Title: General Relativity for Cosmology Lecture - 092623

Speakers: Achim Kempf

Collection: General Relativity for Cosmology

Date: September 26, 2023 - 2:00 PM

URL: https://pirsa.org/23090005

Abstract: Zoom: https://pitp.zoom.us/j/91640855624?pwd=dWVWV2doSnBhUS9JUkhjQVBwY0h0dz09

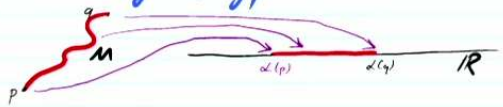
File Edit View Insert Actions Tools Help
Sharing Desktop
Stop Sharing

covariant tensors, i.e., about differential forms?

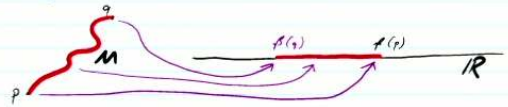
A: Antisymmetry ⇒ special transformation property under chart changes:
~ det (Jacobian)
⇒ suitable for integration:
s-forms have natural integrals in s-dimensional manifolds

Except: Depending on charts, sign of Jacobian may be wrong!

- Could have charts of the type



- or charts of the type



- But, since $\int f(t)\,dt = -\int f(t')\,dt'$ one needs to decide!
because $\frac{dt}{dt'} = -1$ (which is det[Jacobian])

For n-dim mflds, may need several charts.

Definitions:

- A complete collection of charts, i.e., an Atlas, A, is called oriented if among all overlapping charts with coordinates say $x, \tilde{x}$ the Jacobi determinants are positive:

$$\det\left(\frac{\partial \tilde{x}^i}{\partial x^k}\right) > 0$$

- A mfld M is called orientable if it possesses an oriented atlas.

Example: Möbius strips

are not orientable.

- A mfld, M, together with a choice of oriented atlas, A, is called an oriented manifold.
- Then, an arbitrary chart is called positive (or negative) if its Jacobian determinant with charts of the atlas A is positive (or negative).

File Edit View Insert Actions Tools Help
Sharing Desktop
Stop Sharing

charts with coordinates say $x, \tilde{x}$ the Jacobi determinants are positive:

$$\det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) > 0$$

- A mfld M is called orientable if it possesses an oriented atlas.

are not orientable.

- A mfld, M, together with a choice of oriented atlas, A, is called an oriented manifold.
- Then, an arbitrary chart is called positive (or negative) if its Jacobian determinant with charts of the atlas A is positive (or negative).

Definition:

An n-form $\Omega \in \Lambda_m(M)$ is called a volume form if it nowhere vanishes. We will later find a preferred volume form for space-time (using the metric).

Proposition:

M possesses a volume form

$\Updownarrow$

M is orientable

Integration:

- Recall change of cds in integration in $\mathbb{R}^m$:

For $(x^1, \dots, x^m) \to (\tilde{x}^1, \dots, \tilde{x}^m)$:

Riemann or Lebesgue integrals

$$\int_{\mathbb{R}^m} g(x^1, \dots, x^m)\, dx^1 \dots dx^m \overset{(*)}{=} \int_{\mathbb{R}^m} g(x(\tilde{x}))\, \det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right) d\tilde{x}^1 \dots d\tilde{x}^m$$

($g: \mathbb{R}^m \to \mathbb{R}$)

(*) Jacobian determinant is negative if coordinate systems change handedness.

- Now for a general n-dimensional diffable mfld M, consider an n-form ω in a chart:

$$\omega = f(x)\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^m$$

File Edit View Insert Actions Tools Help
Sharing Desktop
Stop Sharing

vanishes. We will later find a preferred volume form for space-time (using the metric).

Proposition:

M possesses a volume form

$\Updownarrow$

M is orientable

Riemann or Lebesgue integrals

$$\int_{\mathbb{R}^m} g(x^1,\dots,x^m)\,dx^1\dots dx^m = \int_{\mathbb{R}^m} g(x(\tilde{x}))\,\det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right) d\tilde{x}^1\dots d\tilde{x}^m \qquad (*)$$

$g:\mathbb{R}^m\to\mathbb{R}$

↑ Jacobian determinant is negative if coordinate systems change handedness.

