Title: TBA

Speakers: Jamie Sikora

Series: Quantum Foundations

Date: August 17, 2023 - 11:00 AM

URL: https://pirsa.org/23080035

Abstract: Abstract: TBD

Zoom Link: https://pitp.zoom.us/j/94487792881?pwd=TU9CTEZGcFBTZXdxaWFFS25rOVlpZz09

When are quantum states antidistinguishable?

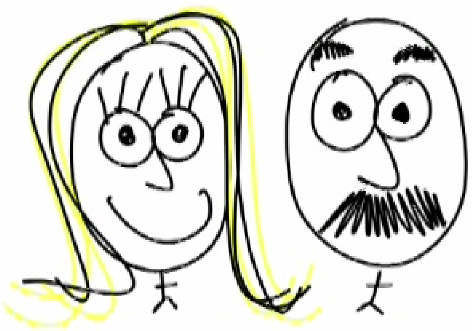
Perimeter
08/17/2023

Jamie Sikora
Virginia Tech

On the arXiv soon. Hopefully.

This talk

- When are states distinguishable?
- When are states antidistinguishable?

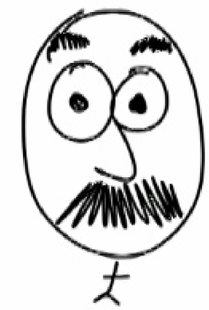


Quantum state discrimination

Alice

$|\psi\rangle$

Bob



Alice prepares

$|\psi\rangle = |0\rangle$

or

$|\psi\rangle = |1\rangle$

Reality check: Alice is sending a particle encoded in state $|\psi\rangle$, she is not sending a vector!

Quantum state discrimination

Alice prepares

$|\psi\rangle = |0\rangle$

or

$|\psi\rangle = |+\rangle$

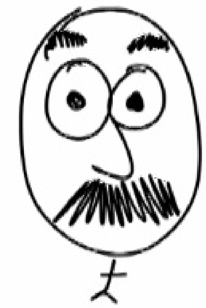
Does Bob have any hope now?

Quantum state discrimination

Alice

$|\psi\rangle$

Bob



Alice prepares

$|\psi\rangle = |0\rangle$

or

$|\psi\rangle = |+\rangle$

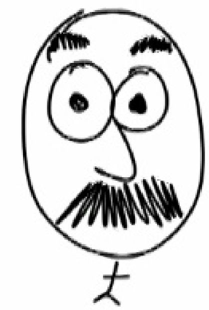
We changed one of the states!

Quantum state discrimination

Alice

$|\psi\rangle$

Bob



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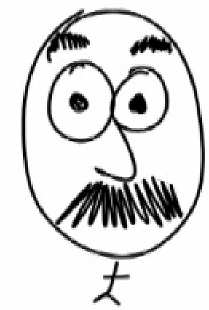


Alice prepares

$|\psi\rangle = |0\rangle$

or

$|\psi\rangle = |+\rangle$

Does Bob have any hope now?

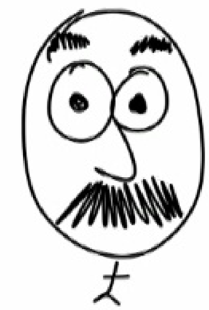
In general, two states are <u>perfectly</u> distinguishable if and only if they are orthogonal to each other.

Quantum state discrimination

Alice

ρ_i

Bob



Alice prepares one of
$\{\rho_1, \ldots, \rho_n\}$
and sends it to Bob

Quantum state discrimination

Alice Bob

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Quantum state discrimination

Alice prepares one of
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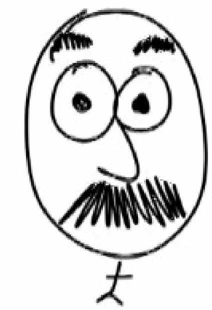
Can Bob guess what "i" is?

Quantum state discrimination

Alice

ρ_i

Bob



Bob can measure with a POVM $\{M_1, \ldots, M_n\}$
and on output j guess the state index is "j"

Alice prepares one of
$\{\rho_1, \ldots, \rho_n\}$
and sends it to Bob

Can Bob guess what "i" is?

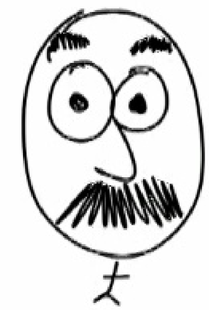
Under what conditions can Bob perfectly guess what "i" is?

