Title: General Relativity for Cosmology Lecture - 092623

Speakers: Achim Kempf

Collection: General Relativity for Cosmology

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Abstract: Zoom: https://pitp.zoom.us/j/91640855624?pwd=dWVWV2doSnBhUS9JUkhjQVBwY0h0dz09

covariant tensors, i.e., about differential forms?

A: Antisymmetry ⇒ special transformation property under chart changes:

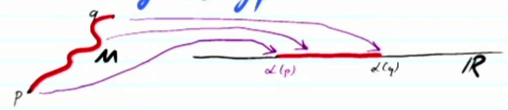
$\sim$ det (Jacobian)

⇒ suitable for integration:

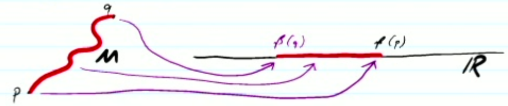
s-forms have natural integrals in s-dimensional manifolds

Except: Depending on charts, sign of Jacobian may be wrong!

- Could have charts of the type



- or charts of the type



- But, since $\int_a^b f(t)\,dt = -\int_b^a f(t')\,dt'$ one needs to decide!

because $\frac{dt}{dt'} = -1$ (which is det[Jacobian])

For n-dim mflds, may need several charts.

Definitions:

- A complete collection of charts, i.e., an Atlas, A, is called oriented if among all overlapping charts with coordinates say $x, \tilde{x}$ the Jacobi determinants are positive:

$$\det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) > 0$$

- A mfld M is called orientable if it possesses an oriented atlas.

Example: Möbius strips

are not orientable.

- A mfld, M, together with a choice of oriented atlas, A, is called an oriented manifold.

- Then, an arbitrary chart is called positive (or negative) if its Jacobian determinant with charts of the atlas A is positive (or negative).

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- A mfld M is called orientable if it possesses an oriented atlas.

Definition:

An n-form $\Omega \in \Lambda_n(M)$ is called a volume form if it nowhere vanishes. We will later find a preferred volume form for space-time (using the metric).

Proposition:

M possesses a volume form

$$\Updownarrow$$

M is orientable

are not orientable.

- A mfld, M, together with a choice of oriented atlas, A, is called an oriented manifold.
- Then, an arbitrary chart is called positive (or negative) if its Jacobian determinant with charts of the atlas A is positive (or negative).

Integration:

- Recall change of cds in integration in $\mathbb{R}^n$:

For $(x^1, \dots, x^n) \to (\tilde{x}^1, \dots, \tilde{x}^n)$:

Riemann or Lebesgue integrals $\to$ $g: \mathbb{R}^n \to \mathbb{R}$

$$\int_{\mathbb{R}^n} g(x^1, \dots, x^n)\, dx^1 \dots dx^n = \int_{\mathbb{R}^n} g(x(\tilde{x}))\, \det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right) d\tilde{x}^1 \dots d\tilde{x}^n \quad (*)$$

Jacobian determinant is negative if coordinate systems change handedness.

- Now for a general n-dimensional diff'able mfld M, consider an n-form ω in a chart:

$$\omega = f(x)\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$$

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Now for a general n-dimensional diffable mfld M, consider an n-form ω in a chart:

$$\omega = f(x)\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$$

Then what is ω in an overlapping, second chart?

$$\omega = f(x(\tilde{x})) \frac{\partial x^1}{\partial \tilde{x}^{i_1}} \frac{\partial x^2}{\partial \tilde{x}^{i_2}} \dots \frac{\partial x^n}{\partial \tilde{x}^{i_n}} \underbrace{d\tilde{x}^{i_1} \wedge d\tilde{x}^{i_2} \wedge \dots \wedge d\tilde{x}^{i_n}}_{\text{totally antisymmetric!}}$$

- terms are nonzero only if contain each number $1,\dots,n$ exactly once, e.g. $d\tilde{x}^1 \wedge d\tilde{x}^3 \wedge d\tilde{x}^2 \wedge d\tilde{x}^4 \wedge d\tilde{x}^5 \wedge \dots \wedge dx^n$.
- Reorder those terms – they are all $d\tilde{x}^1 \wedge d\tilde{x}^2 \wedge \dots \wedge d\tilde{x}^n$ up to a possible factor -1 because $dx^i \wedge dx^j = -dx^j \wedge dx^i$

$$\Rightarrow \quad \omega = f(x(\tilde{x}))\, \det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right) d\tilde{x}^1 \wedge d\tilde{x}^2 \wedge \dots \wedge d\tilde{x}^n$$

Compare with equation $(*)$ above $\Rightarrow$

The following definition of the integral of n-forms in an n-dim. diffable mfld is chart-independent, i.e., is well-defined:

Definition:

Assume M is an oriented n-dim mfld and $\omega \in \Lambda_n(M)$ reads in a chart d: $\omega = f(x)\, dx^1 \wedge \dots \wedge dx^n$.

