Title: General Relativity for Cosmology Lecture - 091423

Speakers: Achim Kempf

Collection: General Relativity for Cosmology

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GR for Cosmology, Achim Kempf. **Lecture 3**

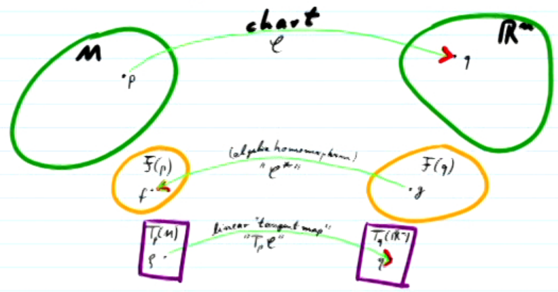
The "physicist's definition of $T_p(M)$"

Recall: We obtain concrete representations for $p \in M$ and $f \in \mathcal{F}(p)$ and $\xi \in T_p(M)$ using a chart $\varphi: M \to \mathbb{R}^n$:

Recall: Def's used pre-composition:

$$\varphi^*[g] = g \circ \varphi$$

$$T_p\varphi[\xi] = \xi \circ \varphi^*$$



Terminology: φ^* is called the "pullback" of φ

$T_p\varphi$ is called the "pullback" of φ^*

Namely:

- Each $p \in M$ has now a concrete image $q \in \mathbb{R}^n$, i.e., it has 'coordinates'.
- Each $f \in \mathcal{F}(p)$ is the image of a concrete function germ $g \in \mathcal{F}(q)$.
- Each $\xi \in T_p(M)$ has now a concrete image $\eta \in T_q(\mathbb{R}^n)$ which we know has the form:

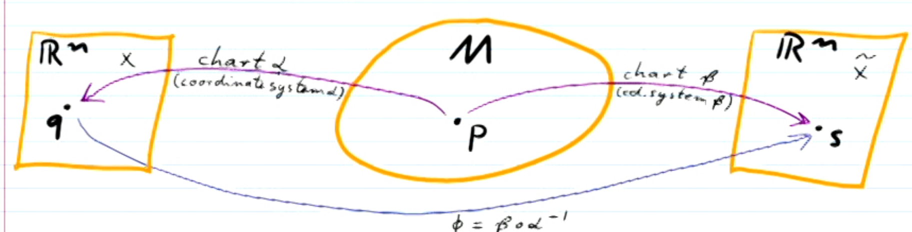
$$\eta = \sum_{i=1}^{n} \eta_i \frac{\partial}{\partial x_i}\bigg|_{x=q}$$

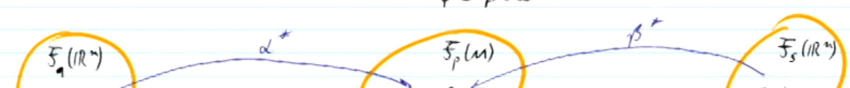
coefficients $\in \mathbb{R}$

Question:

Given a $p \in M$ and a $\xi \in T_p(M)$, how do their coordinates and coefficients change under a change of charts?

$\Rightarrow$ When changing from chart α to chart β:





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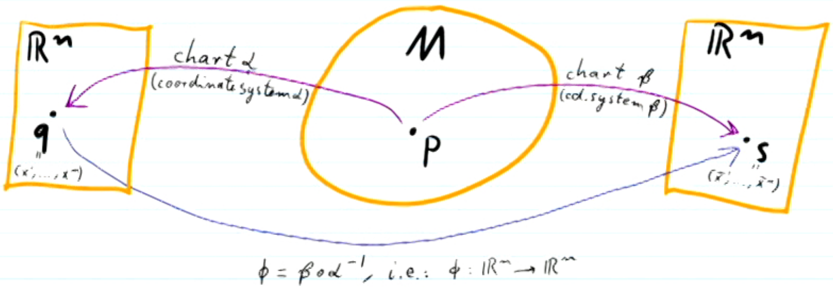