Answer: Bob can perfectly guess the state when $\rho_i \rho_j = 0, \forall i \neq j$

Quantum state discrimination

Alice

Bob



ρ_i

Bob can measure with a POVM $\{M_1, \ldots, M_n\}$ and on output j guess the state index is "j"

Alice prepares one of $\{\rho_1, \ldots, \rho_n\}$ and sends it to Bob

Proof: Math

Answer: Bob can perfectly guess the state when $\rho_i \rho_j = 0, \forall i \neq j$

Quantum state discrimination

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If they are orthogonal, then you can measure in a basis that contains their supports. So they are perfectly distinguishable.

Quantum state discrimination

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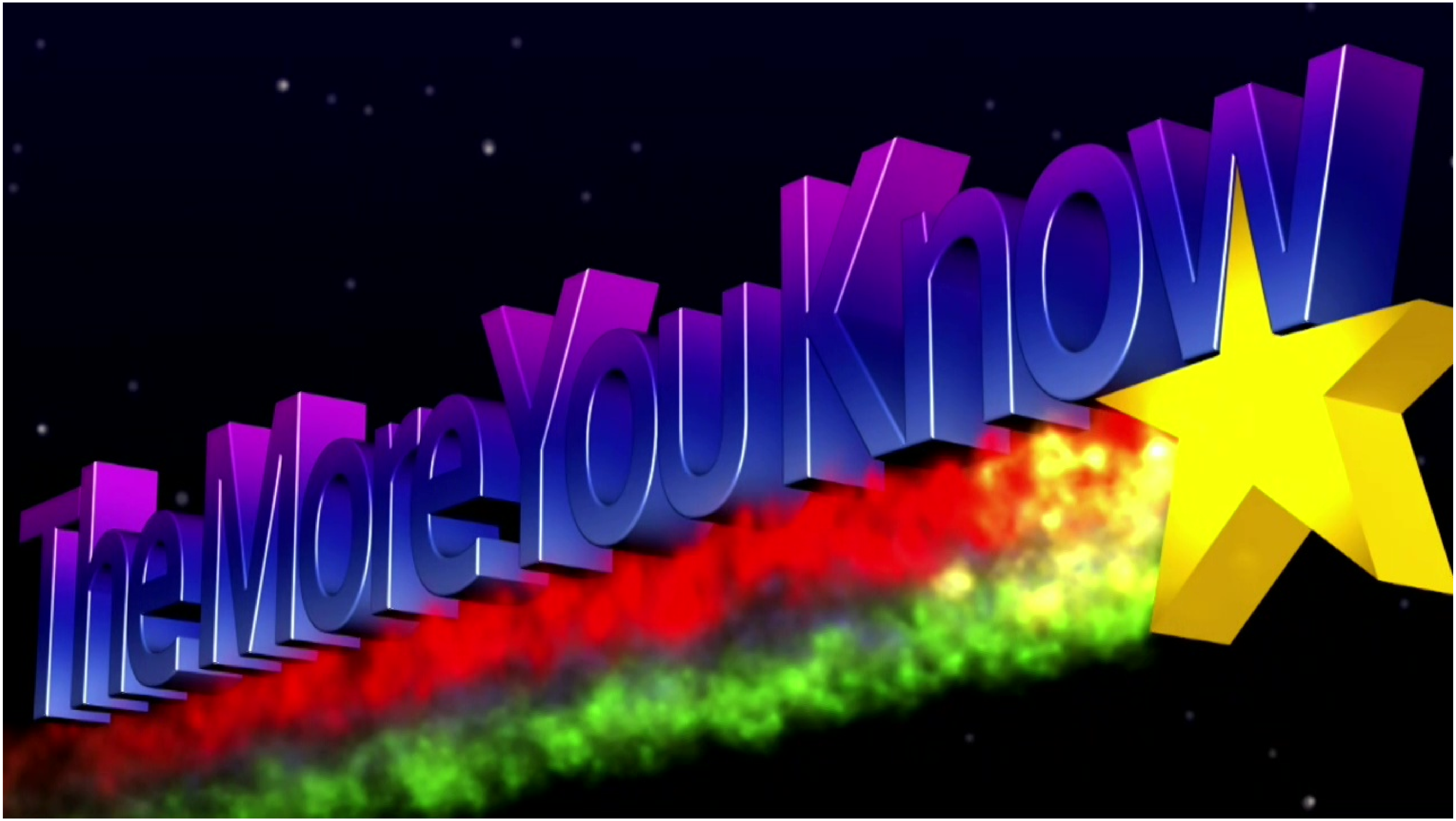
If they are orthogonal, then you can measure in a basis that contains their supports. So they are perfectly distinguishable.

