Title: General Relativity for Cosmology Lecture - 090723

Speakers: Achim Kempf

Collection: General Relativity for Cosmology

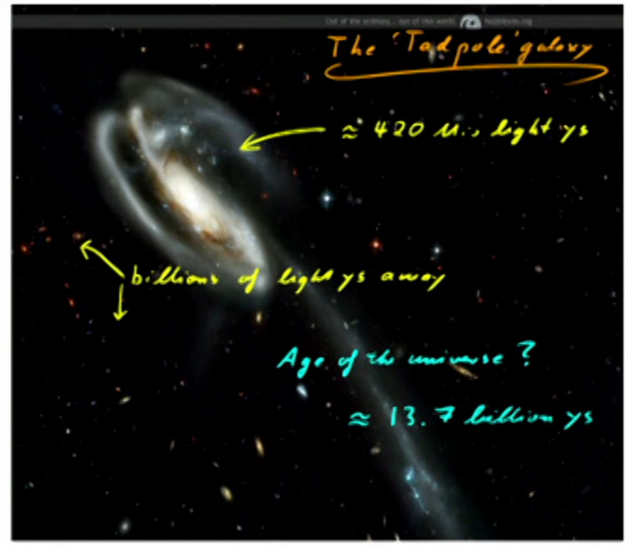
Date: September 07, 2023 - 2:00 PM

URL: https://pirsa.org/23090000

Abstract: Zoom: https://pitp.zoom.us/j/91640855624?pwd=dWVWV2doSnBhUS9JUkhjQVBwY0h0dz09

in F2023

- **Term:** Fall 2023
- **Course codes:** AMATH875/PHYS786
- **Instructor:** Achim Kempf
- **Instructor Office hours:** Wednesdays 12:30-1:30pm in MC6334
- **TA:** Sadat Khajouei (d2sadathosseinikhajouei at uwaterloo.ca)
- **Prerequisites:** A first course in General Relativity or consent of instructor
- **The Lectures**



Regular Time & Venue: Tue + Thu 2-3:20pm, in the Alice Room at Perimeter Institute
Exceptions:
* The class of Thu. Sep 21 will be in the Bob room.
* The class of Tue. Oct. 17 will be in the Space room.

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Contact us

Annoucements

Health precautions (permanent):

- *If you have cold/flu/covid symptoms, don't come to class. Do the right thing, which is taking care of yourself and getting healthy first.*
    - For videos of the lectures scroll down to the bottom of this page.
    - All important updates will always be posted here.

Content

This is an advanced graduate course which develops the math and physics of general relativity from scratch up to the highest level. The going will sometimes be steep but I try to be always careful. The purpose is to prepare for studies in quantum gravity, relativistic quantum information, black hole physics and cosmology. Quick summary of the contents:

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Content

This is an advanced graduate course which develops the math and physics of general relativity from scratch up to the highest level. The going will sometimes be steep but I try to be always careful. The purpose is to prepare for studies in quantum gravity, relativistic quantum information, black hole physics and cosmology. Quick summary of the contents:

- Coordinate-free Differential Geometry, Weyl versus Ricci curvature versus Torsion, Vielbein Formalism, Spin-connections, Form-valued Tensors, Spectral Geometry, some Cohomology.
- Derivations of General Relativity including as a Gauge Theory, Diffeomorphism Invariance vs. Symmetries, Bianchi Identities vs. Local and Global Conservation Laws.
- Penrose Diagrams for Black Holes and Cosmology, Types of Horizons, Energy Conditions and Singularity theorems, Properties and Classification of Exact Solutions.
- Cosmology and Models of Cosmic Inflation

In W24, I am planning to teach the follow-up course Quantum Field Theory for Cosmology AMATH872/PHYS785, which was last taught in W22.

Grades

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Grades

- The grades are currently planned to be based on an essay and a presentation.
  - Essay/presentation topics: TBA

The Lectures: Links to the Videos and Lecture Notes

Here are Video Recordings of the Lectures of a pre-pandemic teaching of this course: PIRSA

Here are the Lecture Notes:

Lecture 1: Overview. Differentiable manifolds.

Lecture 2: Algebra of function germs. Algebraic definition of tangent space.

Lecture 3: Physical and geometric definitions of tangent space. Tangent bundle and its sections.