Then, if one chart suffices:

$$\int_M \omega := \overbrace{\int_{d(M)} f(x)\, dx^1 dx^2 \dots dx^n}^{\text{usual Riemann or Lebesgue integral}}$$

$d(M)$: image of M in $\mathbb{R}^n$

Else: Piece right hand side together from several charts

Note: how to piece together does not matter as long as charts are from the atlas that M is equipped with. That's why orientation is important.

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exactly once, e.g.

- Reorder those terms – they are all

$$d\tilde{x}^1 \wedge d\tilde{x}^2 \wedge \ldots \wedge d\tilde{x}^m$$

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Compare with equation (*) above $\Rightarrow$

and $\omega \in \Lambda_m(M)$ reads in a chart d: $\omega = f(x)\, dx^1 \wedge \ldots \wedge dx^m$.

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Else: Piece right hand side together from several charts

Note: how to piece together does not matter as long as charts are from the atlas that M is equipped with. That's why orientation is important.

Definition: The boundary operator, ∂

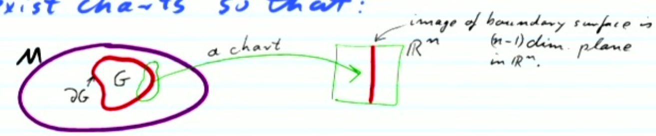
- Assume $G \subset M$ is a region (i.e. an n-dim., open and connected subset) of the n-dim mfld M.

  We denote the $(n-1)$ dim. boundary manifold of G by ∂G:

  $$\partial G := \text{boundary}(G)$$

  ↖ the boundary operator

- We say that ∂G is smooth if locally there exist charts so that:



Proposition: If M is orientable, then so is G. Also, the orientation of G induces an orientation of ∂G.

We finally have all ingredients for one of Math's most important theorems:

Stokes' theorem: If closure $\bar{G}$ of G is a compact n-dim region, then:

$$\boxed{\int_G d\omega = \int_{\partial G} \omega \quad \text{for all } \omega \in \Lambda_{n-1}(M)}$$

Definition: d is also called "co-boundary operator".

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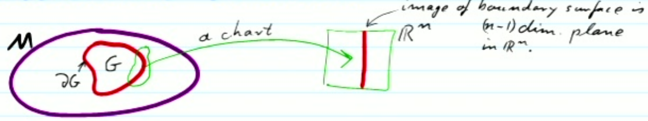

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Definition: d is also called "co-boundary operator".

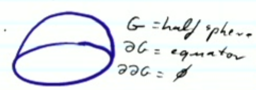
Remark:

- Let us try iterating Stokes!
- Assume $G = \partial H$.
- Then, by Stokes we obtain $0 = 0$:

$$\int_H \underbrace{dd}_{\substack{=0 \text{ always} \\ \text{for algebraic} \\ \text{reasons.}}}\omega \overset{\text{Stokes}}{=} \int_{\partial H} d\omega \overset{\text{Stokes}}{=} \int_{\underbrace{\partial\partial H}_{=0}} \omega$$

$=0$ for geometric reasons because, indeed, boundaries don't possess boundaries:

E.g.



i.e.: Stokes implies $d^2 = 0 \Leftrightarrow \partial^2 = 0$

- Stokes links homology (geometric) to cohomology (algebraic).

Special case I:

Assume: $M = \mathbb{R}$, $G = (a, b)$

Therefore: $\partial G = \{a, b\}$

Then, Stokes' theorem is $\int_G df = \int_{\partial G} f$, namely:

$$\int_a^b \underbrace{df}_{= \frac{df}{dx} dx} = f\Big|_a^b \qquad \text{(fund. thm of calculus)}$$

Then, by Stokes we obtain

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G = half sphere
∂G = equator
$\partial\partial G = \emptyset$

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$$\int_a^b df = f\Big|_a^b \qquad \text{(fund. thm of calculus)}$$

$$df = \frac{df}{dx}\,dx$$

Special case II: "Green's theorem".