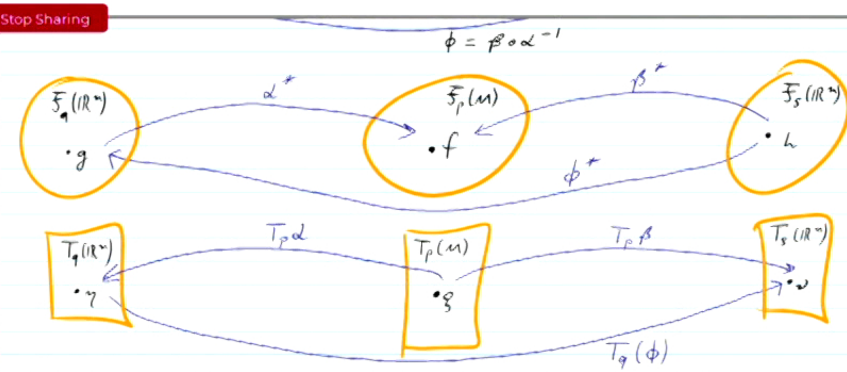

1. Every point $p \in M$ now has 2 images, $q = (x^1,\dots,x^n)$ and $s = (\tilde{x}^1,\dots,\tilde{x}^n)$

$$(\tilde{x}^1,\dots,\tilde{x}^n) = \phi(x^1,\dots,x^n)$$

Concretely: $\tilde{x}^i = \phi^i(x^1,\dots,x^n)$.

2. Every function germ $f \in \mathcal{F}_p(M)$ has 2 pre-images, $g \in \mathcal{F}_q(\mathbb{R}^n)$ and $h \in \mathcal{F}_s(\mathbb{R}^n)$, related by

$f(p) = g(q) = h(s)$ $\quad (\in \mathbb{R})$ and by

$h(\tilde{x}^1,\dots,\tilde{x}^n) = g(x^1,\dots,x^n)$ $\quad (*)$ (in a neighborhood)

3. Every tangent vector $\xi \in T_p(M)$ now has 2 images, $\eta \in T_q(\mathbb{R}^n)$ and $\nu \in T_s(\mathbb{R}^n)$.

By construction: (b/c of precomposition)

$$\eta(g) = \xi(f) = \nu(h) \qquad (\in \mathbb{R})$$

$\Rightarrow$ in particular:

$$\overbrace{\sum_{i=1}^n \eta^i \frac{\partial}{\partial x^i} g(x^1,\dots,x^n)\Big|_{x=q}}^{\eta(g)} = \overbrace{\sum_{j=1}^n \nu^j \frac{\partial}{\partial \tilde{x}^j} \underbrace{h(\tilde{x}^1,\dots,\tilde{x}^n)}_{\substack{\parallel \\ g(x^1,\dots,x^n)}}\Big|_{\tilde{x}=s}}^{\nu(h)} \text{ by } (*)$$

$$= \sum_{\substack{j=1\\k=1}}^n \nu^j \frac{\partial x^k}{\partial \tilde{x}^j}\Big|_{\tilde{x}=s} \frac{\partial}{\partial x^k} g(x^1,\dots,x^n)\Big|_{x=q}$$

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2. Every function germ $f \in \mathcal{F}_p(M)$ has 2 pre-images, $g \in \mathcal{F}_q(\mathbb{R}^m)$ and $h \in \mathcal{F}_s(\mathbb{R}^m)$, related by

$f(p) = g(q) = h(s)$ $\quad (\in \mathbb{R})$ and by

$h(\tilde{x}^1, \dots, \tilde{x}^m) = g(x^1, \dots, x^m)$ $\quad (*)$ (in a neighborhood)

3. Every tangent vector $\xi \in T_p(M)$ now has 2 images, $\eta \in T_q(\mathbb{R}^m)$ and $\nu \in T_s(\mathbb{R}^m)$.

$\Rightarrow$ in particular:

$$\overbrace{\sum_{i=1}^m \eta^i \frac{\partial}{\partial x^i} g(x^1, \dots, x^m)\Big|_{x=q}}^{\eta(g)} = \overbrace{\sum_{j=1}^m \nu^j \frac{\partial}{\partial \tilde{x}^j} \underbrace{h(\tilde{x}^1, \dots, \tilde{x}^m)\Big|_{\tilde{x}=s}}_{\overset{\text{by } (*)}{=}\; g(x^1, \dots, x^m)}}^{\nu(h)}$$

$$= \sum_{\substack{j=1 \\ k=1}}^m \nu^j \frac{\partial x^k}{\partial \tilde{x}^j}\Big|_{\tilde{x}=s} \frac{\partial}{\partial x^k} g(x^1, \dots, x^m)\Big|_{x=q}$$