□ Now for a general n-dimensional diffable mfld M, consider an n-form ω in a chart:

$$\omega = f(x)\,dx^1\wedge dx^2\wedge\dots\wedge dx^m$$

Then what is ω in an overlapping, second chart?

$$\omega = f(x(\tilde{x}))\,\frac{\partial x^{i_1}}{\partial \tilde{x}^1}\frac{\partial x^{i_2}}{\partial \tilde{x}^2}\dots\frac{\partial x^{i_n}}{\partial \tilde{x}^n}\,d\tilde{x}^{i_1}\wedge d\tilde{x}^{i_2}\wedge\dots\wedge d\tilde{x}^{i_n}$$

totally antisymmetric!

- terms are nonzero only if contain each number $1,\dots,n$ exactly once, e.g. $d\tilde{x}^1\wedge d\tilde{x}^3\wedge d\tilde{x}^2\wedge d\tilde{x}^4\wedge d\tilde{x}^5\wedge\dots\wedge dx^m$.
- Reorder those terms – they are all $d\tilde{x}^1\wedge d\tilde{x}^2\wedge\dots\wedge d\tilde{x}^m$ up to a possible factor -1 because $dx^i\wedge dx^j = -dx^j\wedge dx^i$

$$\Rightarrow\quad \omega = f(x(\tilde{x}))\,\det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right) d\tilde{x}^1\wedge d\tilde{x}^2\wedge\dots\wedge d\tilde{x}^m$$

Compare with equation $(*)$ above $\Rightarrow$

The following definition of the integral of n-forms in an n-dim. diffable mfld is chart-independent, i.e., is well-defined:

Definition:

Assume M is an oriented n-dim mfld and $\omega\in\Lambda_n(M)$ reads in a chart α: $\omega = f(x)\,dx^1\wedge\dots\wedge dx^m$.

Then, if one chart suffices:

$$\int_M \omega := \int_{\alpha(M)} f(x)\,dx^1 dx^2\dots dx^m$$

usual Riemann or Lebesgue integral

↖ image of M in $\mathbb{R}^m$

Else: Piece right hand side together from several charts

Note: how to piece together does not matter as long as charts are from the atlas that M is equiped with. That's why orientation is important.

File Edit View Insert Actions Tools Help
Sharing Desktop
Stop Sharing

- Reorder those terms - they are all

$$d\tilde{x}^1 \wedge d\tilde{x}^2 \wedge \dots \wedge d\tilde{x}^m$$

up to a possible factor -1 because $dx^i \wedge dx^j = -dx^j \wedge dx^i$

$$\Rightarrow \quad \omega = f(x(\tilde{x}))\, \det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right) d\tilde{x}^1 \wedge d\tilde{x}^2 \wedge \dots \wedge d\tilde{x}^m$$

Compare with equation (*) above $\Rightarrow$

and $\omega \in \Lambda_m(M)$ reads in a chart φ: $\omega = f(x)\, dx^1 \wedge \dots \wedge dx^m$.

Then, if one chart suffices:

$$\int_M \omega := \int_{\varphi(M)} f(x)\, dx^1 dx^2 \dots dx^m$$

(usual Riemann or Lebesgue integral; $\varphi(M)$: image of M in $\mathbb{R}^m$)

Else: Piece right hand side together from several charts

Note: how to piece together does not matter as long as charts are from the atlas that M is equiped with. That's why orientation is important.

Definition: The boundary operator, ∂

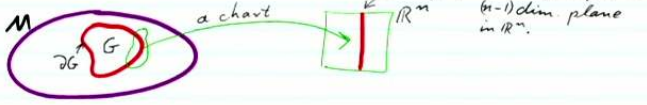
- Assume $G \subset M$ is a region (i.e. an n-dim., open and connected subset) of the n-dim mfld M.

  We denote the $(n-1)$ dim. boundary manifold of G by ∂G:

  (∂: the boundary operator)

  $$\partial G := \text{boundary}(G)$$

- We say that ∂G is smooth if locally there exist charts so that:



Proposition: If M is orientable, then so is G. Also, the orientation of G induces an orientation of ∂G.

We finally have all ingredients for one of Math's most important theorems:

Stokes' theorem: If closure $\bar{G}$ of G is a compact n-dim region, then:

$$\int_G d\omega = \int_{\partial G} \omega \qquad \text{for all } \omega \in \Lambda_{n-1}(M)$$

Definition: d is also called "co-boundary operator".