Lecture 4: Algebra of differential forms. (Anti-) derivations. Exterior derivative. Cohomology. Categories.

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Lecture 3: Physical and geometric definitions of tangent space. Tangent bundle and its sections.

Lecture 4: Algebra of differential forms. (Anti-) derivations. Exterior derivative. Cohomology. Categories.

Lecture 5: Inner derivation. Lie derivative on forms and on general tensors. Infinitesimal diffeomorphisms.

Lecture 6: Orientability and volume. Divergence. Integration. Stokes', Green's and Gauss' theorems.

Lecture 7: Hodge *. Hilbert Space Λ. Co-derivative. d'Alembertian. Klein Gordon and Maxwell equations.

Lecture 8: Affine connection. Absolute covariant derivative. Autoparallels and parallel transport.

Lecture 9: Torsion. Geodesics. Riemann normal coordinates. Curvature. Ricci identity. Bianchi identities.

Lecture 10: Levi-Civita connection. D. Moving frames. Tensor-valued p-forms. Cartan Structure equations.

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Essay

Deadline for submitting the essay in one PDF file not exceeding 10 pages, is 10:00am on 15 December 2021.

Essay topics: TBA.

Advice on how to prepare the Essay:

- Format: title and abstract page/motivation/main parts/summary (or conclusions)/bibliography.
- Bibliography: Again, list all of your sources explicitly. Of course you can use Wikipedia but you should not cite it - because it can change from day to day and because as it not (yet) reliable enough to meet scientific standards. Instead, cite books and papers that you may have found via Wikipedia. Also, it is good style to list items in the bibliography in that sequence in which they are first referred to in the text.
- At most about 10 pages.
- An essay should be a review of existing literature on a given topic. The sources can be textbooks, lecture notes or review articles or original articles or some of each. All and everything that is used

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Literature

We will be using mainly material from the following three texts:

- N. Straumann, General Relativity with Applications to Astrophysics, Springer (2004)
- J. Stewart, Advanced General Relativity, Cambridge (1991)
- S. Hawking, G.F.R. Ellis, The Large Scale Structure of Space-Time, Cambridge (1973)

Note: These three texts are available at the Davis Library.

Recommended general references are also:

- Scott Dodelson, Modern Cosmology, Academic Press, San Diego, (2003)
- A.R. Liddle, D.H. Lyth, Cosmological Inflation and Large-Scale Structure, CUP (2000)
- G.F.R. Ellis and J. Wainwright, Dynamical Systems in Cosmology, CUP (1997)
- R. M. Wald, General Relativity, University of Chicago Press (1984)
- H. Stephani, General Relativity, Cambridge University Press (CUP) (1982)

We will cover Sakharov's "induced gravity" argument. Read the original (very short) paper here:

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- H. Stephani, General Relativity, Cambridge University Press (CUP) (1982)

We will cover Sakharov's "induced gravity" argument. Read the original (very short) paper here:

- Vacuum quantum fluctuations in curved space and the theory of gravitation (PDF)
- Sakharov's induced gravity: a modern perspective (PDF)

Here is a link to very nice lecture notes on Real Analysis (PMATH351) (including an introduction to topology) by my colleague Laurent Marcoux: here

Here are links to general online reviews:

- INSPIRE- library's list of Online Reviews
- Living Reviews in Relativity

You may also wish to have a look at these free only resources: A collection of links

And of course, do work with ChatGPT or GPT4 or similar tools.

