Title: Talk 95 - A Large Holographic Code and its Geometric Flows

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Abstract: The JLMS formula is a cornerstone in our understanding of bulk reconstruction in holographic theories of quantum gravity, best interpreted as a quantum error-correcting code. Moreover, recent work has highlighted the importance of understanding holography as an approximate and perhaps non-isometric code. In this work, we construct an enlarged code subspace for the bulk theory that contains multiple non-perturbatively different background geometries. In such a large holographic code, we carefully derive an approximate version of the JLMS formula from an approximate FLM formula for a class of nice states. We do not assume that the code is isometric, but interestingly find that approximate FLM forces the code to be approximately isometric. Furthermore, we show that the bulk modular Hamiltonian of the entanglement wedge makes important contributions to the JLMS formula and cannot in general be neglected even when the bulk state is semiclassical. Nevertheless, when acting on states with the same background geometry, we find that the modular flow is well approximated by the area flow which takes the geometric form of a boundary-condition-preserving kink transform. We also generalize the results to higher derivative gravity, where area is replaced by the geometric entropy. We conjecture that a Lorentzian definition of the geometric entropy is equivalent to its original, Euclidean definition, and we verify this conjecture in a dilaton theory with higher derivative couplings. Thus we find that the flow generated by the geometric entropy takes the universal form of a boundary-condition-preserving kink transform.

A Large Holographic Code and its Geometric Flows

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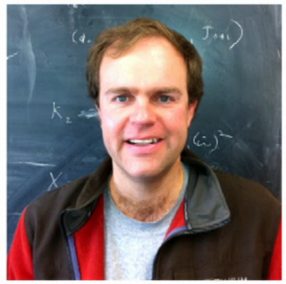
August 4, 2023

It from Qubit 2023, Perimeter Institute

This talk is

- based on upcoming works with Don Marolf and Pratik Rath:



  - XD, Marolf, Rath, "The JLMS Formula, Modular Flow and the Area Operator," to appear.
  - XD, Marolf, Rath, "Geometric Entropies and their Geometric Flow: the Power of Lorentzian Methods," to appear.

- related to [Bousso/Chandrasekaran/Rath/Shahbazi-Moghaddam].

Motivation

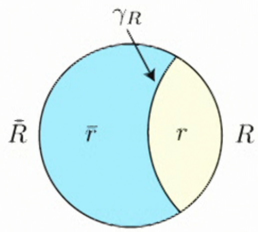
1. What is the relation between modular and area flows of holographic states?

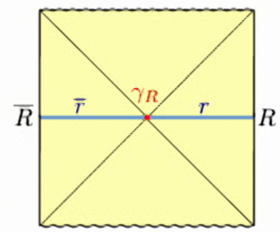
2. How to build a large holographic code that contains different geometries?

Modular flow

- Modular flow has been a useful tool for many purposes (bulk reconstruction, QNEC, ...).

$$|\psi_s^{\text{mod}}\rangle = e^{-iK_{\rho_R}s}|\psi\rangle, \qquad K_{\rho_R} := -\log \rho_R$$



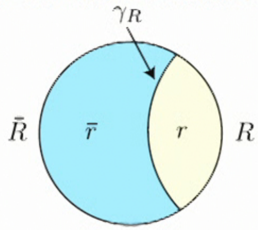


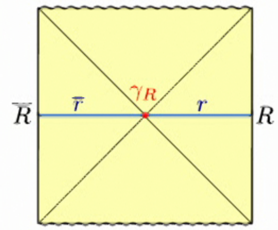
- For a thermal state $\rho_R \propto e^{-\beta H_R}$, modular flow is one-sided (rescaled) time evolution.
- For general states, modular Hamiltonian K_{ρ_R} is complicated and nonlocal.
- Can we approximate modular flow by something simple?
- Our answer will be: *yes, by the area flow in the bulk sometimes.* (We will clarify what 'sometimes' means.)

Area flow

$$|\psi_s^{\text{area}}\rangle = e^{-i\hat{A}s/4G}|\psi\rangle$$

- $\hat{A}$: area operator on HRT surface γ_R.