File Edit View Insert Actions Tools Help

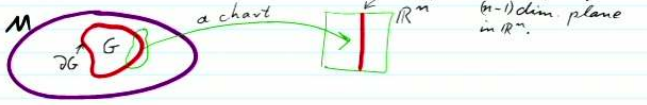
Sharing Desktop ▲ ■ Stop Sharing

We denote the $(n-1)$ dim. boundary manifold of G by ∂G:

the boundary operator

$$\partial G := \text{boundary}(G)$$

- We say that ∂G is smooth if locally there exist charts so that:



important theorems:

Stokes' theorem: If closure $\bar{G}$ of G is a compact n-dim region, then:

$$\int_G d\omega = \int_{\partial G} \omega \quad \text{for all } \omega \in \Lambda_{n-1}(M)$$

Definition: d is also called "co-boundary operator".

Remark:

- Let us try iterating Stokes!
- Assume $G = \partial H$.
- Then, by Stokes we obtain $0 = 0$:

$$\int_H \underbrace{dd}_{=0 \text{ always for algebraic reasons.}} \omega \overset{\text{Stokes}}{=} \int_{\partial H} d\omega \overset{\text{Stokes}}{=} \int_{\underbrace{\partial\partial H}_{=0 \text{ for geometric reasons because, indeed, boundaries don't possess boundaries:}}} \omega$$

E.g. G = half sphere, ∂G = equator, $\partial\partial G = \emptyset$

i.e.: Stokes implies $d^2 = 0 \Leftrightarrow \partial^2 = 0$

- Stokes links homology (geometric) to cohomology (algebraic).

Special case I:

Assume: $M = \mathbb{R}$, $G = (a, b)$

Therefore: $\partial G = \{a, b\}$

Then, Stokes' theorem is $\int_G df = \int_{\partial G} f$, namely:

$$\int_a^b df = f\Big|_a^b \qquad \text{(fund. thm of calculus)}$$

$$= \frac{df}{dx}\, dx$$

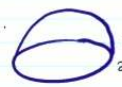
9 / 21

File Edit View Insert Actions Tools Help
Sharing Desktop
Stop Sharing

Then, by Stokes we obtain ...

$$\int_H \underbrace{dd}_{=0 \text{ always for algebraic reasons.}}\omega \overset{\text{Stokes}}{=} \int_{\partial H} d\omega \overset{\text{Stokes}}{=} \int_{\underbrace{\partial\partial H}_{=0}} \omega$$

$=0$ for geometric reasons because, indeed, boundaries don't possess boundaries:

E.g.  G = half sphere, ∂G = equator, $\partial\partial G = \emptyset$

i.e.: Stokes implies $d^2 = 0 \Leftrightarrow \partial^2 = 0$

- Stokes links homology (geometric) to cohomology (algebraic).

Therefore: $\partial G = \{a, b\}$

Then, Stokes' theorem is $\int_G df = \int_{\partial G} f$, namely:

$$\int_a^b df = f\Big|_a^b \qquad \text{(fund. thm of calculus)}$$

$$df = \frac{df}{dx}\,dx$$

Special case II: "Green's theorem".

- $M = \mathbb{R}^2$, $G \subset \mathbb{R}^2$ a region with (closed) boundary curve ∂G.

recall: this is automatic because $\partial\circ\partial = 0$

- Consider an arbitrary 1-form $\omega \in \Lambda_1(M)$:

$$\omega = \omega_1(x)\,dx^1 + \omega_2(x)\,dx^2$$

Then:
$$\begin{aligned} d\omega &= d\omega_1(x)\wedge dx^1 + d\omega_2(x)\wedge dx^2 \\ &= \left(\frac{\partial\omega_1}{\partial x^1}dx^1 + \frac{\partial\omega_1}{\partial x^2}dx^2\right)\wedge dx^1 \\ &\quad + \left(\frac{\partial\omega_2}{\partial x^1}dx^1 + \frac{\partial\omega_2}{\partial x^2}dx^2\right)\wedge dx^2 \\ &= \frac{\partial\omega_1}{\partial x^2}dx^2\wedge dx^1 + \frac{\partial\omega_1}{\partial x^1}dx^1\wedge dx^2 \end{aligned}$$

$$\Longrightarrow d\omega = \left(\frac{\partial\omega_2}{\partial x^1} - \frac{\partial\omega_1}{\partial x^2}\right)dx^1\wedge dx^2$$

Now, Stokes' theorem $\int_G d\omega = \int_{\partial G}\omega$ becomes:

$$\int_G\left(\frac{\partial\omega_2}{\partial x^1} - \frac{\partial\omega_1}{\partial x^2}\right)dx^1 dx^2 = \int_{\partial G}(\omega_1\,dx^1 + \omega_2\,dx^2)$$

Recall: How to evaluate, e.g., the RHS, in practice?

