o.m. pos.}} \times \underbrace{SO(3)}_{\text{orientation in space}} , \quad TQ = Q \times \mathbb{R}^6$$

- history

$$\gamma : \mathbb{R} \longrightarrow Q$$
$$t \longmapsto q^i = \gamma^i(t)$$

at time t, the velocity of the history γ is

$$v^i = \frac{d}{dt}\gamma^i(t) \equiv \dot{\gamma}^i(t)$$

- space of histories

$$\mathcal{H}_Q = C^\infty(\mathbb{R}, Q) \ni \gamma$$

- Lagrangian function

$= \gamma^i(t)$

elocity of

$) \equiv \dot{\gamma}^i(t)$

- space of histories

$$\mathcal{H}_Q = C^\infty(\mathbb{R}, Q) \ni \gamma$$

- Lagrangian function

$$L : TQ \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(q, v, t) \longmapsto L(q, v, t)$$

-

to configuration space $(q,v) \in TQ$

· Action

$$S : \mathcal{H}_Q \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

$$(\gamma, t_0, t_1) \longmapsto S(t_0, t_1, \gamma) = \int_{t_0}^{t_1} dt \; L(\gamma^i(t), \dot{\gamma}^i(t), t)$$

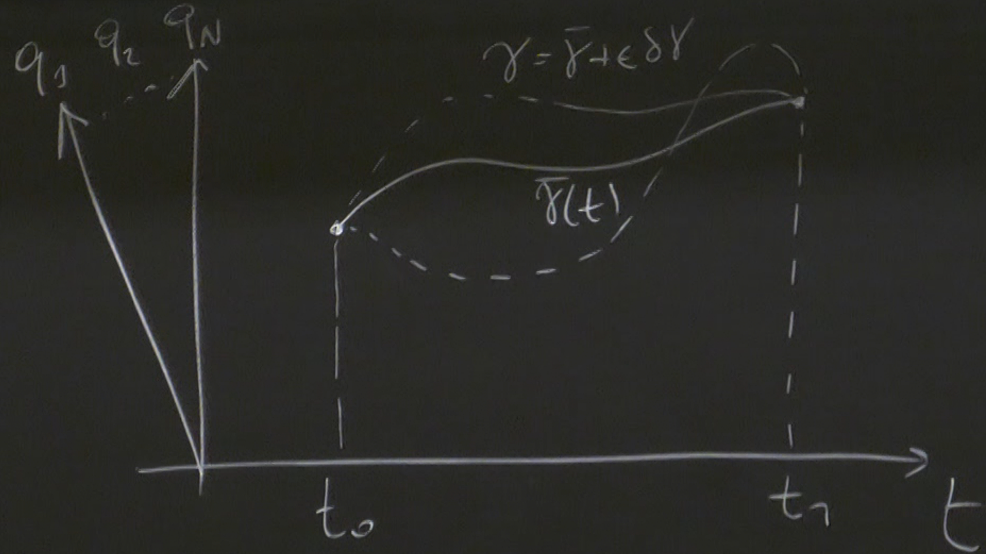
Language: action functional "function of a function"

Notation: $S[q(t)] = \int_{t_0}^{t_1} dt \; L(q, \dot{q}, t)$

Hamilton's Action Principle

The dynamics of any (closed) physical
is fully characterized by a Lagrangian function.
Its physically realized histories are extrema
of the associated action functional,
at fixed boundary conditions (in time).

Hamilt
princip



let me denote $\bar{\gamma}$ a physical history -

$\gamma = \bar{\gamma} + \epsilon \boxed{\delta\gamma}$

$\delta\gamma(t_0) = 0 = \delta\gamma(t_1)$

pos. infinitesimal

one "object"

Hamilton's principle $\boxed{S(\bar{\gamma}+\epsilon\,\delta\gamma, t_0, t_1) = S(\bar{\gamma}, t_0, t_1) + \mathcal{O}(\epsilon^2) \quad \forall\, \delta\gamma(t)}$

