Title: QPV and Geometry of Banach spaces

Speakers: David Perez-Garcia

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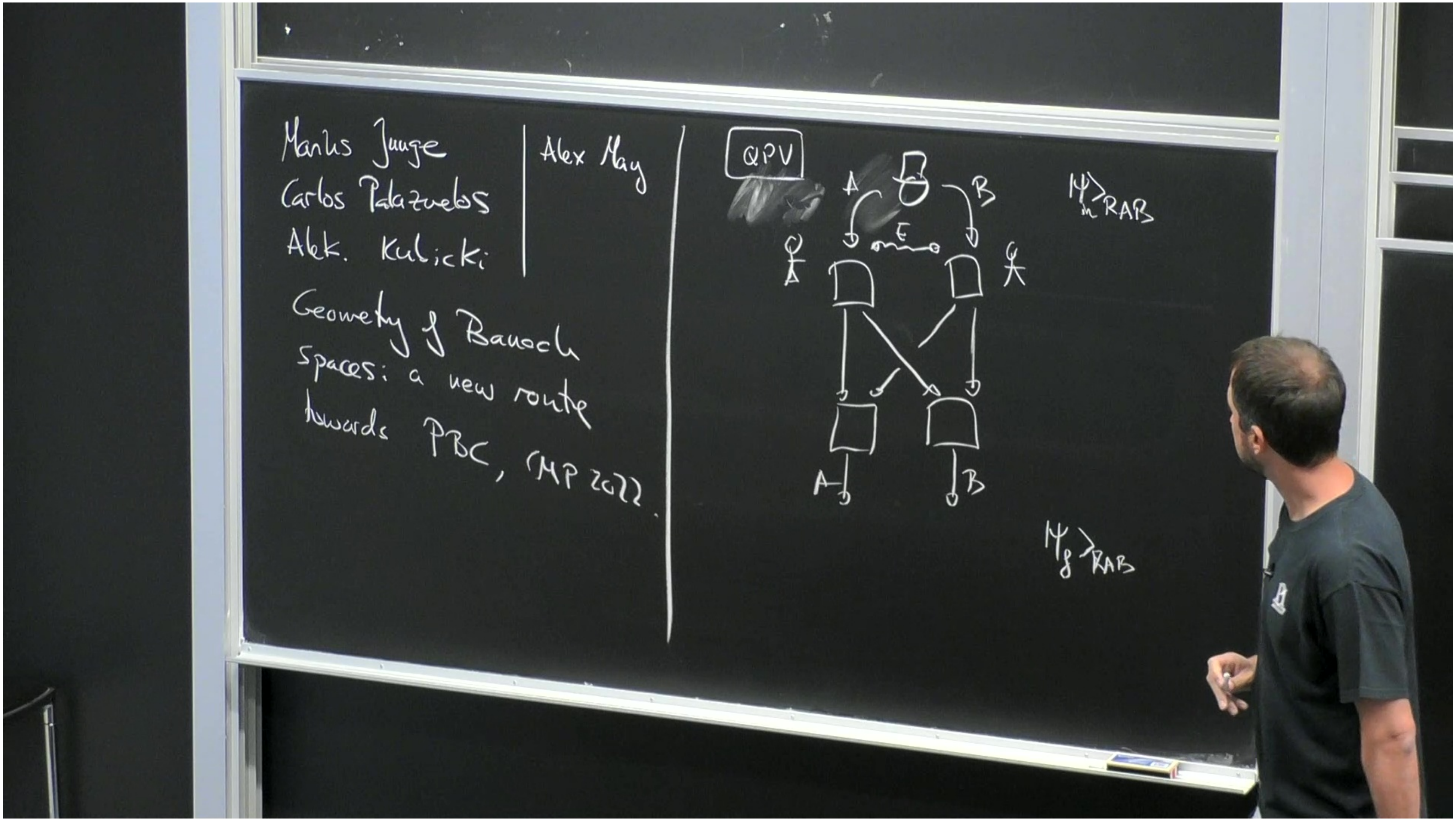