- Choose a chart for ∂G, i.e., a diffable map, invertible map $\partial G \to \mathbb{R}$.
- Its inverse is a path: $\gamma: J \subset \mathbb{R} \to \partial G$, with $\gamma(t) = (x^1(t), x^2(t))$
- Now use $dx^i = \frac{dx^i}{dt}dt$ to obtain an integral over $J \subset \mathbb{R}$

File Edit View Insert Actions Tools Help

Sharing Desktop ■ Stop Sharing

Special case of Green's theorem:

Assume $\omega \in \Lambda_1$ is closed, i.e., $d\omega = 0$, i.e., $\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2} = 0$

Then: $\int_{\partial G} \omega = 0$

Compare: (From the residue theorem)

If a function $w: G \subset \mathbb{C} \to \mathbb{C}$ is holomorphic, i.e., it obeys the Cauchy Riemann equations, then:

$$\int_{\partial G} w(z)\, dz = 0$$

Indeed:

The Cauchy-Riemann equations mean that a diff. form is closed and co-closed. We'll define "co-closedness" later.

Special case III: (exercise)

Similarly, one can show that what is often called the Stokes theorem for $M = \mathbb{R}^3$, namely

"cross product": $\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$

$\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$

$$\int_G \vec{\nabla} \times \vec{w}\, d\mathcal{G} = \int_{\partial G} \vec{w} \cdot d\vec{s}$$

$\vec{\nabla} \times \vec{w}$: vector field

G: a 2-dim submanifold of M

∂G: 1-dim boundary of A

is indeed this special case:

$\omega \in \Lambda_1(G)$ with $\vec{\nabla} \times \vec{w} = d\omega \in \Lambda_2(G)$

Before we can discuss the next example:

How to define the volume of a region $G \subset M$ of a diffable manifold M?

- In $\mathbb{R}^n$, we had: $V = \int_G dx^1 \ldots dx^n$
- In general, we need to choose a Volume form $\Omega \in \Lambda_n$ obeying $\Omega(e) \neq 0 \;\forall e \in G$. Then the (Ω-dependent) volume

Special case IV: Gauss' theorem

To obtain Gauß' theorem we need to define yet a new derivative the **divergence of a vector field.**

Recall: On $\mathbb{R}^n$, the divergence of a vector field, ξ, was defined as

$$\operatorname{div} \xi = \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \xi^i = \xi^i_{,i}$$

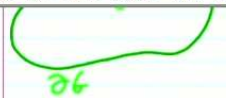
⇝ How to generalize to arbitrary manifolds?

15 / 21

15

File Edit View Insert Actions Tools Help

Sharing Desktop ■ Stop Sharing



- Consider an arbitrary 1-form $\omega \in \Lambda_1(M)$:

$$\omega = \omega_1(x)\,dx^1 + \omega_2(x)\,dx^2$$

Then:
$$\begin{aligned} d\omega &= d\omega_1(x) \wedge dx^1 + d\omega_2(x) \wedge dx^2 \\ &= \left(\frac{\partial \omega_1}{\partial x^1}dx^1 + \frac{\partial \omega_1}{\partial x^2}dx^2\right) \wedge dx^1 \\ &\quad + \left(\frac{\partial \omega_2}{\partial x^1}dx^1 + \frac{\partial \omega_2}{\partial x^2}dx^2\right) \wedge dx^2 \\ &= \frac{\partial \omega_1}{\partial x^2}dx^2 \wedge dx^1 + \frac{\partial \omega_2}{\partial x^1}dx^1 \wedge dx^2 \end{aligned}$$

$$\int_G \left(\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2}\right) dx^1 dx^2 = \int_{\partial G} (\omega_1\,dx^1 + \omega_2\,dx^2)$$

Recall: How to evaluate, e.g., the RHS, in practice?

