Title: Classical Physics Lecture - 090623

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Collection: Classical Physics 2023/24

Date: September 06, 2023 - 9:00 AM

URL: https://pirsa.org/23090027

ACTION PRINCIPLE & LAGRANGIAN MECHANICS

Setup

- generalized coordinates q^i, $i = 1, \dots, N$
  they parametrize <u>configuration space</u> $Q \ni q$
- generalized velocities v^i
- tangent space to configuration space $(q, v) \in TQ$

Ex 1 (Particle in $\mathbb{R}^3$)

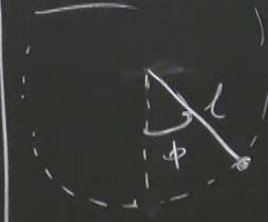
$Q = \mathbb{R}^3$

gen. coords #1 : $\vec{x}$ Cartesian

#2 : (r, θ, φ) spherical coords

$TQ = \underset{\downarrow}{\mathbb{R}^3}_{q} \times \underset{\downarrow}{\mathbb{R}^3}_{v}$

Ex 2 (Pendulum)



$Q = S^1$

$q = \theta \in (0, 2\pi)$

$TQ = \underset{\downarrow}{S^1}_{q} \times \underset{\downarrow}{\mathbb{R}}_{v}$

Ex 3 (Rigid body)

$Q = \underset{\uparrow}{\mathbb{R}^3}_{\text{C.o.m pos.}} \times \underset{\uparrow}{SO(3)}_{\text{orientation in space}}$, $TQ = Q \times \mathbb{R}^6$

- history

$$\gamma : \mathbb{R} \longrightarrow Q$$
$$t \longmapsto q^i = \gamma^i(t)$$

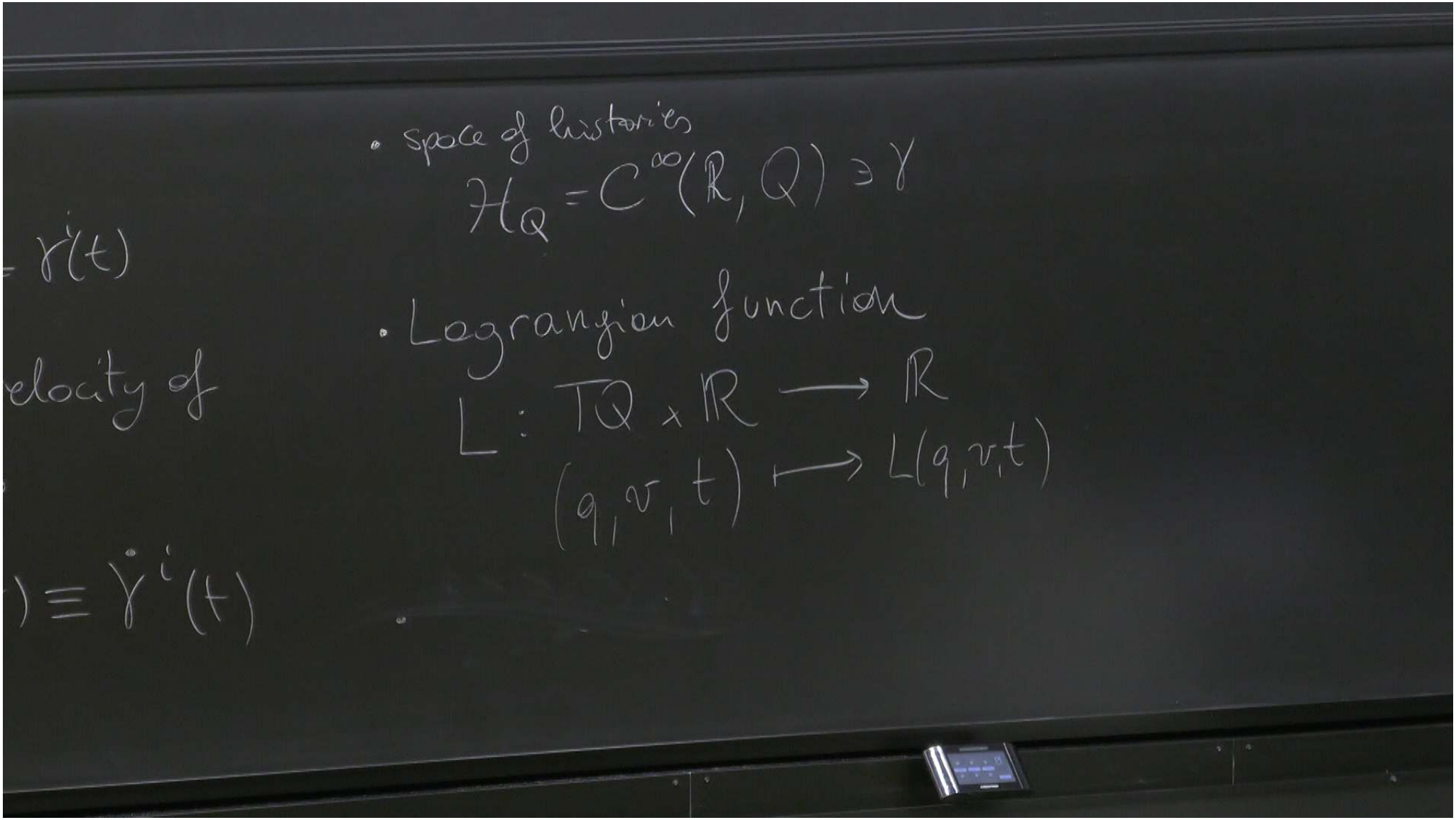
at time t, the velocity of the history γ is