Must be true for all g!

$$\Rightarrow \quad \sum_{i=1}^m \eta^i \frac{\partial}{\partial x^i} = \sum_{\substack{j=1 \\ i=1}}^m \nu^j \frac{\partial x^i}{\partial \tilde{x}^j}\Big|_{\tilde{x}=s} \frac{\partial}{\partial x^i}$$

The $\{\frac{\partial}{\partial x^i}\}$ are linearly independent.

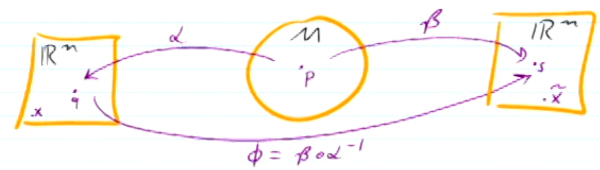
Jacobian matrix $D\phi^{-1}$ of ϕ^{-1} at s.

$$\Rightarrow \quad \eta^i = \sum_{j=1}^m \frac{\partial x^i}{\partial \tilde{x}^j}\Big|_{\tilde{x}=s} \nu^j$$

Jacobian matrix $D\phi$ of ϕ at q.

$$\Rightarrow \text{ conversely: } \quad \nu^i = \sum_{j=1}^m \frac{\partial \tilde{x}^i}{\partial x^j}\Big|_{x=q} \eta^j$$

Summary:



Given $\xi \in T_p(M)$, its images in charts α, β, namely $\eta = \sum_{i=1}^m \eta^i \frac{\partial}{\partial x^i}$ and $\nu = \sum_{i=1}^m \nu^i \frac{\partial}{\partial \tilde{x}^i}$, are related by

Jacobian matrix $D\phi$

$$\nu^i = \underbrace{\sum_{j=1}^m \frac{\partial \tilde{x}^i}{\partial x^j}\Big|_{x=q} \eta^j}_{} = \sum_{j=1}^m \frac{\partial \phi^i(x^1, \dots, x^m)}{\partial x^j}\Big|_{x=q} \eta^j$$

This transformation property can also be used as the starting point for a definition of tangent vectors!

with: $\tilde{x}^i = \phi^i(x^1, \dots, x^m)$

The $\{\frac{\partial}{\partial x^i}\}$ are linearly independent.

$$\Rightarrow \quad \eta^i = \sum_{j=1}^{m} \frac{\partial x^i}{\partial \tilde{x}^j}\Big|_{\tilde{x}=s} \nu^j$$

Jacobian matrix $D\tilde\phi$ of $\tilde\phi$ at s.

$$\Rightarrow \text{ conversely: } \quad \nu^i = \sum_{j=1}^{m} \frac{\partial \tilde{x}^i}{\partial x^j}\Big|_{x=q} \eta^j$$

Jacobian matrix $D\phi$ of ϕ at q.

Given $\xi \in T_p(M)$, its images in charts α, β, namely $\eta = \sum_{i=1}^{m} \eta^i \frac{\partial}{\partial x^i}$ and $\nu = \sum_{i=1}^{m} \nu^i \frac{\partial}{\partial \tilde{x}^i}$, are related by

Jacobian matrix $D\phi$

$$\nu^i = \sum_{j=1}^{m} \frac{\partial \tilde{x}^i}{\partial x^j}\Big|_{x=q} \eta^j = \sum_{j=1}^{m} \frac{\partial \phi^i(x^1,\ldots,x^m)}{\partial x^j}\Big|_{x=q} \eta^j$$

with: $\tilde{x}^i = \phi^i(x^1,\ldots,x^m)$

This transformation property can also be used as the starting point for a definition of tangent vectors!