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Carlos Palazuelos
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Geometry of Banach
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towards PBC, CMP 2022.
QPV
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B
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A
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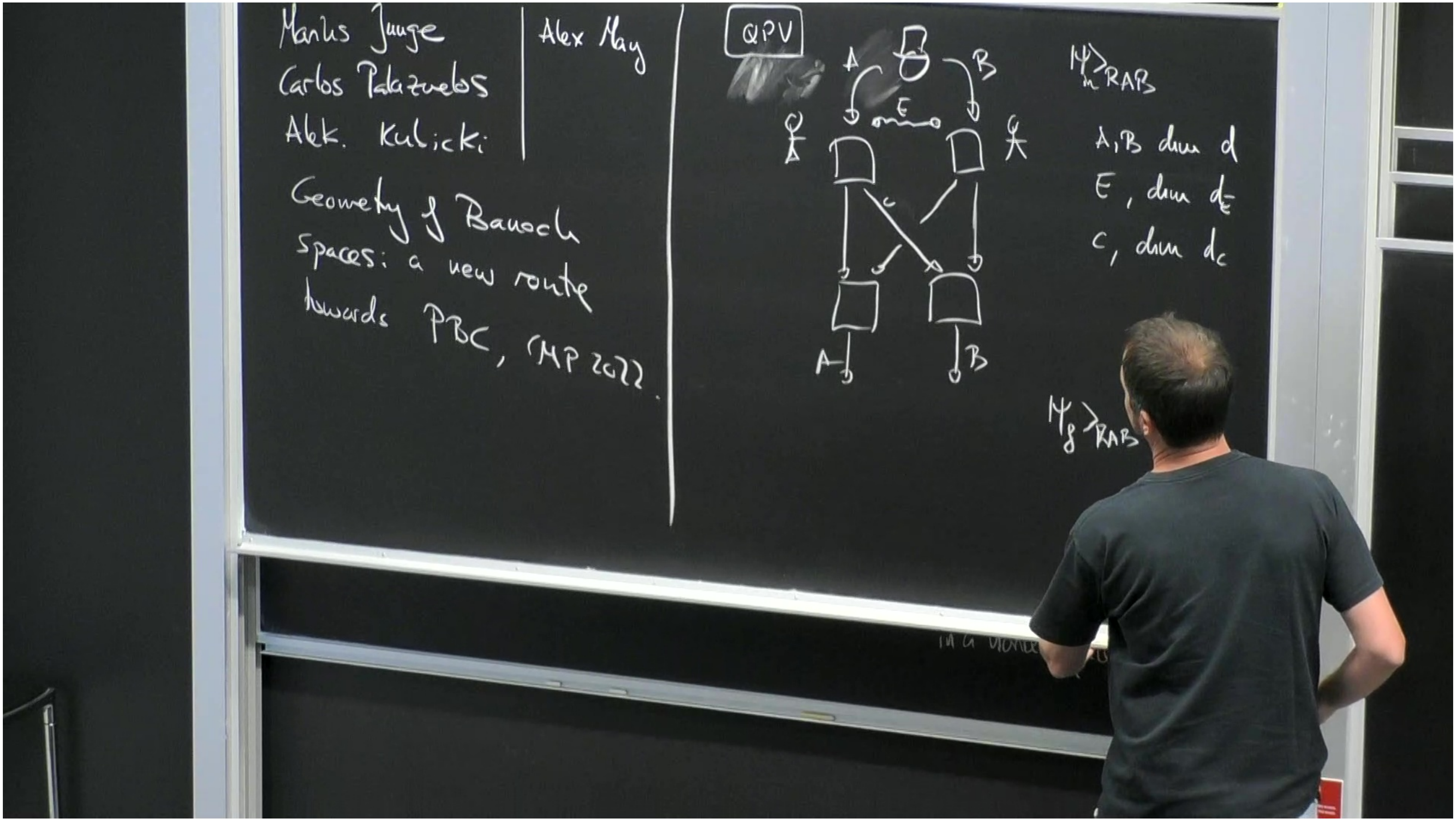
|ψm⟩RAB
|ψg⟩RAB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QPV
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$|\psi\rangle_{RAB}$
A, B dim d
E, dim d_E
c, dim d_c
A
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$|\psi_g\rangle_{RAB}$