- Choose a chart for ∂G, i.e., a diffable map, invertible map $\partial G \to \mathbb{R}$.
- Its inverse is a path: $\gamma : J \subset \mathbb{R} \to \partial G$, with $\gamma(t) = (x^1(t), x^2(t))$
- Now use $dx^i = \frac{dx^i}{dt}\,dt$ to obtain an integral over $J \subset \mathbb{R}$

Special case of Green's theorem:

Assume $\omega \in \Lambda_1$ is closed, i.e., $d\omega = 0$, i.e., $\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2} = 0$

Then: $\int_{\partial G} \omega = 0$

Compare: (From the residue theorem)

If a function $w : G \subset \mathbb{C} \to \mathbb{C}$ is holomorphic, i.e., it obeys the Cauchy Riemann equations, then:

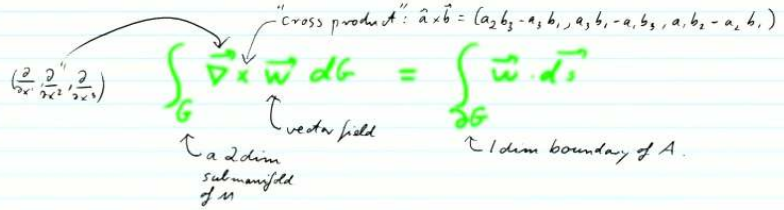
$$\int_{\partial G} w(z)\,dz = 0$$

Indeed:

The Cauchy-Riemann equations mean that a diff. form is closed and co-closed. We'll define "co-closedness" later.

Special case III: (exercise)

Similarly, one can show that what is often called the Stokes theorem for $M = \mathbb{R}^3$, namely



is indeed this special case:

$\omega \in \Lambda_1(G)$ with $\vec{\nabla} \times \vec{w} = d\omega \in \Lambda_2(G)$

13 / 21

13

File Edit View Insert Actions Tools Help

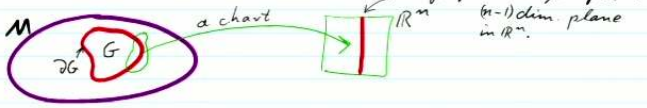
Sharing Desktop | Stop Sharing

We denote the $(n-1)$ dim. boundary manifold of G by ∂G:

the boundary operator

$$\partial G := \text{boundary}(G)$$

- We say that ∂G is smooth if locally there exist charts so that:



important theorems:

Stokes' theorem: If closure $\bar{G}$ of G is a compact n-dim region, then:

$$\int_G d\omega = \int_{\partial G} \omega \quad \text{for all } \omega \in \Lambda_{n-1}(M)$$

Definition: d is also called "co-boundary operator".

Remark:

- Let us try iterating Stokes!
- Assume $G = \partial H$.
- Then, by Stokes we obtain $0 = 0$:

$$\int_H \underbrace{dd\omega}_{=0 \text{ always for algebraic reasons.}} \overset{\text{Stokes}}{=} \int_{\partial H} d\omega \overset{\text{Stokes}}{=} \int_{\underbrace{\partial\partial H}_{=0 \text{ for geometric reasons because, indeed, boundaries don't possess boundaries:}}} \omega$$

i.e.: Stokes implies $d^2 = 0 \Leftrightarrow \partial^2 = 0$

E.g.: G = half sphere, ∂G = equator, $\partial\partial G = \emptyset$

- Stokes links homology (geometric) to cohomology (algebraic).

Special case I:

Assume: $M = \mathbb{R}$, $G = (a, b)$

Therefore: $\partial G = \{a, b\}$

Then, Stokes' theorem is $\int_G df = \int_{\partial G} f$, namely:

$$\int_a^b df = f\Big|_a^b \qquad \text{(fund. thm of calculus)}$$

$$= \frac{df}{dx}\, dx$$

9 / 21

File Edit View Insert Actions Tools Help
Sharing Desktop ■ Stop Sharing

Compare: (From the residue theorem)

If a function $w: G \subset \mathbb{C} \to \mathbb{C}$ is holomorphic, i.e., it obeys the Cauchy Riemann equations, then:

$$\int_{\partial G} w(z)\,dz = 0$$

Indeed:

The Cauchy-Riemann equations mean that a diff.-form is closed and co-closed. We'll define "co-closedness" later.