The "physicist's definition of $T_p(M)$"

Def: A tangent vector $\xi^{(phys)} \in T_p^{(phys)}(M)$ is a map that assigns to each (germ of a) chart a coefficient vector $\in \mathbb{R}^m$, so that if

- $(\eta^1,\ldots,\eta^m)$ is coefficient vector w. resp. to chart α
- $(\nu^1,\ldots,\nu^m)$ is coefficient vector w. resp. to chart β

then:

$$\nu^i = \sum_{j=1}^{m} \frac{\partial \tilde{x}^i}{\partial x^j}\Big|_{x=\alpha(p)} \eta^j \qquad \text{with } \tilde{x}^i = \phi^i(x), \quad \phi = \beta \circ \alpha^{-1}$$

So far, 2 equiv. defs. of $T_p(M)$:

In a chart, α, a tangent vector, $\xi \in T_p(M)$ is:

- algebraically: $\sum_{i=1}^{m} \eta^i \frac{\partial}{\partial x^i}\Big|_{x=\alpha(p)}$

  i.e. it is a directional derivative

  Defining property: Leibniz rule.

- physically: $(\eta^1,\ldots,\eta^m)$

  i.e. it is just the direction vector,

  Defining property: chart change transformation rule

Finally:

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coefficient vector $\in \mathbb{R}^m$, so that if

- $(\gamma^1,\dots,\gamma^m)$ is coefficient vector w. resp. to chart α
- $(\nu^1,\dots,\nu^m)$ is coefficient vector w. resp. to chart β

then:
$$\nu^i = \sum_{j=1}^{m} \frac{\partial \tilde{x}^i}{\partial x^j}\bigg|_{x=\alpha(p)} \gamma^j \qquad \text{with } \tilde{x}^i = \phi^i(x), \quad \phi = \beta \circ \alpha^{-1}$$

i.e. it is a directional derivative

$\boxed{\text{Defining property: Leibniz rule.}}$

- physically: $(\gamma^1,\dots,\gamma^m)$

i.e. it is just the direction vector,

$\boxed{\text{Defining property: chart change transformation rule}}$

Finally:

The "geometric definition of $T_p(M)$":

Idea: Tangent vectors as tangents to paths.

Consider paths in M that pass through p:
$$\gamma : \mathbb{R} \to M$$
$$\gamma(0) = p$$

Note: For any $f: M \to \mathbb{R}$, we obtain:
$$f \circ \gamma : \mathbb{R} \to \mathbb{R}$$

Define: Two diff'able paths, γ_a, γ_b are called equivalent, if for all $f \in \mathcal{F}_p(M)$:
$$\frac{d}{dt}(f \circ \gamma_a)\Big|_{t=0} = \frac{d}{dt}(f \circ \gamma_b)\Big|_{t=0} \qquad (\otimes)$$

Intuition: Two paths γ_a, γ_b are equivalent if they have the same 'velocity' at p.

↑ Note: this includes speed and direction because $\otimes$ must hold for all $f \in \mathcal{F}_p(M)$.

Definition: $T_p(M)^{(geom)}$ is the set of equivalence classes of diff'able paths through p.

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Consider paths in M that pass through p:

$$\gamma : \mathbb{R} \to M$$
$$\gamma(0) = p$$

Note: For any $f : M \to \mathbb{R}$, we obtain:

$$f \circ \gamma : \mathbb{R} \to \mathbb{R}$$

we'll usually mean $T_p^{(alg)}(M)$ when we write $T_p(M)$.

Are $T_p(M)^{(geom)}$ and $T_p(M)^{(alg)}$ equivalent?