- $M = \mathbb{R}^2$, $G \subset \mathbb{R}^2$ a region with (closed) boundary curve ∂G.

recall: this is automatic because $\partial\partial = 0$

- Consider an arbitrary 1-form $\omega \in \Lambda_1(M)$:

$$\omega = \omega_1(x)\,dx^1 + \omega_2(x)\,dx^2$$

Then:
$$\begin{aligned} d\omega &= d\omega_1(x)\wedge dx^1 + d\omega_2(x)\wedge dx^2 \\ &= \left(\frac{\partial\omega_1}{\partial x^1}dx^1 + \frac{\partial\omega_1}{\partial x^2}dx^2\right)\wedge dx^1 \\ &\quad + \left(\frac{\partial\omega_2}{\partial x^1}dx^1 + \frac{\partial\omega_2}{\partial x^2}dx^2\right)\wedge dx^2 \\ &= \frac{\partial\omega_1}{\partial x^2}dx^2\wedge dx^1 + \frac{\partial\omega_2}{\partial x^1}dx^1\wedge dx^2 \end{aligned}$$

$$\Longrightarrow d\omega = \left(\frac{\partial\omega_2}{\partial x^1} - \frac{\partial\omega_1}{\partial x^2}\right)dx^1\wedge dx^2$$

Now, Stokes' theorem $\int_G d\omega = \int_{\partial G}\omega$ becomes:

$$\int_G \left(\frac{\partial\omega_2}{\partial x^1} - \frac{\partial\omega_1}{\partial x^2}\right)dx^1dx^2 = \int_{\partial G}\left(\omega_1 dx^1 + \omega_2 dx^2\right)$$

Recall: How to evaluate, e.g., the RHS, in practice?

- Choose a chart for ∂G, i.e., a diff'able map, invertible map $\partial G \to \mathbb{R}$.
- Its inverse is a path: $\gamma: I \subset \mathbb{R} \to \partial G$, with $\gamma(t) = (x^1(t), x^2(t))$
- Now use $dx^i = \frac{dx^i}{dt}dt$ to obtain an integral over $I \subset \mathbb{R}$

Special case of Green's theorem:

Assume $\omega \in \Lambda_1$ is closed, i.e., $d\omega = 0$, i.e., $\frac{\partial \omega_1}{\partial x^2} - \frac{\partial \omega_2}{\partial x^1} = 0$

Then: $\int_{\partial G} \omega = 0$

Compare: (From the residue theorem)

If a function $w: G \subset \mathbb{C} \to \mathbb{C}$ is holomorphic, i.e., it obeys the Cauchy Riemann equations, then:

$$\int_{\partial G} w(z)\, dz = 0$$

Indeed:

The Cauchy-Riemann equations mean that a diff. form is closed and co-closed. We'll define "co-closedness" later.

Special case III: (exercise)

Similarly, one can show that what is often called the Stokes theorem for $M = \mathbb{R}^3$, namely

"cross product": $\hat{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$

$\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$

$$\int_G \vec{\nabla} \times \vec{w}\, dG = \int_{\partial G} \vec{w} \cdot \vec{ds}$$

vector field

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is indeed this special case:

$\omega \in \Lambda_1(G)$ with $\vec{\nabla} \times \vec{\omega} = d\omega \in \Lambda_2(G)$

Before we can discuss the next example:

How to define the volume of a region $G \subset M$ of a diff'able manifold M?

- In $\mathbb{R}^n$, we had: $V = \int_G dx^1 \dots dx^n$
- In general, we need to choose a Volume form $\Omega \in \Lambda_n$
  obeying $\Omega(p) \neq 0\ \forall p \in G$. Then the ($\Omega$-dependent) volume

Special case IV: Gauss' theorem

To obtain Gauß' theorem we need to define yet a new derivative the divergence of a vector field.

Recall: On $\mathbb{R}^n$, the divergence of a vector field, ξ, was defined as

$$\operatorname{div} \xi = \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \xi^i = \xi^i_{,i}$$

$\rightsquigarrow$ How to generalize to arbitrary manifolds?