"cross product": $\hat{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$

$$\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$$

$$\int_G \vec{\nabla} \times \vec{w}\; dG = \int_{\partial G} \vec{w} \cdot d\vec{s}$$

$\vec{w}$: vector field; G: a 2-dim submanifold of M; ∂G: 1-dim boundary of A.

is indeed this special case:

$w \in \Lambda_1(G)$ with $\vec{\nabla} \times \vec{w} = d w \in \Lambda_2(G)$

Before we can discuss the next example:

How to define the volume of a region $G \subset M$ of a diffable manifold M?

- In $\mathbb{R}^m$, we had: $V = \int_G dx^1 \ldots dx^m$
- In general, we need to choose a Volume form $\Omega \in \Lambda_m$ obeying $\Omega(p) \neq 0 \;\forall p \in G$. Then the ($\Omega$-dependent) volume is defined as:

$$V := \int_G \Omega$$

(We will later use the metric tensor to define a volume form for spacetime.)

Proposition: G orientable $\Leftrightarrow$ $\exists$ volume forms Ω (In fact ∞ many)

Special case IV: Gauss' theorem

To obtain Gauß' theorem we need to define yet a new derivative the divergence of a vector field.

Recall: On $\mathbb{R}^m$, the divergence of a vector field, ξ, was defined as

$$\operatorname{div} \xi = \sum_{i=1}^{m} \frac{\partial}{\partial x^i} \xi^i = \xi^i{}_{,i}$$

⇝ How to generalize to arbitrary manifolds?

Where in this course did we see $\xi^i(x)_{,i}$ before?

Recall:

$$\left(L_\xi \tau\right)^{i_1 \ldots i_r}_{j_1 \ldots j_s}(x) = \tau^{i_1 \ldots i_r}_{j_1 \ldots j_s,k}(x)\,\xi^k(x) - \tau^{i_1 \ldots i_r}_{k j_2 \ldots j_s}(x)\,\xi^k_{,j_1}(x) - \ldots + \tau^{i_1 \ldots i_r}_{j_1 \ldots j_{s-1} k}(x)\,\xi^k_{,j_s}(x) + \ldots \tau^{k i_2 \ldots i_r}_{j_1 \ldots j_s}(x)\,\xi^{i_1}_{,k}(x)$$

$\tau \in T(M)^r_s$

File Edit View Insert Actions Tools Help
Sharing Desktop ■ Stop Sharing

$\int_G$

- In general, we need to choose a **Volume form** $\Omega \in \Lambda_m$ obeying $\Omega(p) \neq 0 \ \forall p \in G$. Then the ($\Omega$-dependent) volume is defined as:

$$V := \int_G \Omega$$

(We will later use the metric tensor to define a volume form for spacetime.)

Proposition: G orientable $\Leftrightarrow$ $\exists$ volume forms Ω (In fact ∞ many)

Recall: On $\mathbb{R}^n$, the divergence of a vector field, ξ, was defined as

$$\operatorname{div} \xi = \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \xi^i = \xi^i_{,i}$$

⇒ How to generalize to arbitrary manifolds?

Where in this course did we see $\xi^i(x)_{,i}$ before?

Recall: $\left(L_\xi \tau\right)^{i_1 \dots i_r}_{j_1 \dots j_s}(x) = \tau^{i_1 \dots i_r}_{j_1 \dots j_s, \kappa}(x)\,\xi^\kappa(x) - \tau^{\kappa, i_2 \dots i_r}_{j_1 \dots j_s}(x)\,\xi^{i_1}_{,\kappa}(x) - \dots + \tau^{i_1 \dots i_{r-1}, \kappa}_{j_1 \dots j_s}(x)\,\xi^{i_r}_{,\kappa}(x) + \dots \tau^{i_1 \dots i_r}_{j_1 \dots j_{s-1}, \kappa}(x)\,\xi^\kappa_{,j_s}(x)$

($\tau \in T(M)^r_s$)

Strategy: If we choose τ to be the volume form, which on flat $\mathbb{R}^m$ we may choose to be $\Omega = 1\, dx^1 \wedge \dots dx^m$, then the first term will drop out on $\mathbb{R}^m$ b/c $1_{,i} = 0$, and so we may be generalizing $\xi^i_{,i}$ on $\mathbb{R}^m$!