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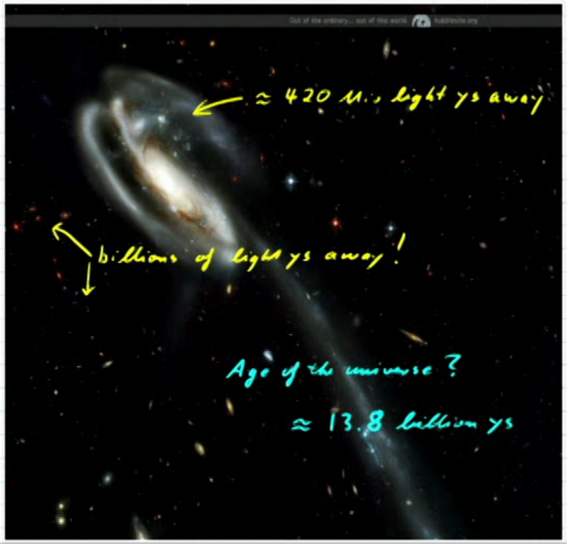
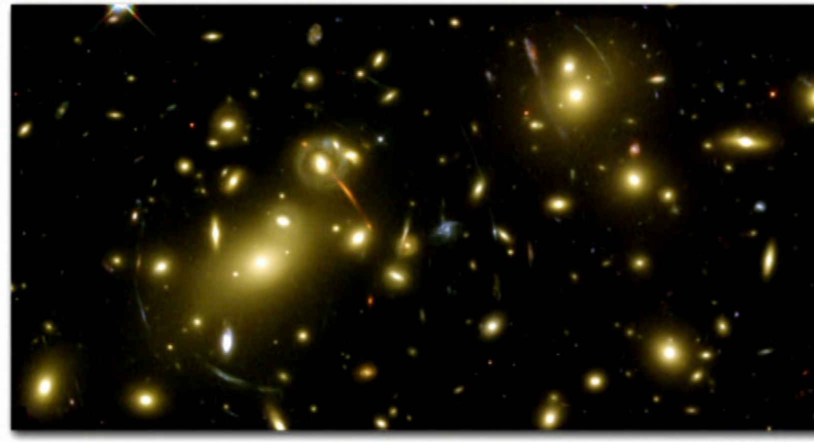
The 'Tadpole' galaxy



Spacetime's curvature can be seen directly:



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How to describe spacetime?

A. Math

Strategy:
- Start with a mere "set" of points (events), M. Then add structure:
- Define open neighborhoods (i.e., a "topology" on M)
- Define "separability" of points (i.e. Hausdorff condition)
- Define "continuity" (preimage of open sets is open)

Other descriptions of curvature? (Why consider others? May be useful for quantum gravity b/c what's on previous page is likely over idealized.)

- Curvature = sum of angles in triangle $\neq \pi$
- Curvature = nontriviality of Pythagoras law
- Curvature = tidal forces. Math of it: Sectional curvatures
- Curvature $\stackrel{?}{=}$ nontrivial sound of object when vibrating

This field is called Spectral Geometry.

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A. Math

Strategy: □ Start with a mere "set" of points (events), M

Then add structure:

- □ Define open neighborhoods (i.e., a "topology" on M)
- □ Define "separability" of points (i.e. Hausdorf condition)
- □ Define "continuity" (preimage of open sets is open)
- □ Define "differentiability" (via chart change diffability)

later: □ Define tangent & tensor spaces

⋮

$\boxed{\text{Curvature = nontriviality of parallel transport}}$

Other descriptions of curvature?

(for quantum gravity b/c what's on previous page is likely over idealized.)

- □ Curvature = sum of angles in triangle $\neq \pi$
- □ Curvature = nontriviality of Pythagoras law
- □ Curvature = tidal forces. Math of it: Sectional curvatures
- □ Curvature $\stackrel{?}{=}$ nontrivial sound of object when vibrating

  This field is called Spectral Geometry.

  Interesting b/c connects mathematical languages of quantum theory (spectra etc) and general relativity.
- □ Curvature $\stackrel{?}{=}$ nontrivial entanglement in vacuum fluctuations

B) Structure and properties of General Relativity?

- □ Equations of motion

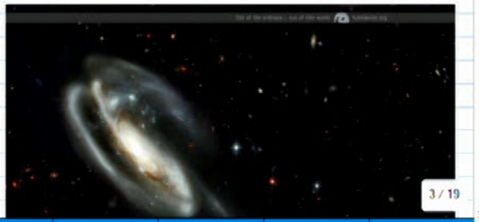
  for scalars, vectors, spinors and curvature
- □ Symmetries

  local and global conservation laws, if any!

C) Applications to Cosmology

- □ Classification of exact solutions
- □ Models of cosmological matter
- □ FRW models, while

  using the tetrad formalism



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B) Structure and properties of General Relativity?

- Equations of motion
  for scalars, vectors, spinors and curvature
- Symmetries
  local and global conservation laws, if any!
- Tetrad formulation, GR as a gauge theory
- Singularities, and their unavoidability

C) Applications to Cosmology

- Classification of exact solutions
- Models of cosmological matter
- FRW models, while
  using the tetrad formalism
  to exercise it. (e.g. for later use in quantum gravity)
- Cosmic inflation
- Black holes

A. Pseudo-Riemannian Differential Geometry

- **Differentiable Manifolds**
  (Riemann ≈ 1850s, Poincaré ≈ 1890s, Whitney ≈ 1930s...)