non-local game $\longleftrightarrow$ $\mathbb{C}^n$

value game $\Longleftrightarrow$ $\|\cdot\|$ on $\mathbb{C}^n$

random family of games G_ε

$$1-\delta \leq \omega(G_\varepsilon) \simeq \mathbb{E}_\varepsilon \|X_\varepsilon\| \leq \frac{\log d_E + \log d_C}{d}$$

value of the game

averages of random vectors in a normed space

Theory of type & cotype ; $X = (\mathbb{C}^n, \|\cdot\|_X)$

$$\mathbb{E}_\varepsilon \Big\| \sum_{i=1}^{n} \varepsilon_i x_i \Big\|_X$$ ← estimates of that.

iid Bernoulli ± 1 prob $\frac{1}{2}$.

$\varepsilon = \{\pm 1\}^n$

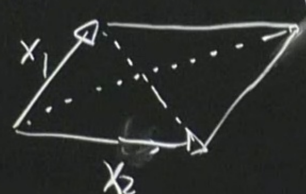
Theory of type & cotype ; $X = (\mathbb{C}^n, \|\cdot\|_X)$

$$\left(\mathbb{E}_\varepsilon \Big\| \sum_{i=1}^{n} \boxed{\varepsilon_i}\, x_i \Big\|_X^2\right)^{1/2} \text{ — estimates of that.}$$

iid Bernoulli ± 1 prob $\frac{1}{2}$.

$\varepsilon = \{\pm 1\}^n$

Hilbert space



$$\frac{1}{\sqrt{2}}\left[\|x_1+x_2\|_H^2 + \|x_1-x_2\|_H^2\right]^{1/2} = \left(\|x_1\|_H^2 + \|x_2\|_H^2\right)^{1/2}$$

$$\left(\mathbb{E}_\varepsilon \Big\| \sum_{i=1}^{n} \varepsilon_i x_i \Big\|_H^2\right)^{1/2} = \left(\sum \|x_i\|_H^2\right)^{1/2}$$

$\forall (x_i)_{i=1}^n \subset \mathcal{H}$

How far is that to be true for a normed space which is not a Hilbert space?

Def: cotype 2 constant of X is the minimal C s.t.

$$C\left(\mathbb{E}_\varepsilon \left\|\sum_i \varepsilon_i x_i\right\|^2\right)^{1/2} \geq \left(\sum \|x_i\|^2\right)^{1/2} \quad \forall x_i \in X.$$

type 2 constant T is minimal T s.t.

$$\left(\mathbb{E}_\varepsilon \left\|\sum \varepsilon_i x_i\right\|^2\right)^{1/2} \leq T\left(\sum \|x_i\|^2\right)^{1/2} \quad \forall x_i$$

non-local game $\longleftrightarrow$ $\mathbb{C}^n$

$\boxed{\text{value game}}$ $\longleftrightarrow$ $\|\cdot\|$ on $\mathbb{C}^n$.