If they are perfectly distinguishable, then there exists a measurement $\{M_1, \ldots, M_n\}$ such that $\langle M_j, \rho_i \rangle = 0$ for all $i \neq j$

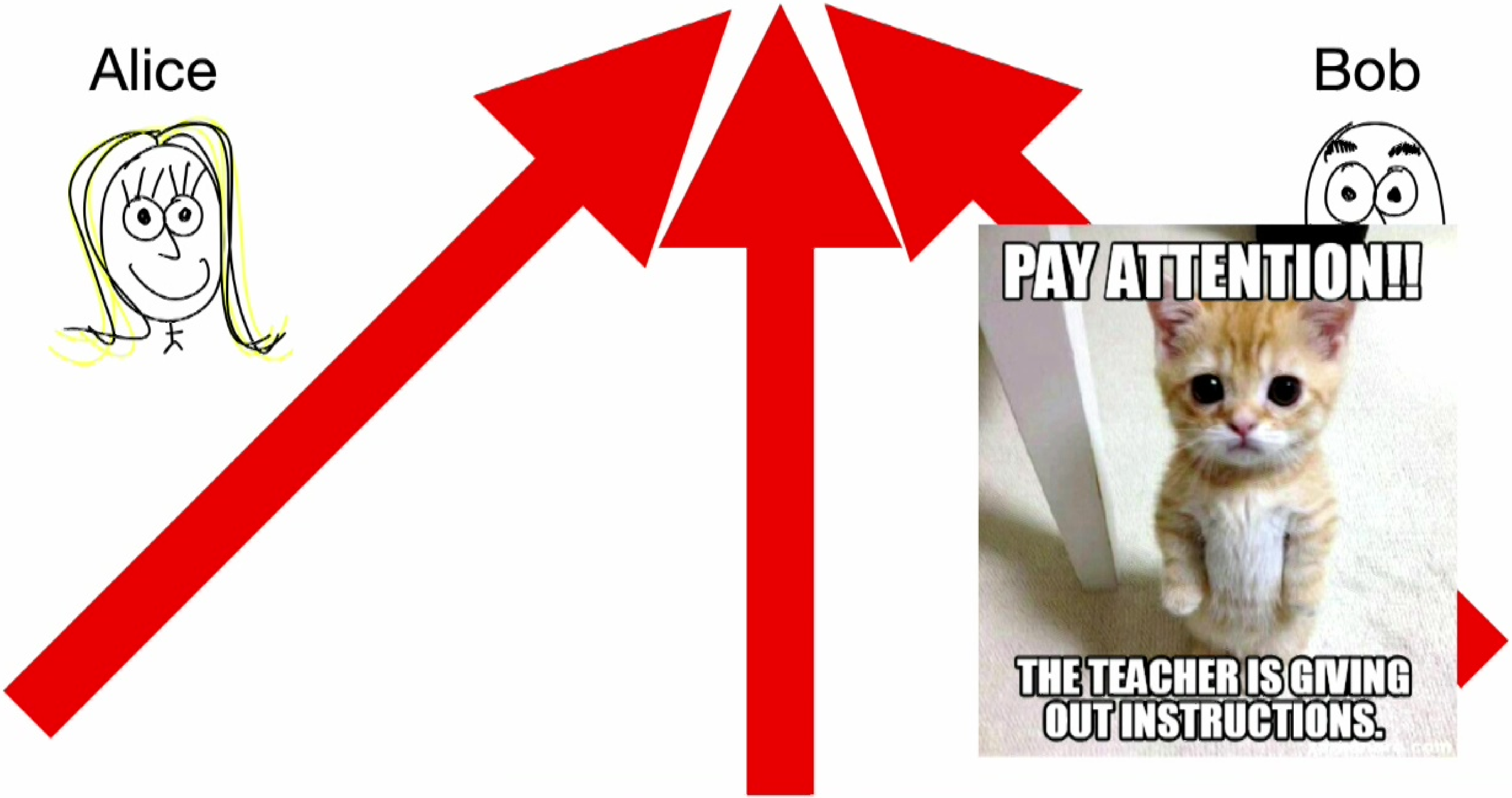
But then for $i \neq j$, we have $\rho_i \rho_j = \rho_i \left(\sum_k M_k \right) \rho_j = \left(\rho_i M_i \right) \rho_j = \rho_i \left(M_i \rho_j \right) = 0$


The More You Know

Why do we care about distinguishing quantum states?