$\hookrightarrow$ often we write $\left.\dfrac{\delta S[q(t)]}{\delta q(t)}\right|_{\bar{q}(t)} = 0$

Let's compute the consequences!

we need: $\dot{\gamma}(t) = \dfrac{d}{dt}\left(\bar{\gamma}(t) + \epsilon\,\delta\gamma(t)\right) = \dot{\bar{\gamma}} + \epsilon \boxed{\dfrac{d}{dt}\delta\gamma(t)}$

$\delta\dot{\gamma}(t) \equiv \dfrac{d}{dt}\delta\gamma(t)$

$$L(\gamma(t), \dot\gamma(t), t) = L(\bar\gamma + \epsilon\delta\gamma, \dot{\bar\gamma} + \epsilon\delta\dot\gamma, t)$$

$$= L(\bar\gamma, \dot{\bar\gamma}, t) + \epsilon \sum_i \frac{\partial L}{\partial q^i}\bigg|_{(\bar\gamma, \dot{\bar\gamma}, t)} \delta\gamma^i(t)$$

$$+ \epsilon \sum_i \frac{\partial L}{\partial v^i}\bigg|_{(\bar\gamma, \dot{\bar\gamma}, t)} \delta\dot\gamma^i(t) + O(\epsilon^2)$$

$$\leadsto \text{"}\delta L = \sum_i \frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot q^i} \delta \dot q^i\text{"}$$

$$\delta S$$

$$S[\bar\gamma + \epsilon\,\delta\gamma] = S[\bar\gamma] + O(\epsilon^2)$$

iff.

$$0 = \delta S = \int_{t_0}^{t_1} dt\ \delta L(\bar\gamma, \dot{\bar\gamma}, t)$$

$$= \int_{t_0}^{t_1} dt\ \sum_i \left(\frac{\partial L}{\partial q^i}\Big|_{\bar\gamma} \delta\gamma^i + \frac{\partial L}{\partial v^i}\Big|_{\bar\gamma} \frac{d}{dt}\delta\gamma^i \right)$$

$$= \int_{t_0}^{t_1} dt\ \sum_i \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial v^i} \right)\Big|_{\bar\gamma} \underbrace{\delta\gamma^i(t)}_{\text{arbitrary}} + \left[\sum_i \frac{\partial L}{\partial v^i} \overbrace{\delta\gamma^i}^{\equiv 0 \text{ @ } t_0, t_1} \right]_{t_0}^{t_1}$$

Hamilton's principle

$\Downarrow$

Euler-Lagrange Eqs

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0$$

at a physical history.

Rmks

1) EL eqs are of the form $\ddot{q}^i = F(q,\dot{q})$

$\uparrow$ accelerations

(2nd order ODE) $\leftarrow$ (Newton)

2) $L = \sum_\alpha \frac{1}{2} m_\alpha \vec{v}_\alpha^{\,2} - V(\vec{x}_\alpha)$

$\Longrightarrow$ Euler Lagrange $\equiv$ Newton $\quad m\vec{a}_\alpha = -\frac{\partial V}{\partial \vec{x}_\alpha}$

$(t_0) = 0 = \delta q(t_1)$

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3) cyclical coords q^1 : $\frac{\partial L}{\partial q^1} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^1} \hat{=} 0 \Leftrightarrow p^1 = \frac{\partial L}{\partial \dot{q}^1}$ is c

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accelerations

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3) cyclical coords q^1: $\quad \frac{\partial L}{\partial q^1} = 0 \Longrightarrow p_1 = \frac{\partial L}{\partial \dot{q}^1}$ is conserved (on shell)