random family of games G_ε

goal

$$1-\delta \leq \underbrace{\omega(G_\varepsilon)}_{\text{value of the game}} \simeq \underbrace{\mathbb{E}_\varepsilon \|X_\varepsilon\|}_{\text{averages of random vectors in a normed space}} \leq \frac{\log d_E + \log d_C}{d}$$

$$X = (\mathbb{C}^n, \|\cdot\|_X)$$

$$X^* = \left\{ \ell : \mathbb{C}^n \to \mathbb{C} \; : \; \|\ell\|_{X^*} = \max_{\|x\|_X \le 1} |\ell(x)| \right\}$$

$$\cong \mathbb{C}^n$$

$$\left(C_2(X) \lesssim T_2(X^*) \right)$$

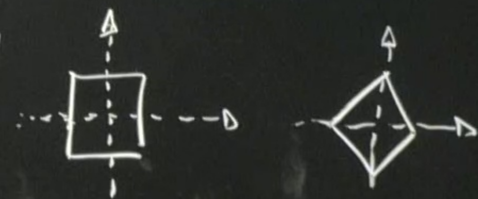
$$\boxed{T_2(X) \le \log(\dim X) \cdot C_2(X^*)}$$

$X = (\mathbb{C}^n, \|\cdot\|_X)$

$X^* = \{ L : \mathbb{C}^n \to \mathbb{C} \; : \; \|L\|_{X^*} = \max_{\|x\|_X \le 1} |L(x)| \}$

$\cong$

$\mathbb{C}^n$



$(C_2(X) \lesssim T_2(X^*))$

$$\boxed{T_2(X) \lesssim \log(\dim X) \cdot C_2(X^*)}$$

volume ratio $\quad X \to B_X = \{ x \in X : \|x\| \le 1 \}$

$$vr(X) = \left[\frac{vol(B_X)}{vol(\mathcal{E}_X)} \right]^{1/\dim X}$$

$\mathcal{E}_X$ is the ellipsoid with largest volume inside B_X

$$X = (\mathbb{C}^n, \|\cdot\|_X)$$

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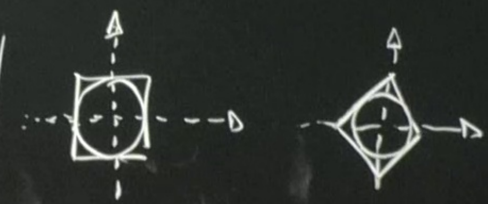

volume ratio $\quad X \to B_X = \{ x \in X : \|x\| \le 1 \}$

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Volume ratio

$$vr(X) = \left[\frac{vol(B_X)}{vol(\mathcal{E}_X)}\right]^{1/\dim X}$$

$\mathcal{E}_X$ is the ellipsoid with largest volume inside B_X

Connection between vr & cotype 2.

* $vr(X) \leq \log(\dim X)\, C_2(X)$ (Bourgain, Milman 80s)
* no reverse inequality in general

but yes if we restrict to subspaces. "Essentially" what is true

$\sup\{vr(Y) : Y \subset X\}$

Crossnorms (Grothendieck 50s)

$X \otimes_\alpha Y$: $\| \ \|_\alpha$ on $X \otimes Y$ s.t.

i) $\|x \otimes y\|_\alpha = \|x\|_X \, \|y\|_Y$

ii) $\|T \otimes S\|_{(X \otimes_\alpha Y)^*} = \|T\|_{X^*} \cdot \|S\|_{Y^*}$

minimum cross norm

$$\varepsilon\Big(\sum_i x_i \otimes y_i\Big) = \sup_{T \in X^*, S \in Y^*} \Big|\sum_i T(x_i) S(y_i)\Big|$$

maximum cross norm

$$\pi(z) = \inf\Big\{ \sum_{i=1}^n \|x_i\| \|y_i\| : z = \sum_{i=1}^n x_i \otimes y_i \Big\}$$

$$\ell_2^d \otimes_\varepsilon \ell_2^d = S_\infty^d$$

$$\ell_2^d \otimes_\pi \ell_2^d = S_1^d$$

Interpolation:

X, Y normed spaces.