Def: The Divergence of a vector field ξ with respect to a volume form, Ω, is defined to be:

$$\operatorname{div}_\Omega \xi := L_\xi(\Omega)$$

(L_ξ: Lie derivative)

- Assume $\Omega = a(x)\, dx^1 \wedge \dots \wedge dx^m$ (volume form) ($a(x) \in \Lambda_0(M)$)
  and $\xi = \xi^i(x) \frac{\partial}{\partial x^i}$ (vector field)
- Then:

$$\operatorname{div}_\Omega \xi = L_\xi \Omega \overset{\text{by Leibnit rule}}{=} \xi(a)\, dx^1 \wedge \dots \wedge dx^m + \dots + a \sum_{i=1}^{m} dx^1 \wedge \dots \wedge L_\xi(dx^i) \wedge \dots \wedge dx^m$$

$\left(\text{recall: } L_\xi(dx^i) = d(\xi(x^i)) = d\left(\xi^j \frac{\partial}{\partial x^j} x^i\right) = d(\xi^j \delta^i_j) = d(\xi^i) = \frac{\partial \xi^i}{\partial x^\tau} dx^\tau\right)$ (only dx^i term survives in wedge product)

$$\Rightarrow \operatorname{div}_\Omega \xi = \left(\xi^i a_{,i} + a\, \xi^i_{,i}\right) dx^1 \wedge \dots \wedge dx^m$$

$$\Rightarrow \boxed{\operatorname{div}_\Omega \xi = \frac{1}{a}(a \xi^i)_{,i}\, \Omega}$$

Notice: If $a \equiv 1$ then $\operatorname{div}_\Omega \xi = \frac{\partial \xi^i}{\partial x^i}$ as expected for the divergence in the simplest case.

Thus: Indeed, if $a(x) = 1 \ \forall x$ then $\operatorname{div}_\Omega \xi = \xi^i_{,i}\, dx^1 \wedge \dots \wedge dx^m$

File Edit View Insert Actions Tools Help
Sharing Desktop
Stop Sharing

and so we may be generalizing $\xi^i_{,i}$ on $\mathbb{R}^n$!

Def: The Divergence of a vector field ξ with respect to a volume form, Ω, is defined to be:

$$\boxed{div_\Omega \xi := L_\xi(\Omega)}$$

↑ Lie derivative

$$div_\Omega \xi = L_\xi \Omega = \xi^i(a)\, dx^1 \wedge \dots \wedge dx^n + \dots$$

$$+ a \sum_{i=1}^{n} dx^1 \wedge \dots \wedge L_\xi(dx^i) \wedge \dots \wedge dx^n$$

$$\left(\text{recall: } L_\xi(dx^i) = d(\xi(x^i)) = d\left(\xi^j \frac{\partial}{\partial x^j} x^i\right) = d(\xi^j \delta^i_j) = d(\xi^i) = \frac{\partial \xi^i}{\partial x^r} dx^r\right)$$

only dx^i term survives in wedge product

$$\Rightarrow \quad div_\Omega \xi = \left(\xi^i a_{,i} + a\, \xi^i_{,i}\right) dx^1 \wedge \dots \wedge dx^n$$

$$\Rightarrow \quad \boxed{div_\Omega \xi = \frac{1}{a}(a\xi^i)_{,i}\, \Omega}$$

Notice: If $a \equiv 1$ then $div_\Omega \xi = \frac{\partial \xi^i}{\partial x^i}$ as expected for the divergence in the simplest case.

Thus: Indeed, if $a(x) = 1 \ \forall x$ then $div_\Omega \xi = \xi^i_{,i}\, dx^1 \wedge \dots \wedge dx^n$.

Now, we can derive Gauss' theorem from Stokes':

- $div_\Omega \xi := L_\xi \Omega \in \Lambda_n(\mathcal{M})$
- $div_\Omega \xi = (d \circ i_\xi + i_\xi \circ d)\,\Omega$

↑ Recall: $d\Omega = 0$ because anti-symmetry doesn't allow $(n+1)$ forms.

$$\Rightarrow \quad \boxed{div_\Omega \xi = d \circ i_\xi(\Omega)}$$

We can now apply Stokes' theorem $\int_G d\nu = \int_{\partial G} \nu$:

$$\int_G d\, i_\xi(\Omega) = \int_{\partial G} i_\xi(\Omega)$$

i.e.:

$$\boxed{\int_G \overbrace{div_\Omega \xi}^{n\text{-form}} = \int_{\partial G} \overbrace{i_\xi(\Omega)}^{(n-1)\text{ form}}}$$

"Gauß' theorem"