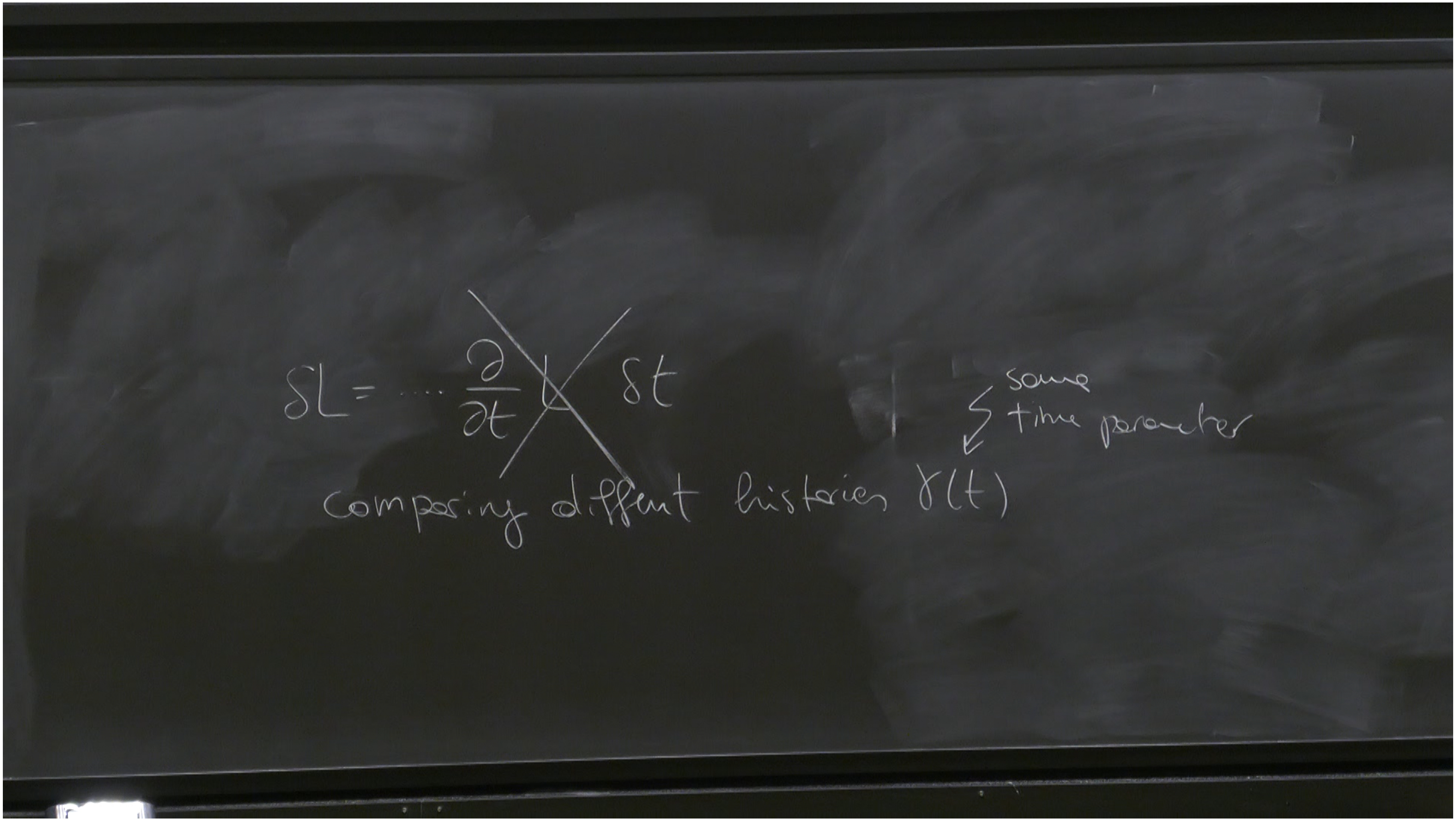
Ham action principle

$q_0 = \gamma(t_0)$ $\rightsquigarrow$ $\bar{\gamma}(t)$ such that $\bar{\gamma}(t_0) = q_0$

$q_1 = \gamma(t_1)$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\bar{\gamma}(t_1) = q_1$

Rmk not guaranteed to exist

Rmk sol.s to EL eqs are uniquely determined by $(q_0, v_0) = (\gamma(t_0), \dot{\gamma}(t_0))$


δL = ... ∂/∂t L δt
some time parameter
comparing diffent histories γ(t)