Def: An n-dimensional topological

Here:

Def: A topological space, M, is a set, together with a specification of subsets U_i, which will be called "open subsets", which must obey $U_i \cap U_j$ is open, and $\bigcup_k U_k$ is open.

Def: A topological space M is called Hausdorff, if it is separat...

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A. Pseudo-Riemannian Differential Geometry

□ Differentiable Manifolds

(Riemann ≈ 1850s, Poincaré ≈ 1890s, Whitney ≈ 1930s...)

Def: An n-dimensional topological Manifold, M, is a Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$.

Example: $\mathbb{R}^n$ with its usual definition of open sets.

Now how is the term "homeomorphic" defined?

For this, we need to define "continuity" first:

Recall: If A, B are topol. spaces, then $f: A \to B$ is called

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Def: A topological space, M, is a set, together with a specification of subsets U_i, which will be called "open subsets", which must obey $U_i \cap U_j$ is open, and $\bigcup_i U_i$ is open.

Def: A topological space M is called Hausdorff, if it is separable, i.e., if $x, y \in M$ and $x \neq y$ then x, y are elements of some disjoint open sets.

$\forall x,y: x \neq y \ \exists\, U_x, U_y$ open $: x \in U_x, y \in U_y$ and $U_x \cap U_y = \{\}$
"for all" "there exist" empty set

Remark: Powerful definition that can be applied very generally.

Why important for us here?

We can now express the idea that a topological Hausdorff space M is continuously parametrizable (as spacetime appears to be)!

Def: Let A, B be topological spaces. Then, a function $f: A \to B$ is called a homeomorphism, if f^{-1} exists and if

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Manifold, M, is a Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$.

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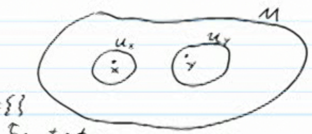

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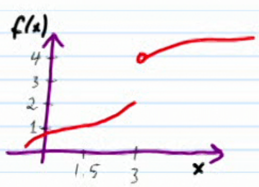
Example: $\mathbb{R}^n$ with its usual definition of open sets.

Now how is the term "homeomorphic" defined?

For this, we need to define "continuity" first:

Recall: If A, B are topol. spaces, then $f: A \to B$ is called continuous if $(\mathcal{U} \subset B$ is open $\Rightarrow \underbrace{f^{-1}(\mathcal{U})}_{= \{x \in A : f(x) \in \mathcal{U}\}} \subset A$ is open$)$

Example:



$f: \mathbb{R} \to \mathbb{R}$

Choose $\mathcal{U} := (1,3)$ open

But $f^{-1}(\mathcal{U}) = (1.5, 3]$ **not** open

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Why important for us here?

We can now express the idea that a topological Hausdorff space M is continuously parametrizable (as spacetime appears to be)!

Def: Let A, B be topological spaces. Then, a function $f: A \to B$ is called a homeomorphism, if f^{-1} exists and if both f and f^{-1} are continuous.

Def: We say that A is locally homeomorphic to B if for all $p \in A$ there exists an open neighborhood U_p of p, $(p \in U_p)$ which is homeomorphic to an open set in B.

We choose $B := \mathbb{R}^n$:

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"for all" "there exist" "empty set"

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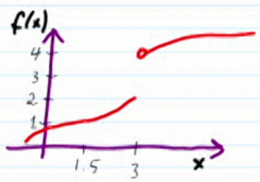

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We choose $B := \mathbb{R}^n$:

Recall:

Def: An n-dimensional topological Manifold, M, is a Hausdorff

Def: A local homeomorphism,

$h: U \to \mathbb{R}^n, \quad U \subset M$

Differentiable Manifolds

(Riemann ≈ 1850s, Poincaré ≈ 1890s, Whitney ≈ 1930s...)

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Recall:

Def: An n-dimensional topological Manifold, M, is a Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$.

Now how is the term

Differentiable Manifold defined?

Def: A local homeomorphism,

$$h: \underset{\text{called "domain"}}{U} \to \mathbb{R}^n, \quad U \subset M$$

is called a chart of M.