$$v^i = \frac{d}{dt}\gamma^i(t) \equiv \dot{\gamma}^i(t)$$

- space of histories

$$\mathcal{H}_Q = C^\infty(\mathbb{R}, Q) \ni \gamma$$

- Lagrangian function


• space of histories
$\mathcal{H}_Q = C^\infty(\mathbb{R}, Q) \ni \gamma$
$= \gamma^i(t)$
• Lagrangian function
elocity of
$L : TQ \times \mathbb{R} \longrightarrow \mathbb{R}$
$(q, v, t) \longmapsto L(q, v, t)$
$) \equiv \dot{\gamma}^i(t)$

to configuration space $(q, v) \in TQ$

- Action

$$S : \mathcal{H}_Q \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

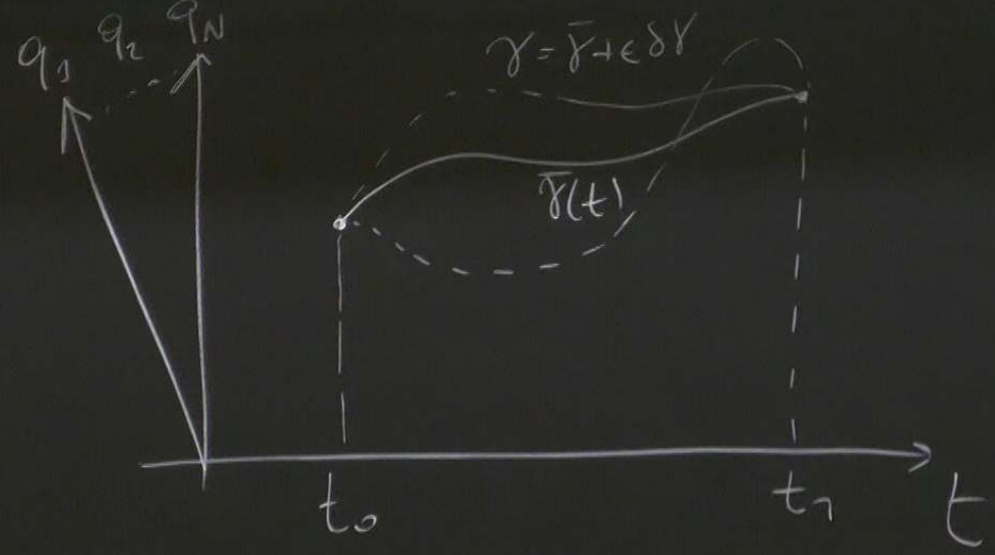
$$(\gamma, t_0, t_1) \longmapsto S(t_0, t_1, \gamma) = \int_{t_0}^{t_1} dt \; L(\gamma^i(t), \dot{\gamma}^i(t), t)$$

Language: action functional (S ↓, γ ↓) "function of a function"

Notation: $S[q(t)] = \int_{t_0}^{t_1} dt \; L(q, \dot{q}, t)$

Hamilton's Action Principle

The dynamics of any (closed) physical
is fully characterized by a Lagrangian function.
Its physically realized histories are extrema
of the associated action functional,
at fixed boundary conditions (in time).



Hamilton principle

let me denote $\bar{\gamma}$ a physical history -

$$\gamma = \bar{\gamma} + \epsilon\,\boxed{\delta\gamma} \qquad \delta\gamma(t_0) = 0 = \delta\gamma(t_1)$$

pos. infinitesimal — one "object"

Hamilton's principle $S(\bar{q}+\epsilon\delta q, t_0, t_1) = S(\bar{q}, t_0, t_1) + O(\epsilon^2) \quad \forall \delta q(t)$

often we write $\left.\dfrac{\delta S[q(t)]}{\delta q(t)}\right|_{\bar{q}(t)} = 0$

$\delta\dot{q}(t) = \dfrac{d}{dt}\delta q(t)$

Let's compute the consequences!

we need: $\dot{\bar{q}}(t) = \dfrac{d}{dt}\left(\bar{q}(t) + \epsilon\,\delta q(t)\right) = \dot{\bar{q}} + \epsilon\dfrac{d}{dt}\delta q(t)$

$$L(\gamma(t),\dot\gamma(t),t) = L(\bar\gamma+\epsilon\delta\gamma,\ \dot{\bar\gamma}+\epsilon\delta\dot\gamma,\ t)$$

$$= L(\bar\gamma,\dot{\bar\gamma},t) + \epsilon\sum_i \left.\frac{\partial L}{\partial q^i}\right|_{(\bar\gamma,\dot{\bar\gamma},t)} \delta\gamma^i(t)$$

$$+ \epsilon\sum_i \left.\frac{\partial L}{\partial v^i}\right|_{(\bar\gamma,\dot{\bar\gamma},t)} \delta\dot\gamma^i(t) + O(\epsilon^2)$$

$$\rightsquigarrow\ "\delta L = \sum_i \frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot q^i}\delta\dot q^i"$$

δS

$$S[\bar{\gamma} + \epsilon \delta\gamma] = S[\bar{\gamma}] + O(\epsilon^2)$$

iff.