∂G

- Consider an arbitrary 1-form $\omega \in \Lambda_1(M)$:

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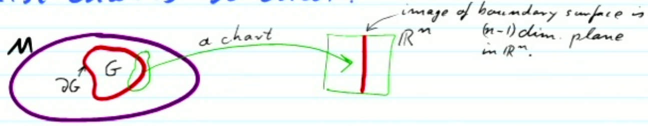

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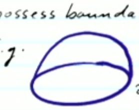
E.g. 

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"cross product": $\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$

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Before we can discuss the next example:

How to define the volume of a region $G \subset M$ of a diff'able manifold M?

- In $\mathbb{R}^n$, we had: $V = \int_G dx^1 \dots dx^n$

- In general, we need to choose a Volume form $\Omega \in \Lambda_n$ obeying $\Omega(p) \neq 0 \ \forall p \in G$. Then the ($\Omega$-dependent) volume is defined as:

$$V := \int_G \Omega$$

(We will later use the metric tensor to define a volume form for spacetime)

Proposition: G orientable $\Leftrightarrow$ $\exists$ volume forms Ω

(In fact ∞ many)

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To obtain Gauß' theorem we need to define yet a new derivative the **divergence of a vector field.**

Recall: On $\mathbb{R}^n$, the divergence of a vector field, ξ, was defined as

$$\mathrm{div}\,\xi = \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \xi^i = \xi^i_{\ ,i}$$

$\leadsto$ How to generalize to arbitrary manifolds?

Where in this course did we see $\xi^i(x)_{,i}$ before? !

Recall: $(L_\xi \tau)^{i_1 \dots i_r}_{j_1 \dots j_s}(x) = \tau^{i_1 \dots i_r}_{j_1 \dots j_s, k}(x)\, \xi^k(x) - \tau^{\bar{i}_1 \dots i_r}_{j_1 \dots j_s}(x)\, \xi^{i_1}_{\ ,\bar{i}_1}(x) - \dots$

$+ \tau^{i_1 \dots i_r}_{\bar{j}_1 \dots j_s}(x)\, \xi^{\bar{j}_1}_{\ ,j_1}(x) + \dots \tau^{i_1 \dots i_r}_{j_1 \dots \bar{j}_s}(x)\, \xi^{\bar{j}_s}_{\ ,j_s}(x)$

$\tau \in T(M)^r_s$

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$\int_G$

▯ In general, we need to choose a Volume form

$\Omega \in \Lambda_n$

obeying $\Omega(p) \neq 0 \; \forall p \in G$. Then the ($\Omega$-dependent) volume is defined as:

$$V := \int_G \Omega$$

(We will later use the metric tensor to define a volume form for spacetime)

Proposition: G orientable $\iff$ $\exists$ volume forms Ω (In fact ∞ many)

Strategy: If we choose τ to be the volume form, which on flat $\mathbb{R}^n$ we may choose to be $\Omega = 1 dx^1 \wedge \ldots dx^n$, then the first term will drop out on $\mathbb{R}^n$ b/c $1_{,i} = 0$, and so we may be generalizing $\xi^i_{,i}$ on $\mathbb{R}^n$!

Def: The Divergence of a vector field ξ with respect to a volume form, Ω, is defined to be:

$$\boxed{div_\Omega \xi := \underbrace{L_\xi}_{\text{Lie derivative}}(\Omega)}$$

Recall: On $\mathbb{R}$, the divergence of a vector field, ξ, was defined as

$$div \xi = \sum_{i=1}^n \frac{\partial}{\partial x^i} \xi^i = \xi^i_{,i}$$

⇝ How to generalize to arbitrary manifolds?

Where in this course did we see $\xi^i(x)_{,i}$ before? !

Recall: $(L_\xi \tau)^{i_1 \ldots i_r}_{j_1 \ldots j_s}(x) = \tau^{i_1 \ldots i_r}_{j_1 \ldots j_s, k}(x) \xi^k(x) - \tau^{\bar{i} \ldots i_r}_{j_1 \ldots j_s}(x) \xi^{i_1}_{,\bar{i}}(x) - \ldots$
$+ \tau^{i_1 \ldots i_r}_{\bar{j} \ldots j_s}(x) \xi^{\bar{j}}_{,j_1}(x) + \ldots \tau^{i_1 \ldots i_r}_{j_1 \ldots \bar{j}}(x) \xi^{\bar{j}}_{,j_s}(x)$