Yes!

really: each equivalence class of diff'able paths through p

- Each path γ defines a linear map $\bar{\gamma}$:

$$\bar{\gamma} : \mathcal{F}(p) \to \mathbb{R}$$
$$\bar{\gamma} : f \to \frac{d}{dt}(f \circ \gamma)\Big|_{t=0}$$

- These $\bar{\gamma}$ obey the Leibniz rule:

$$\bar{\gamma}(fg) = \frac{d}{dt}(f \cdot g)(\gamma(t))\Big|_{t=0} = \frac{d}{dt}\big(f(\gamma(t))\, g(\gamma(t))\big)\Big|_{t=0}$$
$$= \frac{d}{dt} f(\gamma(t))\Big|_{t=0} g(\underbrace{\gamma(0)}_{=p}) + f(\gamma(0)) \frac{d}{dt} g(\underbrace{\gamma(t)}_{=p})\Big|_{t=0}$$
$$= \bar{\gamma}(f)\, g + f\, \bar{\gamma}(g) \checkmark$$

- $\Rightarrow \bar{\gamma}$ is an element of $T_p(M)^{(alg)}$

$$\frac{d}{dt}(f \circ \gamma_a)\Big|_{t=0} \quad \frac{d}{dt}(f \circ \gamma_b)\Big|_{t=0}$$

Intuition: Two paths γ_a, γ_b are equivalent if they have the same 'velocity' at p:

↑ Note: this includes speed and direction because ⊗ must hold for all $f \in \mathcal{F}_p(M)$.

Definition: $T_p(M)^{(geom)}$ is the set of equivalence classes of diff'able paths through p.

The "Cotangent Space" $T_p(M)^*$:

Recall:

Given an n-dimensional vector space V, the set of linear maps $w : V \to \mathbb{R}$ forms also an n-dim. vector space. It is called the "dual space", and denoted V^*.

Definition:

The dual vector space to $T_p(M)$ is called the Cotangent Space, and denoted $T_p(M)^*$.

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$$\bar{\gamma} : f \to \frac{d}{dt}(f \circ \gamma)\Big|_{t=0}$$

- These $\bar{\gamma}$ obey the Leibnitz rule:

$$\bar{\gamma}(fg) = \frac{d}{dt}(f \cdot g)(\gamma(t))\Big|_{t=0} = \frac{d}{dt}\left(f(\gamma(t))\, g(\gamma(t))\right)\Big|_{t=0}$$

$$= \frac{d}{dt} f(\gamma(t))\Big|_{t=0}\, g(\underbrace{\gamma(0)}_{=p}) + f(\gamma(0)) \frac{d}{dt} g(\underbrace{\gamma(t)}_{=p})\Big|_{t=0}$$

$$= \bar{\gamma}(f)\, g + f\, \bar{\gamma}(g) \checkmark$$

- $\Rightarrow \bar{\gamma}$ is an element of $T_p(M)$ (alg)

We notice:

For every (germ of a) function at p, $f \in \mathcal{F}(p)$

one naturally obtains an element "df" $\in T_p(M)^*$

called the "differential of f."

Namely:

$df : T_p(M) \to \mathbb{R}$ is the linear map:

$$df : \xi \to \xi(f)$$

(Note: thus, we can view "d" as a map: $d : \mathcal{F}_p(M) \to T_p(M)^*$. See later...)

also an n-dim. vector space. It is called the "dual space", and denoted V^*.

Definition:

The dual vector space to $T_p(M)$ is called the Cotangent Space, and denoted $T_p(M)^*$.

Concretely: in a cds., i.e., in a chart, the abstract $\xi \in T_p(M)$ and $f \in \mathcal{F}(p)$ correspond to some $\eta \in T_q(\mathbb{R}^n)$ and $g \in \mathcal{F}(q)$.

Then: $\underset{\in\, T_p(M)^*}{dg} : T_q(\mathbb{R}^n) \to \mathbb{R}$

$$dg : \eta \to \eta(g) = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\Big|_{x=q} g(x^1, \ldots, x^n)$$

Recall: Since all $\eta \in T_q(\mathbb{R}^n)$ take the form $\eta = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\Big|_{x=q}$

a basis of $T_q(\mathbb{R}^n)$ is $\left\{ \frac{\partial}{\partial x^i}\Big|_{x=q} \right\}_{i=1}^{n}$

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"df" $\in T_p(M)^*$

called the "differential of f."

Namely:

$df: T_p(M) \to \mathbb{R}$ is the linear map:

$df: \xi \to \xi(f)$

(Note: thus, we can view "d" as a map. $d: \mathcal{F}(M) \to T_p(M)^*$. See later...)