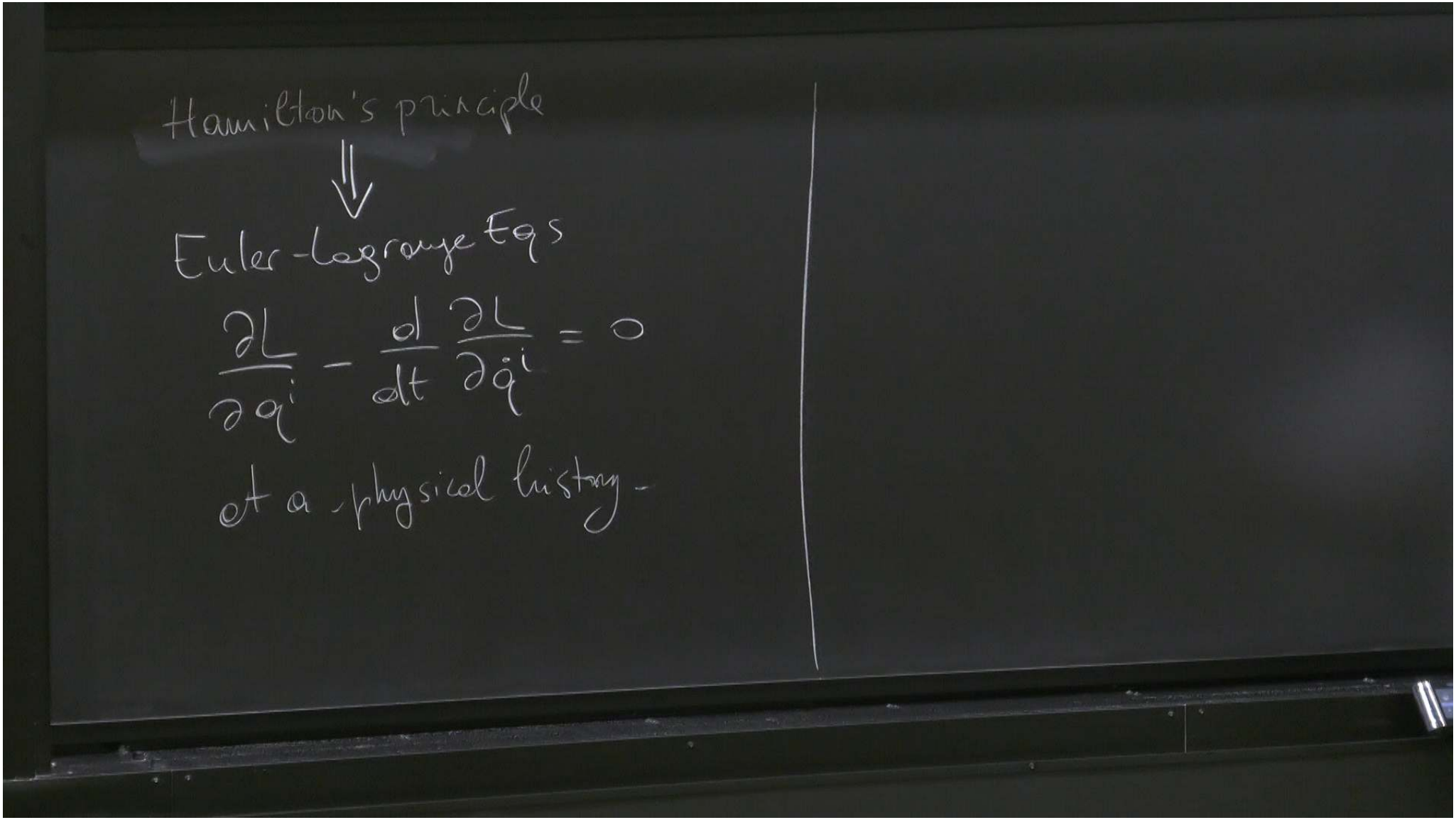
$$0 = \delta S = \int_{t_0}^{t_1} dt \; \delta L(\bar{\gamma}, \dot{\bar{\gamma}}, t)$$