For any point $q \in U$ its image

$$h(q) \in \mathbb{R}^n$$

is a set of n numbers $(x_1, x_2, \dots, x_n)$ called the coordinates of q.

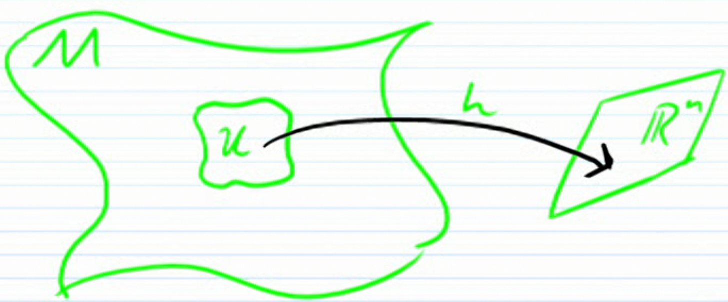
Def: A chart, h, with domain U,

Def: A collection of charts h_α with domains U_α is called an homeomorphic to $\mathbb{R}^n$.

Now how is the term

Differentiable Manifold defined?

Def: A chart, h, with domain $\mathcal{U}$,



is also called a

local coordinate system for $\mathcal{U}$.

For any point $q \in \mathcal{U}$ its image

$h(q) \in \mathbb{R}^n$

is a set of n numbers $(x_1, x_2, \dots, x_n)$ called the coordinates of q.

Def: A collection of charts h_α with domains $\mathcal{U}_\alpha$ is called an atlas if $\bigcup_\alpha \mathcal{U}_\alpha = \mathcal{M}$.

→ What, if we want to change coordinates, i.e. if we want to re-label the points of (e.g. a subset of) the manifold?

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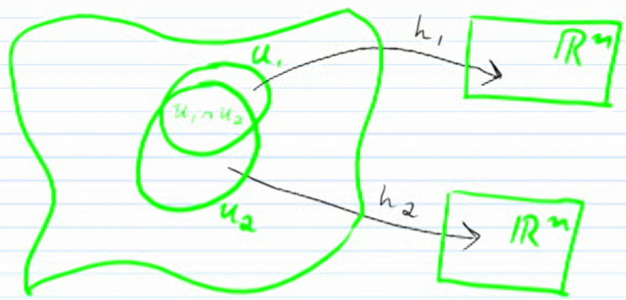
Consider 2 charts h_1, h_2, with intersecting domains $U_1 \cap U_2 \neq \phi$:



Then, $h_{12} = h_2 \circ h_1^{-1}$ is a continuous change of coordinates map $h_{12}: \mathbb{R}^n \to \mathbb{R}^n$.

Notice: For maps $\mathbb{R}^n \to \mathbb{R}^n$ we know what differentiability means!

Strategy: Let us define the diffability of an atlas through the diffability of its chart changes:

Def: An atlas is called C^r differentiable, if all its coordinate changes, $h_{\alpha\beta}$, are C^r diffeomorphisms, i.e., r times continuous differentiable.

Strategy: Enlarge atlas so every point of M is in multiple charts. Then, diffability of M is definable through atlas diffability

Def: Given a C^r differentiable atlas, A, we can generate a maximal

Theorem: (Whitney)

Every C^k structure with $k \geq 1$ is C^k equivalent to a C^∞ structure (i.e. there is always a suitable set of charts).

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Strategy: Enlarge Atlas so every point of $\mathcal{M}$ is in multiple charts. Then, diff'ability of $\mathcal{M}$ is definable through atlas diff'ability.

Def: Given a C^r differentiable atlas, A, we can generate a maximal C^r differentiable atlas, $D(A)$, by adding all charts whose chart changes with charts in A are differentiable.

Def: $D(A)$ is also called a "Differentiable Structure" of class C^r for $\mathcal{M}$.

Def: A differentiable manifold of class C^r is a topol. manifold with a maximal atlas of class C^r, i.e., with a differentiable structure of class C^r.

Theorem: (Whitney)

Every C^k structure with $k \geq 1$ is C^k equivalent to a C^∞ structure (i.e. there is always a suitable set of charts).

I.e. any diff'able structure can be smoothened. Any lack of higher diff'ability is due to unlucky choice of chart.

Def: Since any C^1 manifold is also a C^∞ manifold, we also call diff'able manifolds simply smooth manifolds.