Then: $dg: T_q(\mathbb{R}^n) \to \mathbb{R}$

$$dg: \quad \eta \to \eta(g) = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\Big|_{x=q} g(x^1,..,x^n)$$

Recall: Since all $\eta \in T_q(\mathbb{R}^n)$ take the form $\eta = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\Big|_{x=q}$

a basis of $T_q(\mathbb{R}^n)$ is $\left\{ \frac{\partial}{\partial x^i}\Big|_{x=q} \right\}_{i=1}^{n}$

Question: What is the dual basis in $T_q(\mathbb{R}^n)^*$?

- Consider the coordinate functions: $x^k: \mathbb{R}^n \to \mathbb{R}$.
- Their differentials $dx^k \in T_q(\mathbb{R}^n)^*$ obey:

$$dx^k: T_q(\mathbb{R}^n) \to \mathbb{R}$$

$$dx^k: \frac{\partial}{\partial x^i}\Big|_{x=q} \to \frac{\partial}{\partial x^i} x^k\Big|_{x=q} = \delta_i^k$$

$\Rightarrow$ The dual basis in $T_q(\mathbb{R}^n)^*$ is given by

$$\{dx^k\}_{k=1}^{n}$$

Thus:

Every element $\omega \in T_q(\mathbb{R}^n)^*$ takes the form:

$$\omega = \sum_{i=1}^{n} \underbrace{\omega_i}_{\in \mathbb{R}} dx^i$$

and its action is:

$$\omega: T_q(\mathbb{R}^n) \to \mathbb{R}$$

$$\omega: \sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j} \to \sum_{i=1}^{n} \omega_i\, dx^i\left(\sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j}\right)$$

$$= \underbrace{\sum_{i=1}^{n} \omega_i \sum_{j=1}^{n} \eta^j \underbrace{\frac{\partial}{\partial x^j} x^i}_{=\delta_j^i}}_{=\sum_{i=1}^{n} \omega_i \eta^i}$$

$$\Rightarrow \omega\left(\sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j}\right) = \sum_{i=1}^{n} \omega_i \eta^i \qquad (\mathrm{I})$$

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$$dx^k : T_q(\mathbb{R}^n) \to \mathbb{R}$$

$$dx^k : \left.\frac{\partial}{\partial x^i}\right|_{x=q} \to \left.\frac{\partial}{\partial x^i} x^k\right|_{x=q} = \delta_i^k$$

$\Rightarrow$ The dual basis in $T_q(\mathbb{R}^n)^*$ is given by

$$\{dx^k\}_{k=1}^n$$

In particular: For arbitrary $g \in \mathcal{F}(q)$, its differential $dg \in T_q(\mathbb{R}^n)^*$ must be of the form:

$$dg = \sum_{k=1}^n w_k \, dx^k \text{ with suitable } w_k \in \mathbb{R}.$$

How to calculate them?

We know:

$$dg(\eta) = \eta(g) = \sum_{i=1}^n \eta^i \underbrace{\left.\frac{\partial}{\partial x^i} g(x)\right|_{x=q}}_{= w_i} \quad (\mathrm{II})$$

Compare I, II $\Rightarrow$ $w_i = \left.\frac{\partial}{\partial x^i} g(x)\right|_{x=q}$

$$\Rightarrow \boxed{dg = \sum_{i=1}^n \left(\left.\frac{\partial}{\partial x^i} g(x)\right|_{x=q}\right) dx^i}$$

and its action is:

$$w : T_q(\mathbb{R}^n) \to \mathbb{R}$$

$$w : \sum_{j=1}^n \eta^j \frac{\partial}{\partial x^j} \to \sum_{i=1}^n w_i \, dx^i\left(\sum_{j=1}^n \eta^j \frac{\partial}{\partial x^j}\right)$$

$$= \underbrace{\sum_{i=1}^n w_i \sum_{j=1}^n \eta^j \underbrace{\frac{\partial}{\partial x^j} x^i}_{=\delta_j^i}}_{= \sum_{i=1}^n w_i \eta^i}$$

$$\Rightarrow w\left(\sum_{j=1}^n \eta^j \frac{\partial}{\partial x^j}\right) = \sum_{i=1}^n w_i \eta^i \quad (\mathrm{I})$$