$\tau \in T(M)^r_s$

▯ Assume $\Omega = a(x) \, dx^1 \wedge \ldots \wedge dx^n$ (volume form), $a(x) \in \Lambda_0(M)$

and $\xi = \xi^i(x) \frac{\partial}{\partial x^i}$ (vector field)

▯ Then:

$$div_\Omega \xi = L_\xi \Omega \overset{\text{by Leibnitz rule}}{=} \underbrace{\xi(a)}_{= \xi^i \frac{\partial}{\partial x^i} a(x)} dx^1 \wedge \ldots \wedge dx^n + \ldots$$
$$+ a \sum_{i=1}^n dx^1 \wedge \ldots \wedge L_\xi(dx^i) \wedge \ldots \wedge dx^n$$

(recall: $L_\xi(dx^i) = d(\xi(x^i)) = d(\xi^j \frac{\partial}{\partial x^j} x^i) = d(\xi^j \delta^i_j) = d(\xi^i) = \frac{\partial \xi^i}{\partial x^r} dx^r$) only dx^i term survives in wedge product

$\Rightarrow$ $div_\Omega \xi = (\xi^i a_{,i} + a \xi^i_{,i}) \, dx^1 \wedge \ldots \wedge dx^n$

$\Rightarrow$ $\boxed{div_\Omega \xi = \frac{1}{a} (a \xi^i)_{,i} \, \Omega}$

Notice: If $a \equiv 1$ then $div_\Omega \xi = \frac{\partial \xi^i}{\partial x^i}$ as expected for the divergence in the simplest case.

Thus: Indeed, if $a(x) = 1 \; \forall x$ then $div_\Omega \xi = \xi^i_{,i} \, dx^1 \wedge \ldots \wedge dx^n$

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and so we may be generalizing $\xi^i_{,i}$ on $\mathbb{R}^n$!

Def: The Divergence of a vector field ξ with respect to a volume form, Ω, is defined to be:

$$\boxed{\mathrm{div}_\Omega \xi := L_\xi(\Omega)}$$

Lie derivative

$$\mathrm{div}_\Omega \xi = L_\xi \Omega = \xi(a)\, dx^1 \wedge \dots \wedge dx^n + \dots$$

$$+ a \sum_{i=1}^{n} dx^1 \wedge \dots \wedge L_\xi(dx^i) \wedge \dots \wedge dx^n$$

$$\left(\text{recall: } L_\xi(dx^i) = d(\xi(x^i)) = d\left(\xi^j \tfrac{\partial}{\partial x^j} x^i\right) = d(\xi^j \delta_j^i) = d(\xi^i) = \tfrac{\partial \xi^i}{\partial x^r} dx^r\right)$$

only dx^i term survives in wedge product

$$\Rightarrow \quad \mathrm{div}_\Omega \xi = \left(\xi^i a_{,i} + a\, \xi^i_{,i}\right) dx^1 \wedge \dots \wedge dx^n$$

$$\Rightarrow \quad \boxed{\mathrm{div}_\Omega \xi = \tfrac{1}{a}(a \xi^i)_{,i}\, \Omega}$$

Notice: If $a \equiv 1$ then $\mathrm{div}_\Omega \xi = \frac{\partial \xi^i}{\partial x^i}$ as expected for the divergence in the simplest case.

Thus: Indeed, if $a(x) = 1\ \forall x$ then $\mathrm{div}_\Omega \xi = \xi^i_{,i}\, dx^1 \wedge \dots \wedge dx^n$.

Now, we can derive Gauss' theorem from Stokes':

- $\mathrm{div}_\Omega \xi = L_\xi \Omega \in \Lambda_n(M)$
- $\mathrm{div}_\Omega \xi = (d \circ i_\xi + i_\xi \circ d)\,\Omega$

Recall: $d\Omega = 0$ because anti-symmetry doesn't allow $(n+1)$ forms.

$$\Longrightarrow \quad \boxed{\mathrm{div}_\Omega \xi = d \circ i_\xi(\Omega)}$$

We can now apply Stokes' theorem $\int_G dv = \int_{\partial G} v$:

$$\int_G d\, i_\xi(\Omega) = \int_{\partial G} i_\xi(\Omega)$$

i.e.:

$$\boxed{\int_G \overbrace{\mathrm{div}_\Omega \xi}^{n\text{-form}} = \int_{\partial G} \overbrace{i_\xi(\Omega)}^{(n-1)\text{ form}} \qquad \text{"Gauß' theorem"}}$$