- It's fun!
- It's the simplest form of quantum machine learning
- Many quantum tasks can be rephrased in this way (quantum algorithms, for example)
- Quantum cryptography

Quantum state antidiscrimination

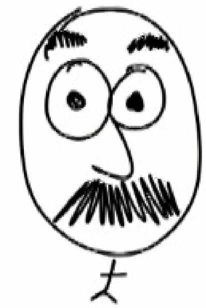


Quantum state **anti**discrimination

Alice

$|\psi_i\rangle$

Bob



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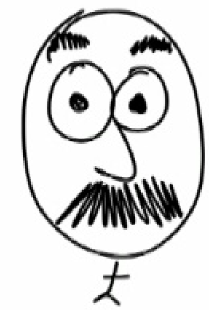

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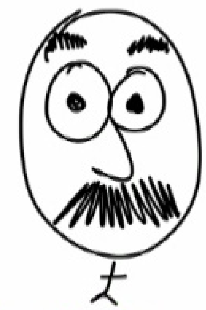


Can Bob guess what state was **not** sent?

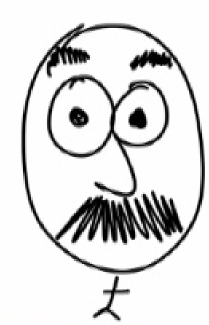
Alice prepares one of
$\{\,|\psi_1\rangle, \ldots, |\psi_n\rangle\,\}$
and sends it to Bob

Quantum state **anti**discrimination

Alice

Bob

$|\psi_i\rangle$



Alice prepares one of $\{|\psi_1\rangle, \ldots, |\psi_n\rangle\}$ and sends it to Bob

Can Bob guess what state was **not** sent?

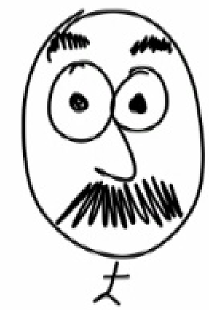
Quantum state antidiscrimination

EXAMPLE

Alice

$|\psi_1\rangle$

Bob



Alice prepares one of
$\{\,|\psi_1\rangle, \ldots, |\psi_n\rangle\}$
and sends it to Bob

Quantum state antidiscrimination

EXAMPLE

Alice prepares one of $\{ |\psi_1\rangle, \ldots, |\psi_n\rangle \}$ and sends it to Bob

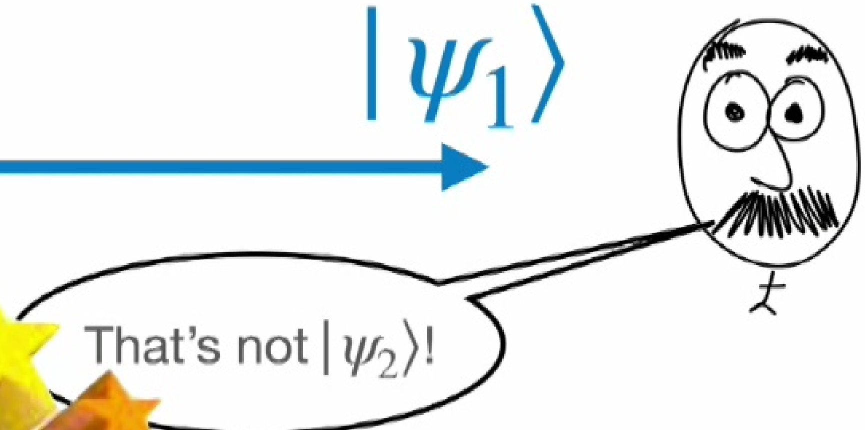
Quantum state antidiscrimination

EXAMPLE

Alice

$|\psi_1\rangle$

Bob



Alice pre

$\{|\psi_1\rangle, ..$

and send

Quantum state antidiscrimination

Definition: A set of states $\{ |\psi_1\rangle, \ldots, |\psi_n\rangle \}$ is antidistinguishable when Bob has a perfect strategy for guessing which state was NOT sent.

Quantum state antidiscrimination

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Compare

Definition: A set of states $\{ |\psi_1\rangle, \ldots, |\psi_n\rangle \}$ is distinguishable when Bob has a perfect strategy for guessing which state was sent.

Quantum state antidiscrimination

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Definition: A set of states $\{ |\psi_1\rangle, \ldots, |\psi_n\rangle \}$ is distinguishable when Bob has a perfect strategy for guessing which state was sent.

We understand distinguishability, but what can we show about antidistinguishability?