Exercise: (the "pull back" map)

Assume that $\rho \in T_p(M)^*$, under two charts α, β, as above, corresponds to $w \in T_q(\mathbb{R}^n)^*$ and $\mu \in T_s(\mathbb{R}^n)^*$ with:

$$w = \sum_{i=1}^n w_i \, dx^i \quad \text{and} \quad \mu = \sum_{i=1}^n \mu_i \, d\tilde{x}^i$$

Show that $\mu_i = \sum_{j=1}^n \left.\frac{\partial x^j}{\partial \tilde{x}^i}\right|_{\tilde{x}=s} w_j$

Notice that this is the inverse of the Jacobian matrix of $\beta \circ \alpha^{-1}$ at q

Remark: The physicist's definition of $T_p(M)^*$ uses this.

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Some notation and terminology:

- Elements of $T_p(\mathcal{M})$ are called contravariant vectors
- Elements of $T_p(\mathcal{M})^*$ are called covariant vectors
- One often writes symbolically

$$\xi = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i}\Big|_p \quad \text{for } \xi \in T_p(\mathcal{M})$$

$$\omega = \sum_{i=1}^{n} \omega_i \, dx^i \quad \text{for } \omega \in T_p(\mathcal{M})^*$$

even without specifying a particular chart.

We are now ready to define tensors:

Def: A tensor, t, of rank (r,s) is an element of

$$T_p(\mathcal{M})^r_s := \underbrace{T_p(\mathcal{M})\otimes \ldots \otimes T_p(\mathcal{M})}_{r \text{ factors}} \otimes \underbrace{T_p(\mathcal{M})^* \otimes \ldots \otimes T_p(\mathcal{M})^*}_{s \text{ factors}}$$

In a chart:

$$t = \sum_{\substack{i_1 \ldots i_r = 1 \\ j_1 \ldots j_s}}^{n} \underset{\substack{\cap \\ \mathbb{R}}}{t^{i_1 \ldots i_r}_{j_1 \ldots j_s}} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}$$

Under chart change: (physicists, incl. Einstein, defined tensors this way)

$$\tilde{t}^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \sum_{\substack{k_1 \ldots k_r \\ j_1 \ldots j_s = 1}}^{n} \frac{\partial \tilde{x}^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial \tilde{x}^{i_r}}{\partial x^{k_r}} \frac{\partial x^{l_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{l_s}}{\partial \tilde{x}^{j_s}} \, t^{k_1 \ldots k_r}_{l_1 \ldots l_s}$$

Thus: $T_p(\mathcal{M}) = T_p(\mathcal{M})^1$ and $T_p(\mathcal{M})^* = T_p(\mathcal{M})_1$, i.e.:

- a tangent vector is a tensor of rank $(1,0)$
- a cotangent vector is a tensor of rank $(0,1)$

Finally: From local to global!

Def: We call $T(\mathcal{M}) := \bigcup_{p \in \mathcal{M}} (p, T_p(\mathcal{M}))$, the **Tangent bundle**.

a "base point" (p); a "fibre" ($T_p(\mathcal{M})$)

Note: $T(\mathcal{M})$ is itself a manifold. It is $2n$-dimensional.

Def: $T(\mathcal{M})$ is then also called the "Total Space".

Remark: One obtains other Fibre bundles by choosing other standard fibers.

E.g.:
- Co-tangent bundle $T^*(\mathcal{M})$
- (r,s)-tensor bundle $T^r_s(\mathcal{M})$
- Bundles for isospinors (vector bundles) and gauge groups (principle bundles)

Def: The map $\pi: T(\mathcal{M}) \to \mathcal{M}$

$\pi: (p, T_p(\mathcal{M})) \to p$ (i.e.: $\pi^{-1}(p) = T_p(\mathcal{M})$)

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- One often writes symbolically

$$\xi = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i}\Big|_p \quad \text{for } \xi \in T_p(M)$$

$$\omega = \sum_{i=1}^{n} \omega_i \, dx^i \quad \text{for } \omega \in T_p(M)^*$$

even without specifying a particular chart.