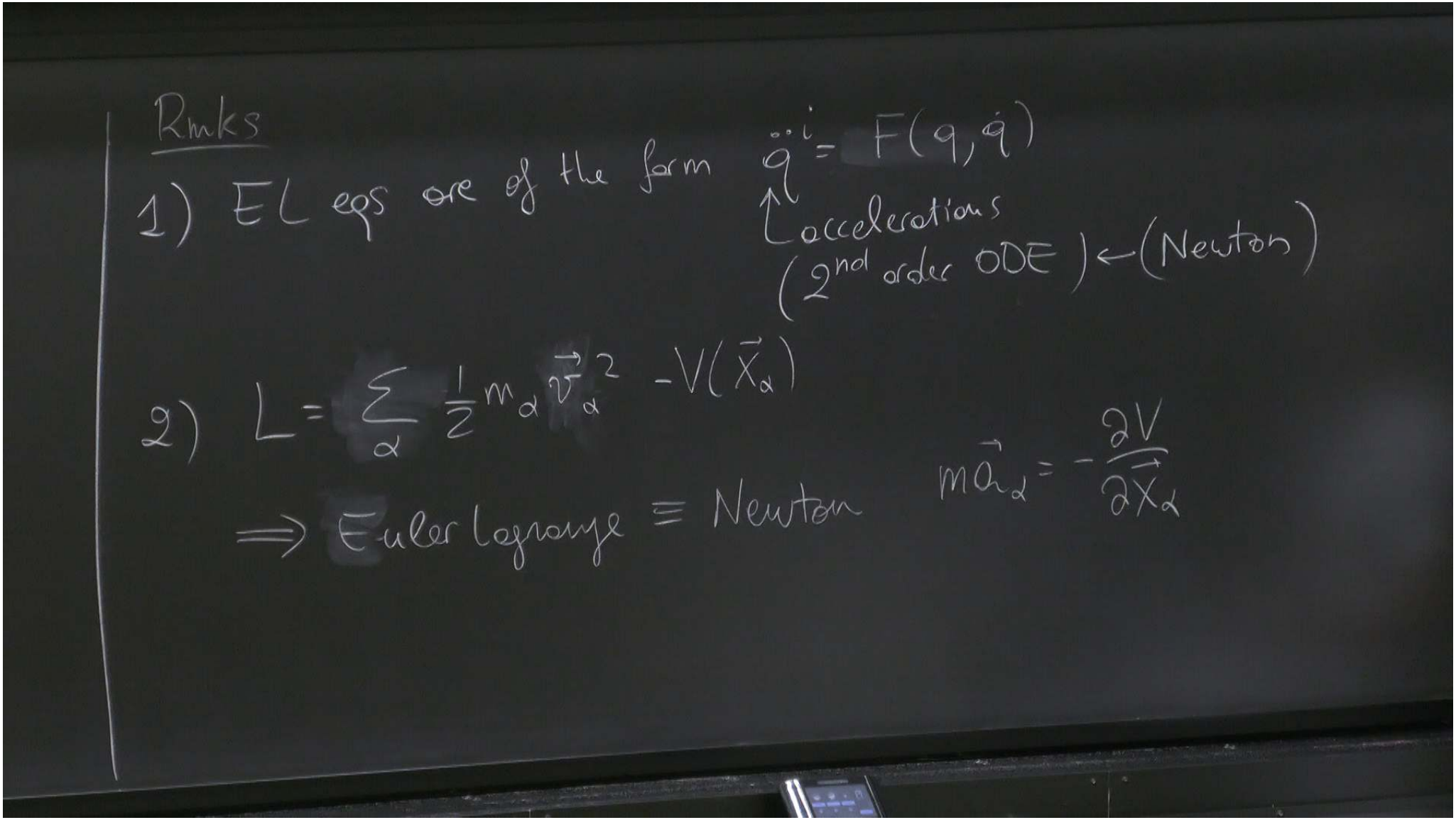
$$= \int_{t_0}^{t_1} dt \sum_i \left(\frac{\partial L}{\partial q^i}\bigg|_{\bar{\gamma}} \delta\gamma^i + \frac{\partial L}{\partial v^i}\bigg|_{\bar{\gamma}} \frac{d}{dt}\delta\gamma^i \right)$$

$$= \int_{t_0}^{t_1} dt \sum_i \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial v^i} \right)\bigg|_{\bar{\gamma}} \delta\gamma^i(t) + \left[\sum_i \frac{\partial L}{\partial v^i} \delta\gamma^i \right]_{t_0}^{t_1}$$

$\equiv 0$ @ t_0, t_1

$\delta\gamma^i(t)$: arbitrary


Hamilton's principle
⇓
Euler-Lagrange Eqs
$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0$
at a physical history


Rmks
1) EL eqs are of the form $\ddot{q}^i = F(q, \dot{q})$
accelerations
(2nd order ODE) ← (Newton)
2) $L = \sum_\alpha \frac{1}{2} m_\alpha \vec{v}_\alpha^{\,2} - V(\vec{X}_\alpha)$
$\Rightarrow$ Euler Lagrange $\equiv$ Newton $m\vec{a}_\alpha = -\frac{\partial V}{\partial \vec{X}_\alpha}$

$(t_0) = 0 = \delta f(t_1)$

Kmks

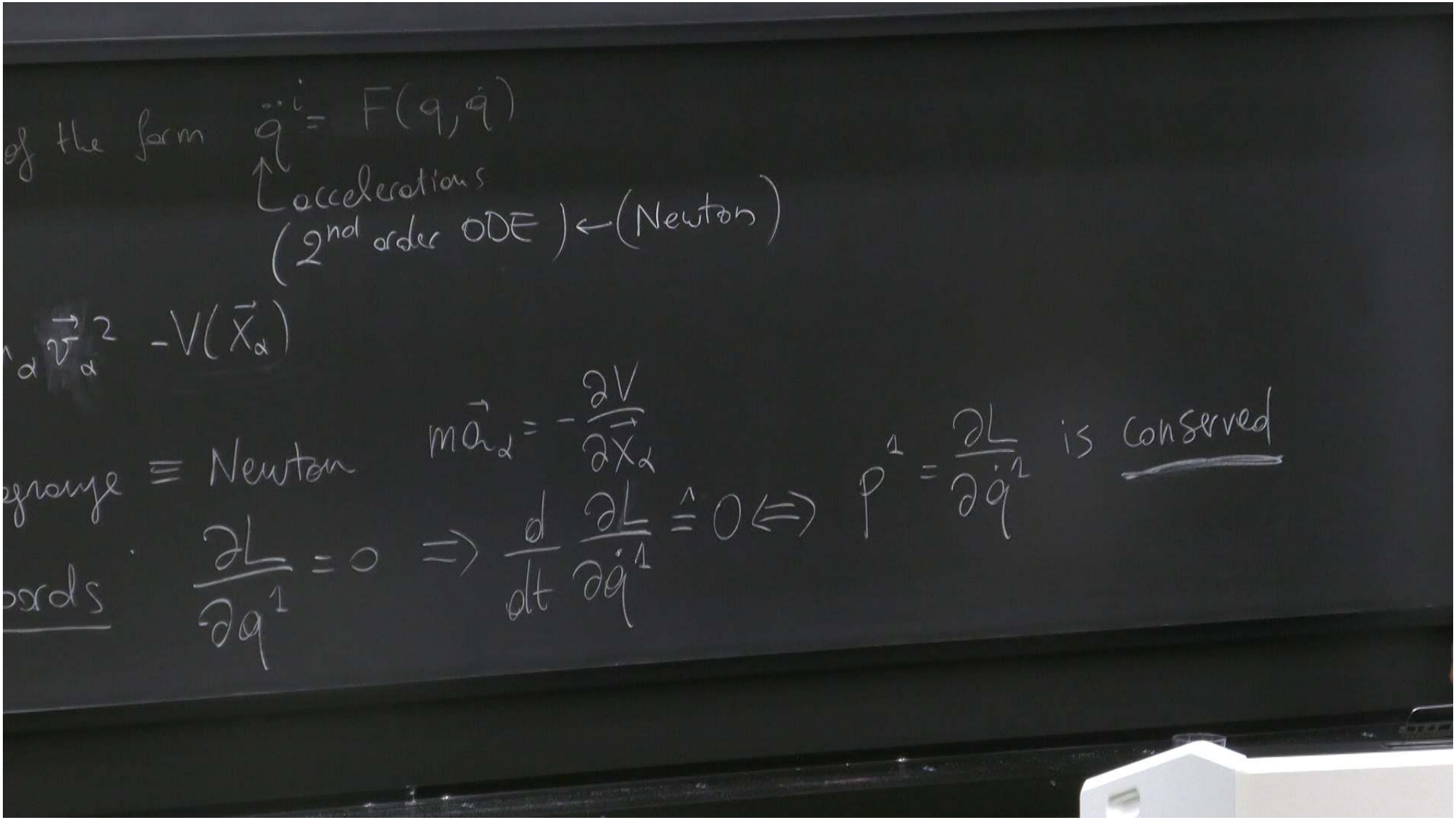
1) EL eqs are of the form $\ddot{q}^i = F(q, \dot{q})$
↑ accelerations
(2nd order ODE) ← (Newton)