Example: The three blue states are antidistinguishable (but not orthogonal)

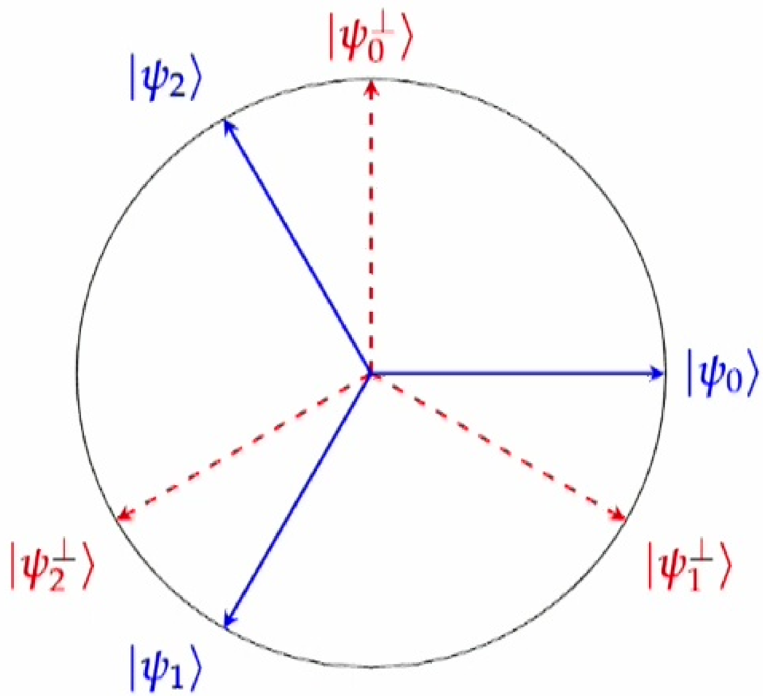

Figure 1: The trine states $\{|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle\}$ on the unit circle in $\mathbb{R}^2$, indicated in solid blue above, are antidistinguishable as witnessed by the measurement $M_0 = \frac{2}{3}|\psi_0^\perp\rangle\langle\psi_0^\perp|$, $M_1 = \frac{2}{3}|\psi_1^\perp\rangle\langle\psi_1^\perp|$, $M_2 = \frac{2}{3}|\psi_2^\perp\rangle\langle\psi_2^\perp|$, where $\{|\psi_0^\perp\rangle, |\psi_1^\perp\rangle, |\psi_2^\perp\rangle\}$ are indicated in dashed red.

Quantum state antidiscrimination

Conjecture (Havlíček, Barret)

$\{ |\psi_1\rangle, \ldots, |\psi_n\rangle \} \subset \mathbb{C}^n$ is antidistinguishable when $|\langle\psi_i|\psi_j\rangle| \leq (n-2)/(n-1), \forall i \neq j$.

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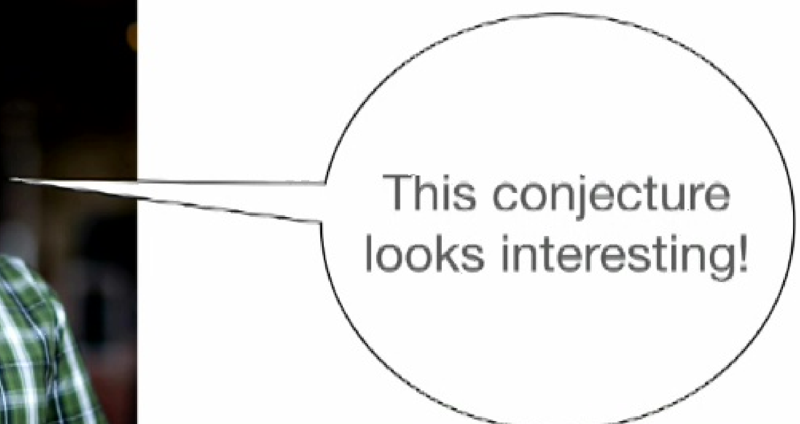
Is this true?

2019



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Is this true?

2021

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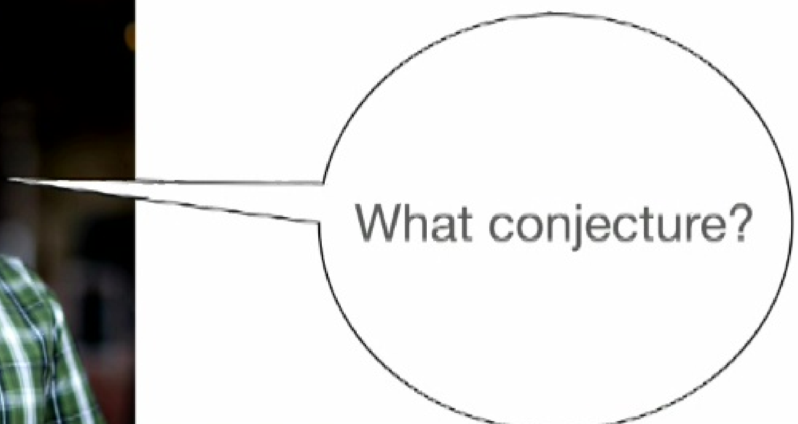
2021



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Counterexample (Russo, S.) (Phys. Rev. A **107**, L030202)

There exists 4 states $\{ |\psi_1\rangle, \ldots, |\psi_4\rangle \} \subset \mathbb{C}^4$ that satisfy $|\langle\psi_i|\psi_j\rangle| \leq 0.645... < 2/3, \forall i \neq j$ that are not antidistinguishable

SDPs!

