

Title: General Relativity for Cosmology Lecture - 091923

Speakers: Achim Kempf

Collection: General Relativity for Cosmology

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GR for Cosmology, Achim Kempf Lecture 4

Differential forms (also called "exterior differential forms")

Why? Every p-dim. integration is an integration over a differential p-form!

Preparation:

Consider the cotangent space $T_p^*(M)$ at p :

Each $\omega \in T_p^*(M)$ is a lin. map:

$$\omega: T_p(M) \rightarrow \mathbb{R}$$

$$\omega: \xi \rightarrow \omega(\xi)$$

Each such ω is a covariant tensor, of rank (0, 1)

More generally, consider the covariant tensors of rank (0, r):

Recall: $T_p(M)_r := \overbrace{T_p^*(M) \otimes \dots \otimes T_p^*(M)}^{r \text{ factors}}$

Each $\nu \in T_p(M)_r$ is a multi-linear map:

$$\nu: T_p(M)^r \rightarrow \mathbb{R}$$

In particular, if $\xi_1, \dots, \xi_r \in T_p(M)$ then:

$$\nu: \xi_1, \dots, \xi_r \rightarrow \nu(\xi_1, \dots, \xi_r)$$

Definition: If $r > 1$ and $\nu \in T_p(M)_r$, then we define the "anti-symmetric part of ν " as the image $\tilde{\nu} = A(\nu)$ of ν under the linear antisymmetrization map A :

$$\begin{aligned} \tilde{\nu}(\xi_1, \dots, \xi_r) &= A(\nu)(\xi_1, \dots, \xi_r) \\ &= \frac{1}{r!} \sum_{\sigma \in S_r} \text{sgn}(\sigma) \nu(\xi_{\sigma(1)}, \dots, \xi_{\sigma(r)}) \end{aligned}$$

the sign (± 1) of the permutation σ

\leftarrow group of all $r!$ permutations of $(1, 2, \dots, r)$

Why consider these?

Concretely:

Consider $\nu := df \otimes dg$, for $f, g \in \mathcal{F}_p(M)$

Then $\nu(\xi_1, \xi_2) = df(\xi_1) dg(\xi_2)$ (which is $\xi_1(f) \xi_2(g)$)

Apply A :

$$\tilde{\nu}(\xi_1, \xi_2) = A\nu(\xi_1, \xi_2) = \frac{1}{2} (df(\xi_1) dg(\xi_2) - df(\xi_2) dg(\xi_1))$$

\Rightarrow We can also write:

Definition:

For $r > 1$ and $v \in T_p(M)_r$, then we define the "anti-symmetric part of v " as the image $\tilde{v} = A(v)$ of v under the linear antisymmetrization map A :

$$\tilde{v}(\xi_1, \dots, \xi_r) = A(v)(\xi_1, \dots, \xi_r) = \frac{1}{r!} \sum_{\sigma \in S_r} \text{sgn}(\sigma) v(\xi_{\sigma(1)}, \dots, \xi_{\sigma(r)})$$

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Why consider these?

They will be key for integration! Only antisym. cov. tensors transform under chart change so as to match the Jacobian determinant arising in integrals when changing charts.

Proposition: A is a projector, i.e., it obeys

$$A \circ A = A$$

Check in above example:

$$\begin{aligned} A \circ A(df \otimes dg) &= A\left(\frac{1}{2} df \otimes dg - \frac{1}{2} dg \otimes df\right) \\ &= \frac{1}{2} \left(\frac{1}{2} df \otimes dg - \frac{1}{2} dg \otimes df\right) - \frac{1}{2} \left(\frac{1}{2} dg \otimes df - \frac{1}{2} df \otimes dg\right) \\ &= \frac{1}{2} (df \otimes dg - dg \otimes df) \\ &= A(df \otimes dg) \end{aligned}$$

Definition:

For $r > 1$ we define the space of differential r -forms (or 'exterior' r -forms) $\Lambda_r(p)$ at $p \in M$ as the subspace of totally anti-symmetric tensors

of rank $(0, r)$:

$$\Lambda_r(p) := A T_p(M)_r$$

a projector on a vector space
a vector space

\implies So if $v \in \Lambda_r(p)$ then $\tilde{v} = A(v) = v$

Definition: For $r=0$ we define the set of differential 0-forms at $p \in M$ as:

$$\Lambda_0(p) := \mathbb{R} \quad (\text{for 0-forms on the entire manifold we will have } \Lambda_0 := \mathbb{F}(M))$$

For $r=1$ we define the set of differential 1-forms (or "Pfaffian forms") at $p \in M$ through:

$$\Lambda_1(p) := T_p^*(M)$$

Strategy now:

Define multiplication \rightarrow obtain algebra \rightarrow obtain derivations

Properties of \wedge :

linear:

$$(\omega + \nu) \wedge \eta = \omega \wedge \eta + \nu \wedge \eta$$

$$(a\omega) \wedge \nu = \omega \wedge (a\nu) = a(\omega \wedge \nu) \quad \text{if } a \in \mathbb{R}$$

associative:

$$(\omega \wedge \nu) \wedge \eta = \omega \wedge (\nu \wedge \eta)$$

"graded" commutative:

$$\omega \wedge \nu = (-1)^{rs} \nu \wedge \omega \quad \text{if } \omega \in \Lambda_r, \nu \in \Lambda_s$$

(Eg. for $dx^i \in \Lambda_1(p)$)

wedge product:

Def: If $\omega \in \Lambda_s(p), \nu \in \Lambda_r(p)$, and $r, s \neq 0$ then the wedge product \wedge yields a new differential form:

$$\wedge : \Lambda_r(p) \times \Lambda_s(p) \rightarrow \Lambda_{r+s}(p)$$

$$\wedge : (\omega, \nu) \rightarrow \omega \wedge \nu = \frac{(s+r)!}{s!r!} A(\omega \otimes \nu)$$

a normalization factor

Def: For $c \in \Lambda_0, \omega \in \Lambda_s$ we have $c \wedge \omega = c\omega$

Note: $dx^i \wedge dx^i = 0 \quad \forall i$

Example: For dx^i, dx^j we obtain:

$$dx^i \wedge dx^j = (dx^i \otimes dx^j - dx^j \otimes dx^i)$$

We can use \wedge to build bases of $\Lambda_r(p)$:

Assume: $\{\theta^i\}_{i=1}^m$ is a basis of $\Lambda_1 = T_p^*(M)$.

(for example $\theta^i = dx^i$)

Then: $\{\theta^{i_1} \wedge \theta^{i_2} \wedge \dots \wedge \theta^{i_r}\}_{1 \leq i_1 < i_2 < \dots < i_r \leq m}$

Exercise: show this

is a basis of $\Lambda_r(p)$ for $r \geq 1$.

Therefore:

$$\dim(\Lambda_r(p)) = \binom{m}{r} = \frac{m!}{r!(m-r)!}$$

for $r=1$ we define the set of differential 1-forms (or "Pfaffian forms") at $p \in M$ through:

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(Eg. for $dx^i \in \Lambda_1(p)$:
 $dx^i \wedge dx^i = -dx^i \wedge dx^i$
 since $r=s=1$.)

$$\wedge : \Lambda_r(p) \times \Lambda_s(p) \rightarrow \Lambda_{r+s}(p)$$

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is a basis of $\Lambda_r(p)$ for $r \geq 1$.

Therefore:

$$\dim(\Lambda_r(p)) = \binom{m}{r} = \frac{m!}{r!(m-r)!}$$

\Rightarrow no diff. forms of degree $r > m$!