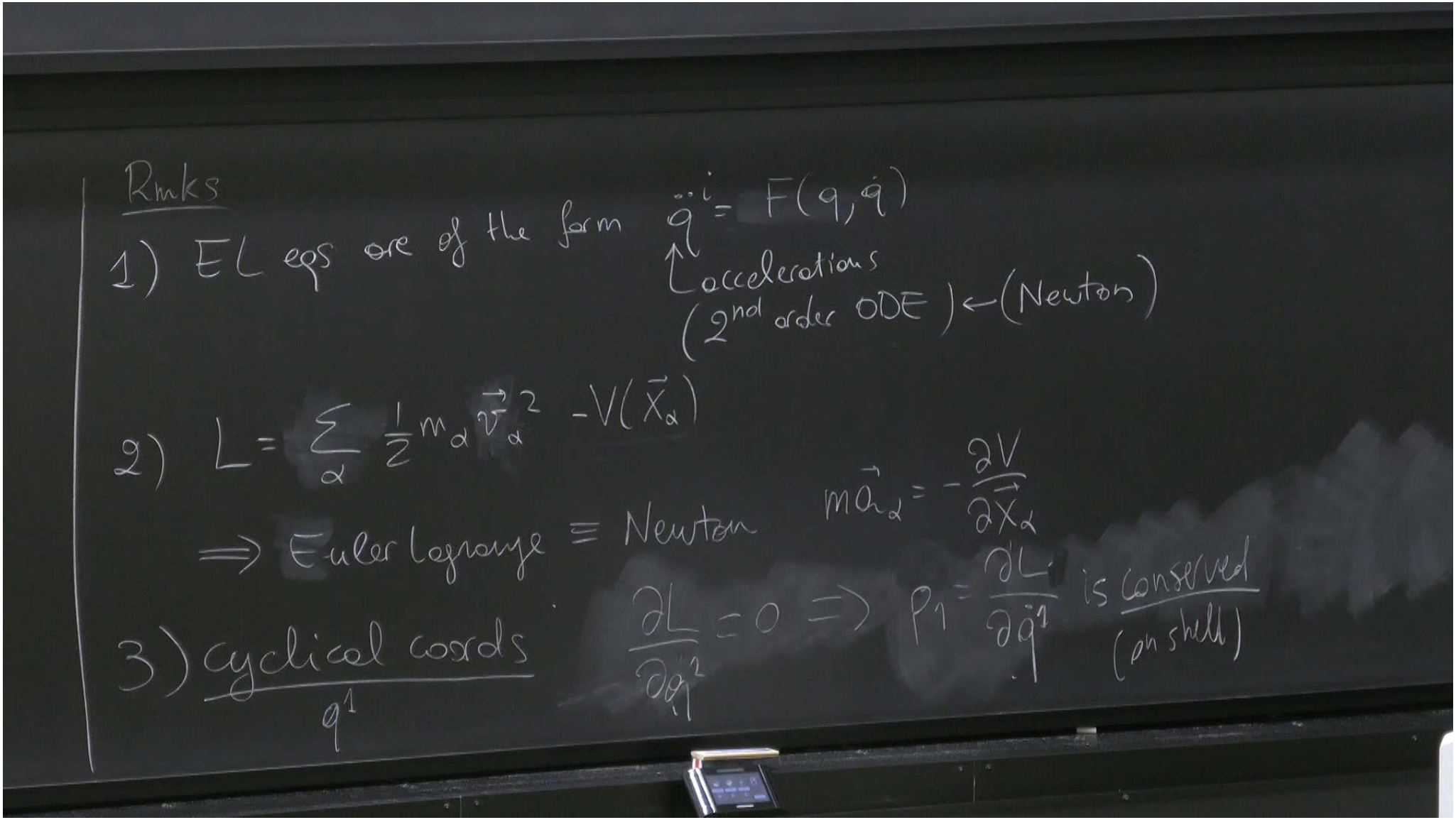
2) $L = \sum_\alpha \frac{1}{2} m_\alpha \vec{v}_\alpha^{\,2} - V(\vec{X}_\alpha)$