Suppose Alice sends ρ_i to Bob and suppose he measures it with the POVM $\{M_1, \ldots, M_n\}$

Suppose Bob interprets the outcome M_j as "this state is NOT ρ_j"

The *sum of errors* can be written as the following SDP

$$\alpha = \text{minimize: } \sum_{i=1}^{n} \langle M_i, \rho_i \rangle$$

$$\text{subject to: } \sum_{i=1}^{n} M_i = I$$

$$M_1, \ldots, M_n \succeq 0$$

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Lemma: $\alpha = 0$ if and only if $\{\rho_1, \ldots, \rho_n\}$ is antidistinguishable

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$$\beta = \text{maximize: } \mathrm{Tr}(Y)$$

$$\text{subject to: } Y \preceq \rho_i, \forall i$$

SDPs!

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$$\beta = \text{maximize: } \mathrm{Tr}(Y)$$

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Lemma: $\alpha = \beta$. So, $\beta = 0$ if and only if $\{\rho_1, \ldots, \rho_n\}$ is antidistinguishable.

SDPs!

$$\alpha = \text{minimize: } \sum_{i=1}^{n} \langle \psi_i | M_i | \psi_i \rangle$$

$$\text{subject to: } \sum_{i=1}^{n} M_i = I$$

$$M_1, \ldots, M_n \succeq 0$$

$$\alpha = \text{maximize: } \mathrm{Tr}(Y)$$

$$\text{subject to: } Y \preceq |\psi_i\rangle\langle\psi_i|, \forall i$$

The pure state version

<u>Lemma</u>

$\{ |\psi_1\rangle, \ldots, |\psi_n\rangle \}$ is antidistinguishable

if and only if

there does NOT exist Y satisfying $Y \preceq |\psi_i\rangle\langle\psi_i|, \forall i$ with positive trace

This gives us a <u>certificate</u> of non-antidistinguishability

Totally legit algorithm

while (pandemic)

generate random $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle \in \mathbb{C}^4$

calculate $\max_{i \neq j} |\langle \psi_i | \psi_j \rangle|$

calculate SDP value

if numbers are good, stop and write paper

print dual solution as certificate

Quantum state antidiscrimination

Conjecture (Havlíček, Barret)

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SUPER SMASH BROS.
Nathaniel Johnston
Mount Allison University
(Quantum, Computer guy)
Joins the Battle!


MATH
THE
AVENGERS

Quantum state antidiscrimination

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Theorem (Johnston, Russo, S.) in progress…

$\{|\psi_1\rangle, \ldots, |\psi_n\rangle\}$ is antidistinguishable when

$$\sum_{i,j} |\langle\psi_i|\psi_j\rangle|^2 \leq \frac{n^2}{2}$$

Proof: Math (we'll come back to this…)

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Corollary

$\{\,|\psi_1\rangle, \ldots, |\psi_n\rangle\}$ is antidistinguishable when

$$|\langle\psi_i|\psi_j\rangle| \leq \frac{1}{\sqrt{2}}\sqrt{\frac{n-2}{n-1}}, \forall i \neq j.$$

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This is tight for all n

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This is tight for $n = 4$

Quantum state antidiscrimination

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Theorem (Johnston, Russo, S. (in progress), Bandyopadhyay, Jain, Oppenheim, Perry)

$\{\,|\psi_1\rangle, \ldots, |\psi_n\rangle\}$ cannot be antidistinguishable when

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Proof: If the above condition happens, you can find a dual SDP solution that certifies that it cannot be antidistinguishable