$[X,Y]_{1/2}$ an intermediate space

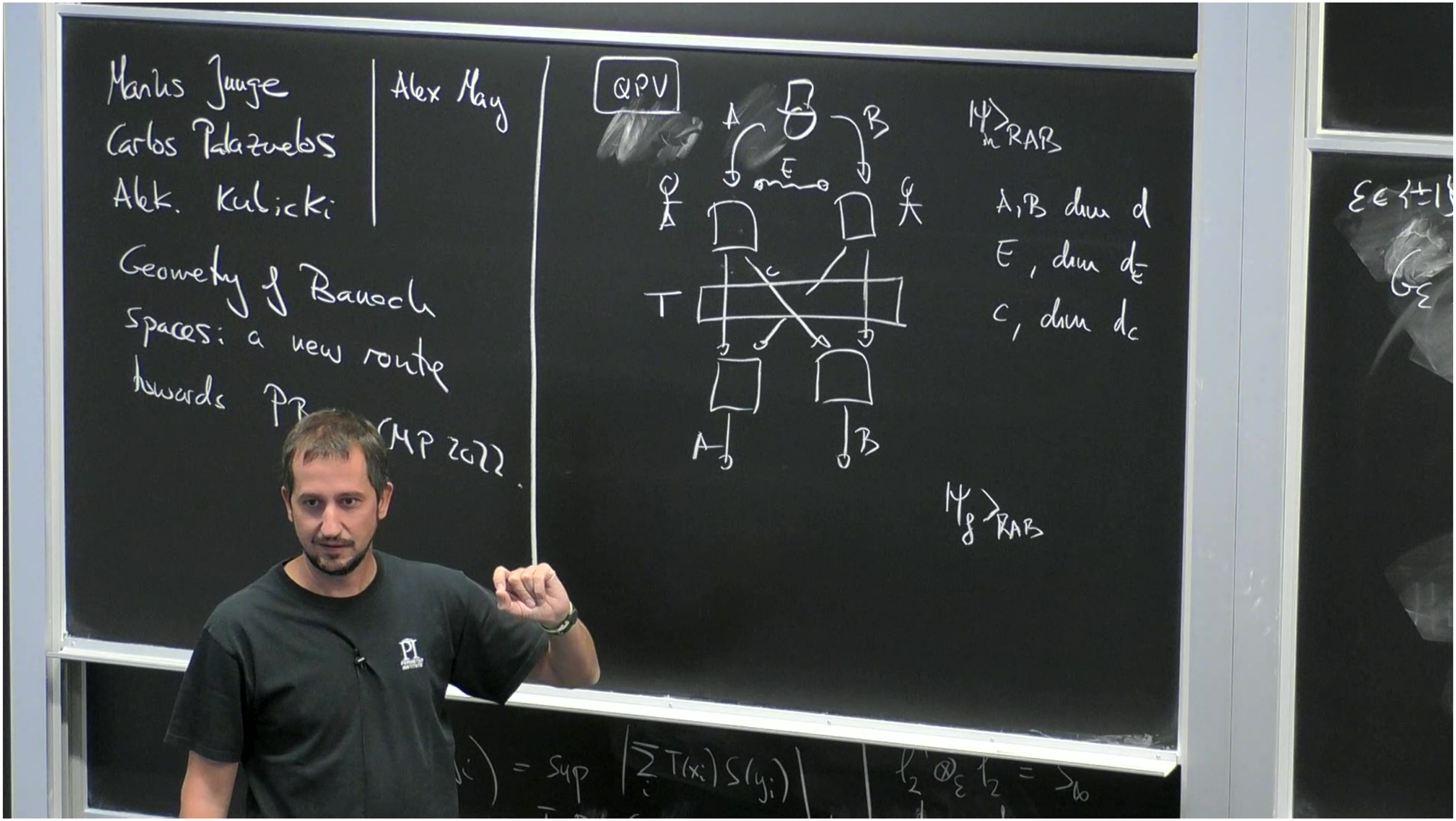
$T: \mathbb{C}^d \to \mathbb{C}^d$

$\|T\|_{X\to X} = \max_{\|x\|\le 1} \|T(x)\|$

$\|T\|_{[X,Y]_{1/2}\to [X,Y]_{1/2}} \le \|T\|_{X\to X}^{1/2} \cdot \|T\|_{Y\to Y}^{1/2}$