$\Rightarrow$ Euler Lagrange $\equiv$ Newton $\quad m\vec{a}_\alpha = -\frac{\partial V}{\partial \vec{X}_\alpha}$

3) cyclical coords $\quad \frac{\partial L}{\partial q^1} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^1} \hat{=} 0 \Leftrightarrow p^1 = \frac{\partial L}{\partial \dot{q}^1}$ is

q^1


of the form $\ddot{q}^i = F(q, \dot{q})$
accelerations
$(2^{nd}$ order ODE $) \leftarrow$ (Newton)
$\alpha \vec{v}_\alpha^{\,2} - V(\vec{X}_\alpha)$
$m\vec{a}_\alpha = -\frac{\partial V}{\partial \vec{X}_\alpha}$
grange $\equiv$ Newton
oxrds
$\frac{\partial L}{\partial q^1} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^1} \doteq 0 \Leftrightarrow p^1 = \frac{\partial L}{\partial \dot{q}^1}$ is conserved


Rmks
1) EL eqs are of the form $\ddot{q}^i = F(q, \dot{q})$
accelerations
(2^{nd} order ODE) ← (Newton)
2) $L = \sum_\alpha \frac{1}{2} m_\alpha \vec{v}_\alpha^{\,2} - V(\vec{X}_\alpha)$
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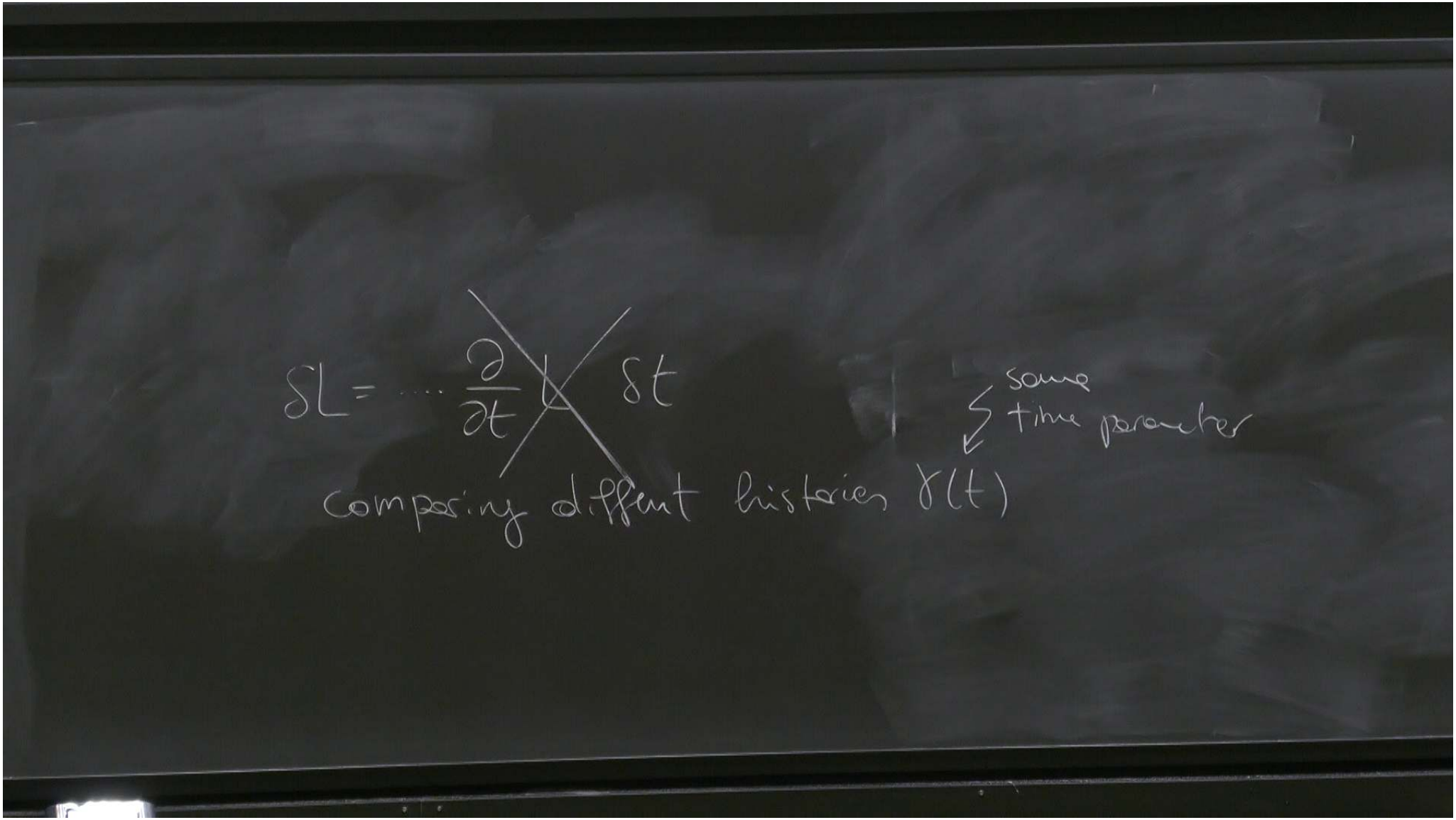
3) cyclical coords q^1
$\frac{\partial L}{\partial q^1} = 0 \Rightarrow p_1 = \frac{\partial L}{\partial \dot{q}^1}$ is conserved (on shell)