- linear:

$$(\omega + \nu) \wedge \eta = \omega \wedge \eta + \nu \wedge \eta$$

$$(a\omega) \wedge \nu = \omega \wedge (a\nu) = a(\omega \wedge \nu) \text{ if } a \in \mathbb{R}$$

- associative:

$$(\omega \wedge \nu) \wedge \eta = \omega \wedge (\nu \wedge \eta)$$

- "graded" commutative:

$$\omega \wedge \nu = (-1)^{rs} \nu \wedge \omega \text{ if } \omega \in \Lambda_r, \nu \in \Lambda_s$$

(Eg. for $dx^i \in \Lambda_1(p)$:
 $dx^i \wedge dx^i = -dx^i \wedge dx^i$
 since $r=s=1$.)

Assume: $\{\theta^i\}_{i=1}^m$ is a basis of $\Lambda_1 = T_p(M)^*$
 (for example $\theta^i = dx^i$)

Then: $\{\theta^{i_1} \wedge \theta^{i_2} \wedge \dots \wedge \theta^{i_r}\}_{1 \leq i_1 < i_2 < \dots < i_r \leq m}$
 Exercise: show this is a basis of $\Lambda_r(p)$ for $r \geq 1$.

Therefore:

$$\dim(\Lambda_r(p)) = \binom{m}{r} = \frac{m!}{r!(m-r)!}$$
 \Rightarrow no diff. forms of degree $r > m$!

Example: For $p \in M = \mathbb{R}^3$ we have bases:

- $\Lambda_1 = \text{span}(dx^1, dx^2, dx^3)$
- $\Lambda_2 = \text{span}(dx^1 \wedge dx^2, dx^1 \wedge dx^3, dx^2 \wedge dx^3)$
- $\Lambda_3 = \text{span}(dx^1 \wedge dx^2 \wedge dx^3)$

Definition:

$\Lambda(p) := \bigoplus_{i=0}^m \Lambda_i(p)$ equipped with the multiplication \wedge , is an associative algebra, called the exterior algebra or the Grassmann algebra over $T_p(M)$.
 ($\dim(\Lambda) = 2^m$)

Generalization to fields:

- A differential form field is a mapping that associates to each $p \in M$ an element: $\omega(p) \in \Lambda(p)$

It is usually also called simply a differential form and denoted ω .

- These fields form the Grassmann algebra of differential forms, $\Lambda(M)$.

Recall:

Given an algebra, it is often useful to consider **derivations of the algebra**, i.e., to consider linear maps that obey the Leibniz rule. (similar to how we defined tangent vectors as derivations of $F(M)$)

Here: For the algebra $\Lambda(M)$, let us consider the **exterior** and the **inner** derivations:

Definition:

A linear map $\Phi: \Lambda(M) \rightarrow \Lambda(M)$ is called an **anti-derivation of degree k** , if for all $\alpha \in \Lambda^s(M)$, $\beta \in \Lambda(M)$:

$$\Phi: \Lambda_s(M) \rightarrow \Lambda_{s+k}(M) \quad \text{for all } s \text{ and}$$

$$\Phi: \alpha \wedge \beta \rightarrow \Phi(\alpha) \wedge \beta + (-1)^s \alpha \wedge \Phi(\beta)$$

↑
"Anti-Leibniz rule"

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$$\Phi: \alpha \wedge \beta \rightarrow \Phi(\alpha) \wedge \beta + \alpha \wedge \Phi(\beta)$$

for all $\alpha, \beta \in \Lambda(M)$.

Leibniz rule

Also:

Proposition: (as we will show constructively)

Because of the Leibniz rule and linearity, any (anti-)derivation

$$\Phi: \Lambda(M) \rightarrow \Lambda(M)$$

is already fully determined by its action only on $\Lambda_0(M)$ and on a basis of $\Lambda_1(M)$.

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The exterior derivative

The exterior derivative,

$$d: \Lambda(M) \rightarrow \Lambda(M)$$

is the anti-derivation of degree $k=1$ which is defined through:

- a) $d: \Lambda_0(M) \rightarrow \Lambda_1(M)$
 $d: f \rightarrow df$ } action of d on $\Lambda_0(M)$
- b) $d: dx^i \rightarrow 0$ for all i . } action of d on a basis of $\Lambda_1(M)$!