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We'll come back to this…

Corollary

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This is tight for $n = 4$

$n \geq 5$ is tricky…

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This is tight for all n

$$|\langle\psi_i|\psi_j\rangle| > (n-2)/(n-1), \forall i \neq j.$$

Optimal examples

For $n = 3$

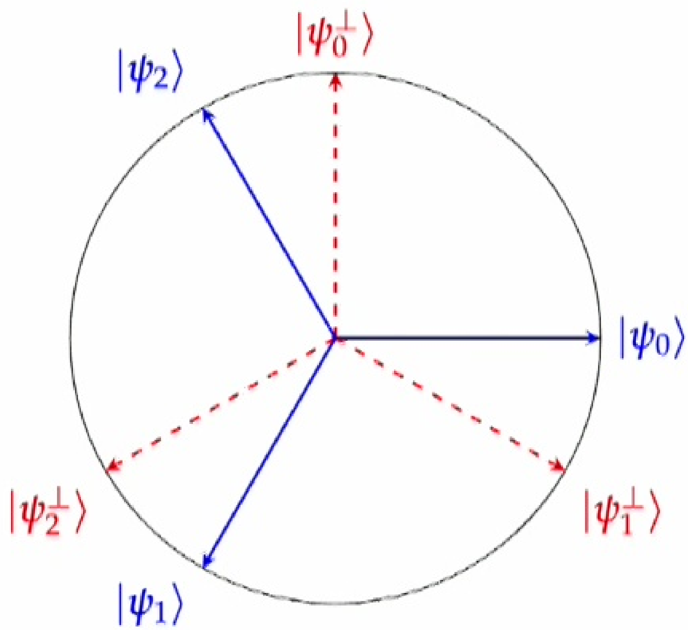

Figure 1: The trine states $\{|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle\}$ on the unit circle in $\mathbb{R}^2$, indicated in solid blue above, are antidistinguishable as witnessed by the measurement $M_0 = \frac{2}{3}|\psi_0^\perp\rangle\langle\psi_0^\perp|$, $M_1 = \frac{2}{3}|\psi_1^\perp\rangle\langle\psi_1^\perp|$, $M_2 = \frac{2}{3}|\psi_2^\perp\rangle\langle\psi_2^\perp|$, where $\{|\psi_0^\perp\rangle, |\psi_1^\perp\rangle, |\psi_2^\perp\rangle\}$ are indicated in dashed red.

Optimal examples

For $n \geq 3$, we need a new definition…

Definition: The Gram matrix of a set of states $\{\,|\psi_0\rangle, |\psi_1\rangle, \ldots, |\psi_{n-1}\rangle\,\}$ is given as

$$G := \begin{pmatrix} 1 & \langle\psi_0|\psi_1\rangle & \langle\psi_0|\psi_2\rangle & \cdots & \langle\psi_0|\psi_{n-1}\rangle \\ \langle\psi_1|\psi_0\rangle & 1 & \langle\psi_1|\psi_2\rangle & \cdots & \langle\psi_1|\psi_{n-1}\rangle \\ \langle\psi_2|\psi_0\rangle & \langle\psi_2|\psi_1\rangle & 1 & \cdots & \langle\psi_2|\psi_{n-1}\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle\psi_{n-1}|\psi_0\rangle & \langle\psi_{n-1}|\psi_1\rangle & \langle\psi_{n-1}|\psi_2\rangle & \cdots & 1 \end{pmatrix}$$

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This matrix is Hermitian positive semidefinite

Lemma: We only need to look at the Gram matrix, not the states themselves.

Optimal examples

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Fake proof:

If the states $\{ |\psi_1\rangle, \ldots, |\psi_n\rangle\}$ are antidistinguishable,
then there exists a measurement $\{M_1, \ldots, M_n\}$ such that $\langle\psi_i|M_i|\psi_i\rangle = 0$

But, then $\{U|\psi_1\rangle, \ldots, U|\psi_n\rangle\}$ is also antidistinguishable as witnessed by the measurement $\{UM_1U^*, \ldots, UM_nU^*\}$.

Since antidistinguishability is unitarily invariant, all we need to look at is the Gram matrix

Optimal examples

Theorem (Johnston, Russo, S. (in progress), Bandyopadhyay, Jain, Oppenheim, Perry)

$\{\, |\psi_1\rangle, \ldots, |\psi_n\rangle \}$ cannot be antidistinguishable when

$$\sum_{i,j} |\langle \psi_i | \psi_j \rangle| > n(n-1)$$

Optimal example:

$$G := \begin{pmatrix} 1 & \gamma & \gamma & \cdots & \gamma \\ \gamma & 1 & \gamma & \cdots & \gamma \\ \gamma & \gamma & 1 & \cdots & \gamma \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma & \gamma & \gamma & \cdots & 1 \end{pmatrix}$$

The set of states is antidistinguishable if and only if $\gamma \leq (n-2)/(n-1)$

Optimal examples

Corollary

$\{\,|\psi_1\rangle, \ldots, |\psi_n\rangle\}$ cannot be antidistinguishable when

$$|\langle\psi_i|\psi_j\rangle| > (n-2)/(n-1),\ \forall i \neq j\,.$$

Optimal example:

$$G := \begin{pmatrix} 1 & \gamma & \gamma & \cdots & \gamma \\ \gamma & 1 & \gamma & \cdots & \gamma \\ \gamma & \gamma & 1 & \cdots & \gamma \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma & \gamma & \gamma & \cdots & 1 \end{pmatrix}$$

The set of states is antidistinguishable if and only if $\gamma \leq (n-2)/(n-1)$

Quantum state antidiscrimination

It also solves all
of our problems.
Thanks.