Ham action principle

$q_0 = \gamma(t_0)$
$q_1 = \gamma(t_1)$ $\rightsquigarrow$ $\bar{\gamma}(t)$ such that $\bar{\gamma}(t_0) = q_0$, $\bar{\gamma}(t_1) = q_1$

Rmk not guaranteed to exist

Rmk Sols to EL eqs are uniquely determined by $(q_0, v_0) = (\gamma(t_0), \dot{\gamma}(t_0))$


$\delta L = \cdots \frac{\partial}{\partial t} \delta t$
some time parameter
composing diffent histories $\gamma(t)$

Noether's 1st theorem

Consider a function s of the form:

$$TQ \times \mathbb{R} \longrightarrow TQ$$
$$(q, v, t) \longmapsto (q, s(q, v, t))$$

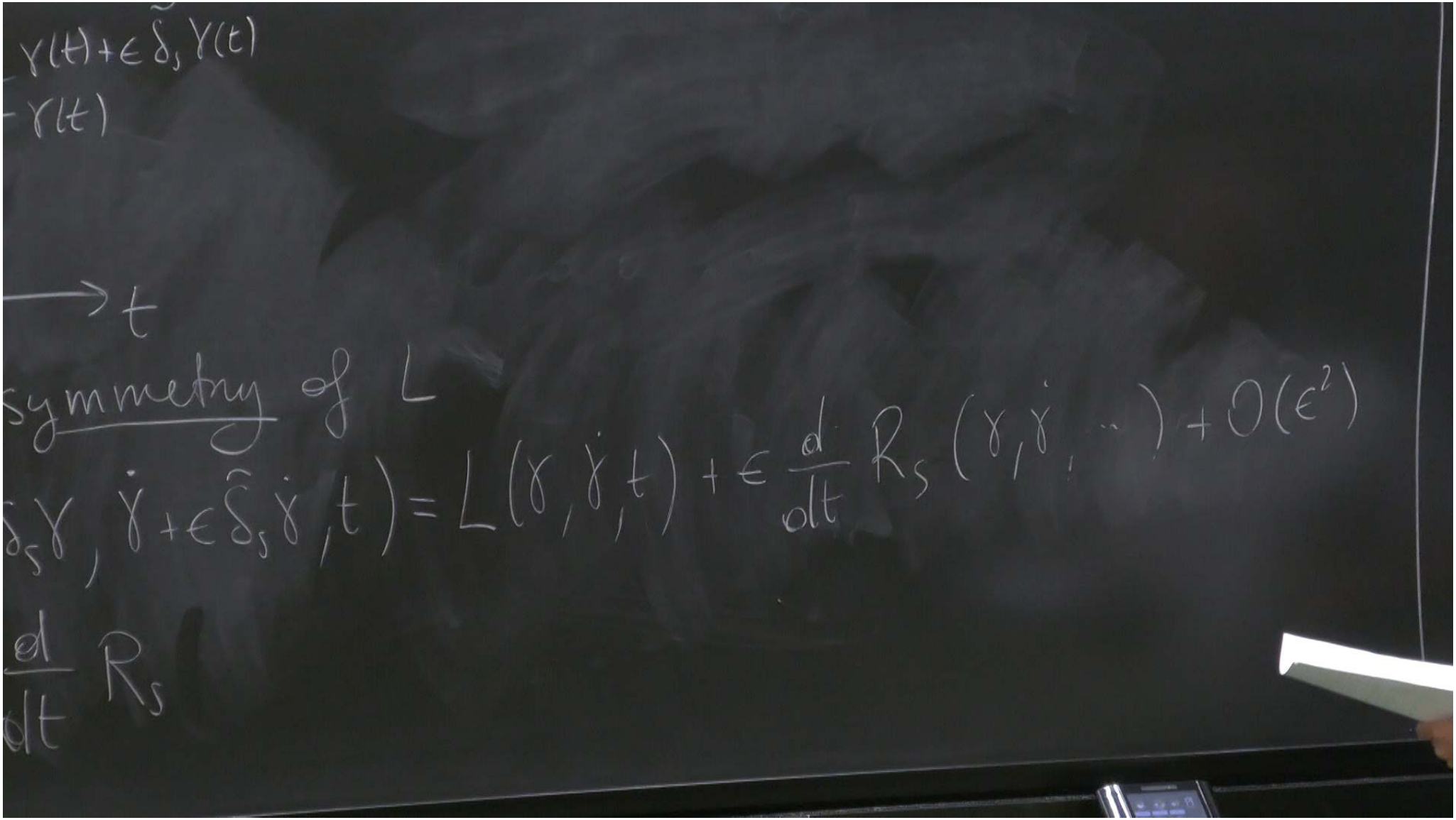
Given a history, I use s to define the following 1st order variation:

$$\gamma(t) \rightsquigarrow \gamma(t) + \epsilon \tilde{\delta}_s \gamma \qquad \tilde{\delta}_s \gamma(t) = s(\gamma(t), \dot{\gamma}(t), t)$$

Def We call s a symmetry of L

iff $L(\gamma+\epsilon\delta_s\gamma, \dot{\gamma}+\epsilon\tilde{\delta}_s\dot{\gamma}, t) = L(\gamma,\dot{\gamma},t) + \epsilon\frac{d}{dt}R_s(\gamma,\dot{\gamma},\ldots$

iff $\tilde{\delta}_s L = \frac{d}{dt}R_s$


$\gamma(t) + \epsilon \tilde{\delta}_s \gamma(t)$
$\gamma(t)$
$\longrightarrow t$
symmetry of L
$L(\delta_s \gamma, \dot{\gamma} + \epsilon \dot{\tilde{\delta}}_s \gamma, t) = L(\gamma, \dot{\gamma}, t) + \epsilon \frac{d}{dt} R_s(\gamma, \dot{\gamma}, \ldots) + O(\epsilon^2)$
$\frac{d}{dt} R_s$