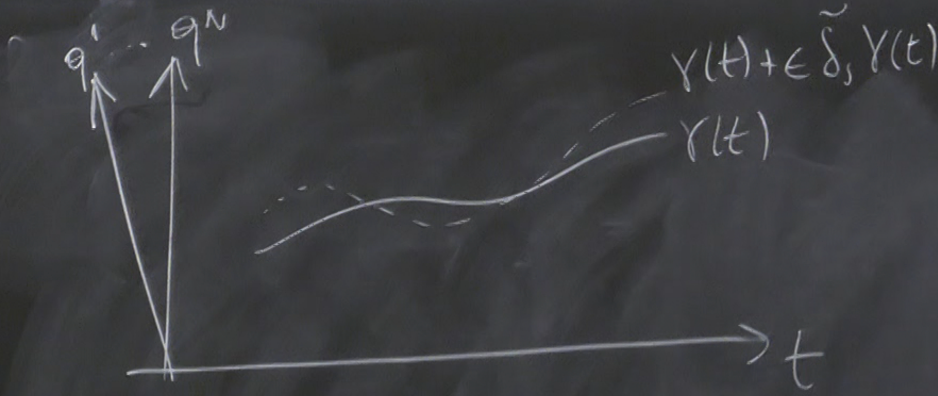
Noether's 1st theorem

consider a function s of the form:

$$TQ \times \mathbb{R} \longrightarrow TQ$$
$$(q, v, t) \longmapsto (q, s(q, v, t))$$

Given a history, I use s to define the following 1st order variation:

$$\gamma(t) \rightsquigarrow \gamma(t) + \epsilon \tilde{\delta}_s \gamma \qquad \tilde{\delta}_s \gamma(t) = s(\gamma(t), \dot{\gamma}(t), t)$$



Def We call s a symmetry of L

iff $\quad L(\gamma + \epsilon \delta_s \gamma, \dot{\gamma} + \epsilon \hat{\delta}_s \dot{\gamma}, t) = L(\gamma, \dot{\gamma}, t) + \epsilon \frac{d}{dt} R_s(\gamma, \dot{\gamma}, \ldots$

iff $\quad \tilde{\delta}_s L = \frac{d}{dt} R_s$

$\ldots \gamma(t) + \epsilon \hat{\delta}_s \gamma(t)$

$\ldots \gamma(t)$

$\longrightarrow t$

symmetry of L

$$\ldots \delta_s \gamma, \dot{\gamma} + \epsilon \hat{\delta}_s \dot{\gamma}, t) = L(\gamma, \dot{\gamma}, t) + \epsilon \frac{d}{dt} R_s(\gamma, \dot{\gamma}, \ldots) + O(\epsilon^2)$$

$$\frac{d}{dt} R_s$$

$\epsilon^2)$

Theorem

If s is a symmetry of L, then

$$Q_s(t) := \sum_i \frac{\partial L}{\partial \dot{q}^i} \tilde{\delta}_s q^i - R_s$$

is conserved on-shell:

$$\frac{d}{dt} Q_s(t) \hat{=} 0 .$$

[Noether 1]

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Ex:

cyclical coordinate q^1:

$$\tilde{\delta}_s q^i(t) = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}$$

$$\begin{cases} q^1(t) \to q^1(t) + \epsilon \\ q^{i\neq 1}(t) \to q^{i \neq 1}(t) \end{cases}$$

$$\Longrightarrow \frac{\partial L}{\partial q^1} = 0 \quad \text{iff} \quad \tilde{\delta}_s L = 0, \quad R_s = 0$$

q^1:

if $i = 1$

if $i \neq 1$

$$Q_s(t) = \sum_i \frac{\partial L}{\partial \dot{q}^i} \tilde{\delta}_s q^i - R_s$$

$R_s \equiv 0$

$\tilde{\delta}_s q^i = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}$

$$= \frac{\partial L}{\partial \dot{q}^1} \text{ is conserved}$$

$\tilde{\delta}_s L = 0$, $R_s = 0$

Hamilton's Ac

The dynamics

is fully chara

Its physical

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