Quantum state antidiscrimination

What if the Gram matrix is circulant?

Quantum state antidiscrimination

It also solves all
of our problems.
Thanks.

Optimal examples

For the **lower bound**, we need a new definition…

Definition: The matrix is circulant if it looks like:

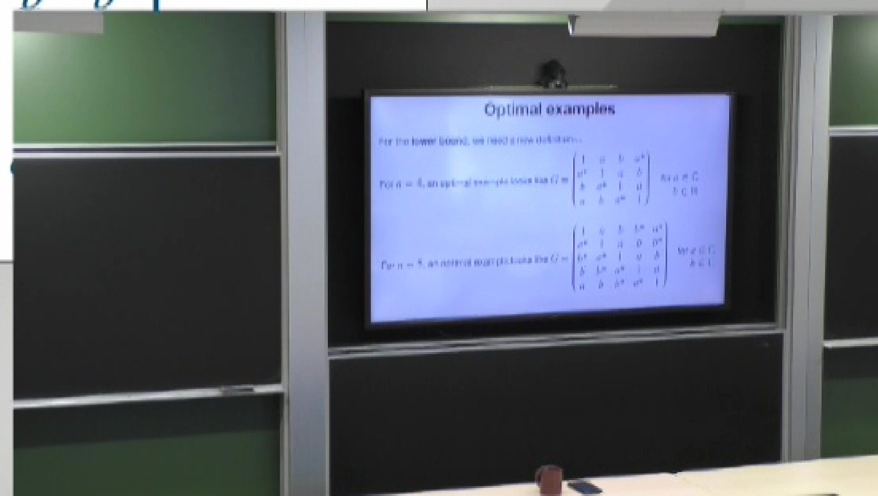
$$G = \begin{pmatrix} g_0 & g_1 & g_2 & \cdots & g_{n-2} & g_{n-1} \\ g_{n-1} & g_0 & g_1 & \cdots & g_{n-3} & g_{n-2} \\ g_{n-2} & g_{n-1} & g_0 & \cdots & g_{n-4} & g_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g_2 & g_3 & g_4 & \cdots & g_0 & g_1 \\ g_1 & g_2 & g_3 & \cdots & g_{n-1} & g_0 \end{pmatrix}$$

Optimal examples

For the **lower bound**, we need a new definition…

For $n = 4$, an optimal example looks like $G = \begin{pmatrix} 1 & a & b & a^* \\ a^* & 1 & a & b \\ b & a^* & 1 & a \\ a & b & a^* & 1 \end{pmatrix}$ for $a \in \mathbb{C}$, $b \in \mathbb{R}$

For $n = 5$, an optimal example looks like $G = \begin{pmatrix} 1 & a & b & b^* & a^* \\ a^* & 1 & a & b & b^* \\ b^* & a^* & 1 & & \\ b & b^* & a^* & & \\ a & b & b^* & & \end{pmatrix}$



So, about that math…

To have $\alpha = 0$, we require that N_i to have the entire i^{th} row and column to be all 0 entries

Such a matrix is called "(n-1)-incoherent"

Lemma: $\{\,|\psi_1\rangle, \ldots, |\psi_n\rangle\}$ is antidistinguishable if and only if G is (n-1)-incoherent

$$\alpha = \text{minimize: } \sum_{i=1}^{n} \langle i \,|\, N_i \,|\, i \rangle$$

$$\text{subject to: } \sum_{i=1}^{n} N_i = G$$

$$N_1, \ldots, N_n \geq 0$$

This is interesting (to Nathaniel)

So, about that math…

Lemma: If the eigenvalues of G satisfy $\sqrt{\lambda_{\max}} \leq \sum_{rest} \sqrt{\lambda_i}$ then it's antidistinguishable

Corollary: If $||G||_F^2 \leq \frac{n}{2}$ then it's antidistinguishable

Lemma: If G is circulant, then $\sqrt{\lambda_{\max}} \leq \sum_{rest} \sqrt{\lambda_i}$ if and only if it's antidistinguishable

Corollary: If G is circulant, then $||G||_F^2 \leq \frac{n}{2}$ is necessary and sufficient for antidistinguishability

Open questions

- Is there an interpretation of these results?
- Are there GPTs with antidistinguishable flavours?
- Does this connect to the PBR theorem at all?
- Is there a nice application of having such examples?