Theorem

If s is a symmetry of L, then

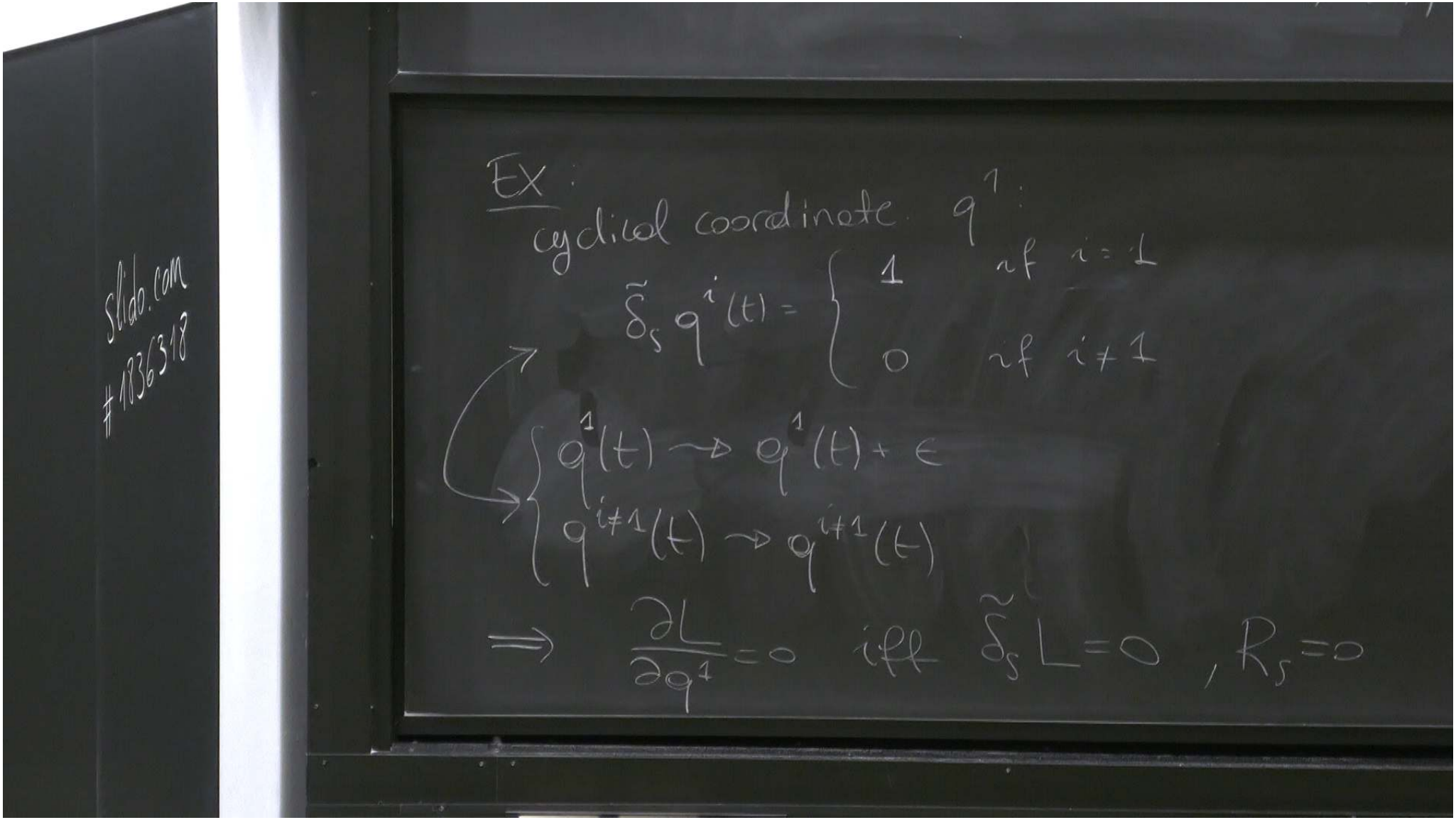
$$Q_s(t) := \sum_i \frac{\partial L}{\partial \dot{q}^i} \tilde{\delta}_s q^i - R_s$$

is conserved on-shell.

$$\frac{d}{dt} Q_s(t) \stackrel{\wedge}{=} 0 .$$

[Noether 1]

ϵ^2)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Ex:
cyclical coordinate q^1:
$\tilde{\delta}_s q^i(t) = \begin{cases} 1 & \text{if } i=1 \\ 0 & \text{if } i\neq 1 \end{cases}$
$q^1(t) \to q^1(t) + \epsilon$
$q^{i\neq 1}(t) \to q^{i\neq 1}(t)$
$\Rightarrow \frac{\partial L}{\partial q^1} = 0$ iff $\tilde{\delta}_s L = 0$, $R_s = 0$

q^1:

if $i = 1$

if $i \neq 1$

$$Q_S(t) = \sum_i \frac{\partial L}{\partial \dot{q}^i} \tilde{\delta}_S q^i - R_S$$

$$= \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}$$

$$= \frac{\partial L}{\partial \dot{q}^1} \quad \text{is conserved}$$

$$\tilde{\delta}_S L = 0 \,, \quad R_S = 0$$

Hamilton's Ac

The dynamics
is fully chara
Its physical
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