In a chart:

We had: $d: f(x) \rightarrow df(x) = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$

Now we have more generally:

$$\beta = \sum_{i_1 < \dots < i_s} \beta_{i_1, \dots, i_s}(x) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_s} \in \Lambda_s(M)$$

↑ recall: $f \wedge w = fw$ when $f \in \Lambda_0$ and $w \in \Lambda$

Q: So how to carry out $d: \beta \rightarrow d\beta$?

A: By applying the anti-Leibniz rule:

$$d\beta = \sum_{i_1 < \dots < i_s} \frac{\partial \beta_{i_1, \dots, i_s}(x)}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_s}$$

Terms of the form $a(x^1, \dots, x^n)$
 = 0 because when applying Leibniz rule to $d(dx^{i_1} \wedge \dots \wedge dx^{i_n})$ we eventually arrive at $d(dx^i) = 0 \dots$

Proposition: $d: \Lambda(M) \rightarrow \Lambda(M)$ obeys:

$$d \circ d = 0$$

Proof: $d(d\beta) = \sum_{\substack{i_1 < \dots < i_n \\ i, k}} \frac{\partial \beta_{i_1, \dots, i_n}(x)}{\partial x^i \partial x^k} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_n} \wedge dx^k$
 $= 0$
 (Sym. in i, k / i.e. antisym. in i, k)
 $\Rightarrow \sum = 0$

Now assume $\gamma \in \Lambda_1(M)$ is an arbitrary (i.e. not necessarily gradient) covariant vector field:

$$\gamma = \sum_{i=1}^3 \gamma_i(x) dx^i \in \Lambda_1(\mathbb{R}^3)$$

Then:

$$\beta := d\gamma = \sum_{i,j} \frac{\partial \gamma_i(x)}{\partial x^j} dx^j \wedge dx^i$$

$i=j$ does not occur because $dx^i \wedge dx^i = 0$

$$= \sum_{i < j} \left(\frac{\partial \gamma_j(x)}{\partial x^i} - \frac{\partial \gamma_i(x)}{\partial x^j} \right) dx^j \wedge dx^i$$

from $dx^j \wedge dx^i = -dx^i \wedge dx^j$

Example:

For $M = \mathbb{R}^3$ and $f \in \mathcal{F}(M)$ we have:

$$df = \sum_{i=1}^3 \frac{\partial f}{\partial x^i} dx^i$$

e.g.: electric potential

Notice: $(\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial f}{\partial x^3})$ is the "Gradient field ∇f of f "

Example: Electric field $E = d\phi$ from potential ϕ .

$$= - \left(\frac{\partial \gamma_1}{\partial x^2} - \frac{\partial \gamma_2}{\partial x^1} \right) dx^1 \wedge dx^2$$

$$- \left(\frac{\partial \gamma_2}{\partial x^3} - \frac{\partial \gamma_3}{\partial x^2} \right) dx^2 \wedge dx^3$$

$$- \left(\frac{\partial \gamma_3}{\partial x^1} - \frac{\partial \gamma_1}{\partial x^3} \right) dx^3 \wedge dx^1$$

Notice: $(\beta_1(x), \beta_2(x), \beta_3(x))$ are the components of the curl
 $\beta = \nabla \times \gamma$
 It is called a "pseudo vector field"
 It is really a 2-form field in 3 dim.