$[S_1^d, S_\infty^d]_{1/2} = S_2^d = \ell_2^{d^2}$

CAUTION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$\varepsilon \in \{\pm 1\}$
G_ε
$= \sup \left| \sum_i T(x_i) S(y_i) \right|$
$\ell_2 \otimes_\varepsilon \ell_2 = S_\infty$

$\varepsilon \in \{\pm 1\}^{d^2}$

G_ε

$$|\psi_{in}\rangle = \frac{1}{d}\sum_{ij} |ij\rangle_R\, |i\rangle_A\, |j\rangle_B$$

$$|\psi_f\rangle = \frac{1}{d}\sum_{ij} \varepsilon_{ij}\, |ij\rangle_R\, |i\rangle_A\, |j\rangle_B$$

$$\omega(G_\varepsilon) \underset{\text{Thm 1}}{\le} \frac{1}{d^2} \Big\| \sum_{ij} \varepsilon_{ij} T \otimes |ij\rangle\langle ij| \Big\|_X$$

$$X = \left[S_1^{d,d_c} \otimes_{[\varepsilon,\pi]_{1/2}} S_1^{d,d_c} \right] \otimes_\varepsilon \left[S_1^{d,d_c} \otimes_{[\varepsilon,\pi]_{1/2}} S_1^{d,d_c} \right]$$

$\varepsilon \in \{\pm 1\}^{d^2}$

G_ε

$$|\psi_{in}\rangle = \frac{1}{d}\sum_{ij} |ij\rangle_R |i\rangle_A |j\rangle_B$$

$$|\psi_\varepsilon\rangle = \frac{1}{d}\sum_{ij} \varepsilon_{ij} |ij\rangle_R |i\rangle_A |j\rangle_B$$

$$\omega(G_\varepsilon) \le \frac{1}{d^2} \Big\| \sum_{ij} \varepsilon_{ij} T \otimes |ij\rangle\langle ij| \Big\|_X$$

Thm 1

$$X = \left[S_1^{d,d_c} \otimes_{[\varepsilon,\pi]_{1/2}} S_1^{d,d_c} \right] \otimes_\varepsilon \left[S_1^{d,d_c} \otimes_{[\varepsilon,\pi]_{1/2}} S_1^{d,d_c} \right]$$

$$\mathbb{E}_\varepsilon (G_\varepsilon) \le \frac{1}{d^2} \mathbb{E}_\varepsilon \| \quad \|_X \le T_2(X) \frac{1}{d^2} \cdot \Big(\sum_{ij} \| T \otimes |ij\rangle\langle ij| \|_X^2 \Big)^{1/2}$$

volume ratio $\quad \mathcal{X} \to B_{\mathcal{X}} = \{x \in \mathcal{X} : \|x\| \leq 1\}$

$$vr(\mathcal{X}) = \left[\frac{vol(B_{\mathcal{X}})}{vol(\mathcal{E}_{\mathcal{X}})}\right]^{1/\dim \mathcal{X}}$$

$\mathcal{E}_{\mathcal{X}}$ is the ellipsoid with largest volume inside $B_{\mathcal{X}}$

$$\leq \frac{T_2(\mathcal{X})}{d^2}\, d = \frac{T_2(\mathcal{X})}{d}$$

Thm 2:

$\|T \otimes |ij\rangle x_{ij}\|_{\mathcal{X}} \leq 1$

$$\leq \frac{\log d_c}{d^{1/4}}$$

Conjecture: $T_2(\mathcal{X}) \leq d^{3/4} \cdot \log d_c$

* $vr(\mathcal{X}^*) = d^{3/4}$
* $T_2\left(S_1^{d,d_c} \otimes_{[\varepsilon,\pi]_{1/2}} S_1^{d,d_c}\right) \leq d^{3/4} \log d_c$

"evidences"