We are now ready to define tensors:

$\mathbb{R}$

Under chart change: (physicists, incl. Einstein, defined tensors this way)

$$\tilde{t}^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \sum \frac{\partial \tilde{x}^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial \tilde{x}^{i_r}}{\partial x^{k_r}} \frac{\partial x^{l_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{l_s}}{\partial \tilde{x}^{j_s}} \, t^{k_1 \ldots k_r}_{l_1 \ldots l_s}$$

Thus: $T_p(M) = T_p(M)'$ and $T_p(M)^* = T_p(M)$, i.e.:

- a tangent vector is a tensor of rank $(1,0)$
- a cotangent vector is a tensor of rank $(0,1)$

Finally: From local to global!

Def: We call $T(M) := \bigcup_{p \in M} (p, T_p(M))$, the **Tangent bundle**.

(a "base point" — p; a "fibre" — $T_p(M)$)

Note: $T(M)$ is itself a manifold. It is $2n$-dimensional.

Def: $T(M)$ is then also called the "Total Space".

Def: M is also called the "Base Space".

Recall that all $T_p(M)$ are n-dimensional real vector spaces, i.e., are isomorphic to $\mathbb{R}^n$.

Def: We therefore call $\mathbb{R}^n$ the "Standard Fibre".

Remark: One obtains other Fibre bundles by choosing other standard fibers.

E.g.:
- Co-tangent bundle $T^*(M)$
- (r,s)-tensor bundle $T^r_s(M)$
- Bundles for isospinors (vector bundles) and gauge groups (principle bundles)

Def: The map $\pi: T(M) \to M$

$$\pi : (p, T_p(M)) \to p \qquad (\text{i.e.: } \pi^{-1}(p) = T_p(M))$$

is called the "Bundle Projection".

Def: A Section, σ, is a map, $\sigma: M \to T(M)$, which is a continuous right inverse of π:

$$\pi(\sigma(x)) = x \quad \forall x \in M \qquad (\text{i.e.: } \pi \circ \sigma = id)$$

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Notice: The graph of a "field" is a section of its fibre bundle.

Recall: The graph of a function $f: A \to B$ is:
$\{(a, f(a))\}_{a \in A}$

Def: ◻ A tangent vector field is a map $\xi: \underset{\in M}{p} \to \underset{\in T_p(M)}{\xi_p}$

In a chart: $\xi = \sum_{i=1}^{n} \xi^i(x) \frac{\partial}{\partial x^i}$

◻ A cotangent vector field is a map $\omega: \underset{\in M}{p} \to \omega_p \in T_p^*(M)$

In a chart: $\omega = \sum_{i=1}^{n} \omega_i(x)\, dx^i$

◻ Similarly, tensor fields: $t: \underset{\in M}{p} \to \underset{\in T_p(M)_s^r}{t_p}$

In a chart: $t = \sum t_{j_1 \ldots j_s}^{i_1 \ldots i_r}(x) \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_r}} dx^{j_1} \cdots dx^{j_s}$

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So far, not a concern with GR, but it does come up with gauge theories.

Why then fibre bundles? To capture global nontriviality.

Fibre bundles are required to be locally trivial:
M can be covered with neighborhoods U_r, so that

means: there exists a differentiable isomorphism

or other standard fibre for other fibre bundles.

$$\pi^{-1}(U_r) \simeq U_r \times \mathbb{R}^n$$

But fibre bundles are allowed to be globally nontrivial:

For a suitable vector bundle B, we can have
$\pi^{-1}(U_1) \simeq U_1 \times \mathbb{R}^n$
$\pi^{-1}(U_2) \simeq U_2 \times \mathbb{R}^n$
but in the overlap regions, the two isomorphisms may differ $\Rightarrow B \neq M \times \mathbb{R}^n$

(The isomorphisms may differ by elements of $Gl_n(\mathbb{R})$, the "structure group" here)

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