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$$\gamma = \sum_{i=1}^3 \gamma_i(x) dx^i \in \Lambda^1(\mathbb{R}^3)$$

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from $dx^i \wedge dx^i = -dx^i \wedge dx^i$

$$\underbrace{\left(\frac{\partial \gamma_i}{\partial x^j} - \frac{\partial \gamma_j}{\partial x^i} \right)}_{\beta_{ij}} dx^j \wedge dx^i$$

$$= \left(\frac{\partial \gamma_1}{\partial x^2} - \frac{\partial \gamma_2}{\partial x^1} \right) dx^1 \wedge dx^2$$

$$- \left(\frac{\partial \gamma_2}{\partial x^3} - \frac{\partial \gamma_3}{\partial x^2} \right) dx^2 \wedge dx^3$$

Notice: $(\beta_1(x), \beta_2(x), \beta_3(x))$ are the components of the curl

$\beta = \nabla \times \gamma$
 It is called a "pseudo vector field"
 It is really a 2-form field in 3 dim.

Recall:

Gradient vector fields are curl free: $\nabla \times (\nabla f) = 0$

This is a special case of

$$d \circ d = 0$$

because if $\beta = d\gamma$ then:

$$d\beta = d^2\gamma = 0$$

Definition:

A differential form w is called closed if:

$$dw = 0$$

E.g.: We saw that $\beta := d\gamma$ is closed. Is this example typical?

A differential form w is called exact if there exists a v so that

$$w = dv \quad (v \text{ is like an anti-derivative!})$$

Gradient vector fields
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□ A differential form w is called exact
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How are closedness and exactness related?

This actually depends on the global topology
of the manifold! (because anti-derivatives are in a sense global)

Simplest case: Assume M is contractible



i.e., $\exists F: [0,1] \times M \rightarrow M$

so that $F(0, x) = x \quad \forall x$

$F(1, x) = x_0 \quad \forall x$

↑ some fixed $p \in M$



Poincaré lemma:

On any contractible manifold:

$$\nabla \text{ exact} \iff \nabla \text{ closed}$$

E.g.

□ \mathbb{R}^m is contractible

□ $\mathbb{R}^m \setminus \{p\}$ is not contractible

↑ some arbitrary point $p \in M$

In general: We only have

$$\neq \text{ exact} \Rightarrow \neq \text{ closed}$$

which is because $d^2 = 0$.

\Rightarrow We obtain a tool for classifying the "global topology" of differentiable manifolds (checking for holes, handles etc.)

How? Take a diffable manifold and calculate the vector space of closed but not exact differential forms.

A look at the bigger picture:

Why are cohomology theories useful?

It is much easier to check if two abelian groups are isomorphic than to check if two differentiable manifolds are diffeomorphic.

What are the abelian groups in the case of de Rham cohomology?

They are the vector spaces of closed but not exact differential forms.

A look at the even bigger picture

This method is a special case of a cohomology theory, called "De Rham Cohomology"

Q: What is a cohomology theory? (roughly)

A: A cohomology theory is a map G

$$G: \{\text{differentiable manifolds}\} \rightarrow \{\text{abelian groups}\}$$

which is such that if:

$$G(M) \neq G(N) \Rightarrow M \neq N$$

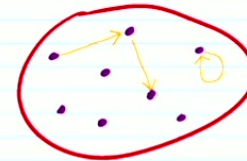
\uparrow
i.e. not isomorphic as abelian groups

\uparrow
i.e. no diffeomorphism exists

Comment: All cohomology theories are:

"Natural Transformations" between two "Categories."

Def: A category is a set of objects and morphisms:



Axioms:

$\square \exists A \rightarrow B \text{ and } B \rightarrow C \Rightarrow \exists A \rightarrow C$

\square Associativity:

$$(\rightarrow \bullet \rightarrow \bullet \rightarrow \bullet) \rightarrow \bullet \rightarrow \bullet = \rightarrow \bullet \rightarrow (\bullet \rightarrow \bullet \rightarrow \bullet) \rightarrow \bullet$$

$\square \forall A \rightarrow A$

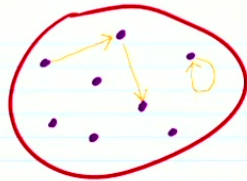
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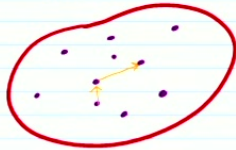
Examples:



Category of vector spaces Vec :

Objects: all vector spaces

Morphisms: linear transformations

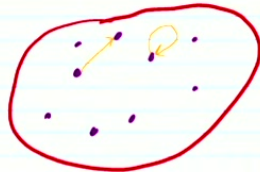


Category of associative algebras Alg :

Objects: all assoc. algebras

Morphisms: algebra homomorphisms

But also:



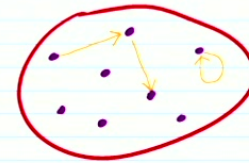
Category of categories Cat :

Objects: all categories

Morphisms: natural transformations also called functors.

"Natural Transformations" between two "Categories"

Def: A category is a set of objects and morphisms:



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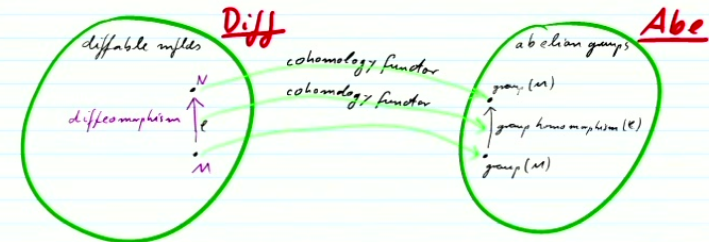
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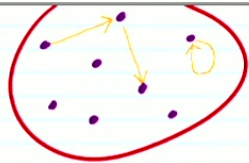
A cohomology theory is a morphism between two objects in Cat , namely $Diff$ and Abe :



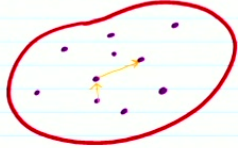
Crucial: $group(M)$ not homomorph $group(N) \Rightarrow M$ not diffeomorph to N

Note: This is what category theory was originally developed for.

Examples:

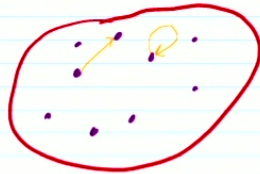


Category of vector spaces
Objects: all vector spaces
Morphisms: linear transformations



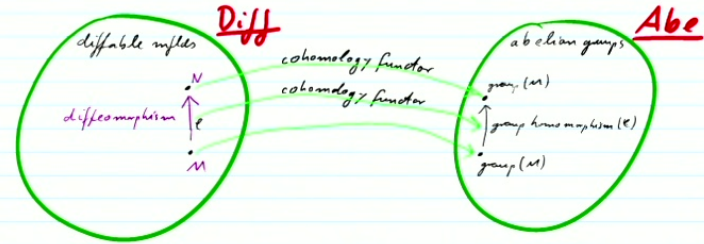
Category of associative algebras
Objects: all assoc. algebras
Morphisms: algebra homomorphisms

But also:



Category of categories
Objects: all categories
Morphisms: natural transformations also called functors.

Cohomology theory is a morphism between two objects in Cat, namely Diff and Abe:



Crucial: $\text{group}(M) \text{ not homomorph } \text{group}(N) \Rightarrow M \text{ not diffeomorph to } N$
 Note: This is what category theory was originally developed for.

Example: K-theory

- This cohomology theory uses the fact that:
- There is only one way to put a vector bundle on a contractible mfd but others can have many non-isomorphic vector bundles.
- Recall that, e.g., for a suitable vector bundle B :



$\pi^{-1}(U_1) \cong U_1 \times \mathbb{R}^n$
 $\pi^{-1}(U_2) \cong U_2 \times \mathbb{R}^n$
 But: $B \neq M \times \mathbb{R}^n$

Q: What are the abelian groups here?

A: The vector bundles form an abelian group through Whitney's generalisation of direct sum \oplus .

In this course: We'll focus on the local properties of the manifolds, such as curvature.

\rightarrow We mention cohomology theory issues only on the side in this course.

Next: The inner derivation of $\Lambda(M)$.