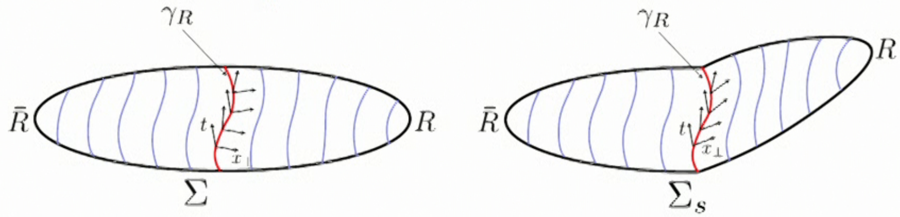




- $\hat{A}$ is well-defined for semiclassical bulk states.
- Viewing holography as a quantum error-correcting (QEC) code, $\hat{A}$ is well-defined within the code subspace.

Classically, the area flow:

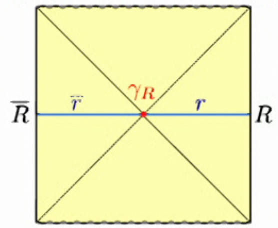
1. Takes the initial data g_{ij}, K_{ij} on a Cauchy surface containing γ_R.
2. Adds a delta function $2\pi s \delta_{\gamma_R}$ to $K_{\perp\perp}$. ($\perp$: orthogonal to γ_R)

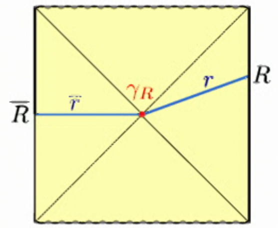


3. Evolves new initial data to find the area-flowed state.

Area flow

- Simplest example: AdS black hole dual to thermofield double state. Evolving new initial data gives the same AdS Schwarzschild solution, as long as we identify the Cauchy surface with a 'kinked' surface:





- In more general situations, the new solution is

1. always the same in the entanglement wedges r and $\overline{r}$,
2. different in the future/past wedges,
3. but well-defined by evolving new initial data.

Approximate JLMS

- The standard derivation of JLMS is based on the FLM formula:
  [Faulkner-Lewkowycz-Maldacena]

$$S(\rho_R) = \frac{A}{4G} + S(\rho_r) + \mathcal{O}(\epsilon_1)$$

- But FLM is only approximate, and the derivation of JLMS may fail due to small eigenvalues of ρ_r.
- To see this, recall we use the "first law":

$$S(\rho + \delta\rho) - S(\rho) = \mathrm{Tr}(K_\rho \delta\rho) + \mathcal{O}(\delta\rho^2)$$

  Ignoring $\mathcal{O}(\delta\rho^2)$ terms, we then find JLMS.
- But $\mathcal{O}(\delta\rho^2) \supset \mathrm{Tr}(\rho^{-1}\delta\rho^2) + \cdots$.
- Thus small eigenvalues of ρ_r may cause a large correction to JLMS.
  [Kudler-Flam/Rath]
- One might try to reduce the $\mathcal{O}(\delta\rho^2)$ terms by reducing $\delta\rho$, but this amplifies the ϵ_1 error.

Plan

1. Motivation

2. Building a large holographic code

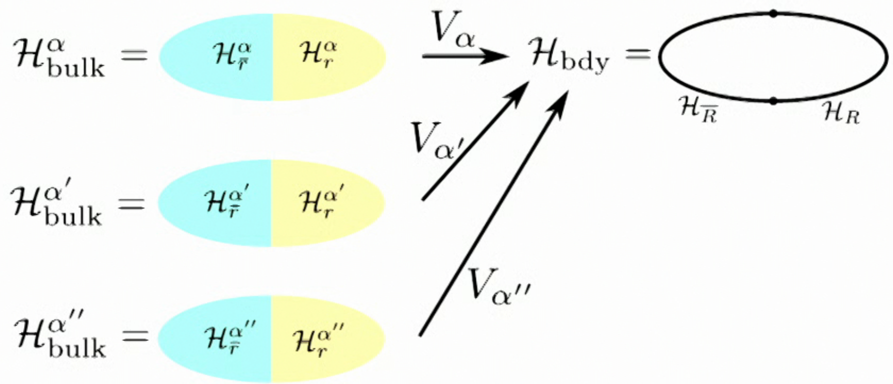
3. Modular flow $\approx$ area flow

4. Generalization to higher derivative gravity and a new Lorentzian derivation of gravitational entropy formula

Building a large holographic code

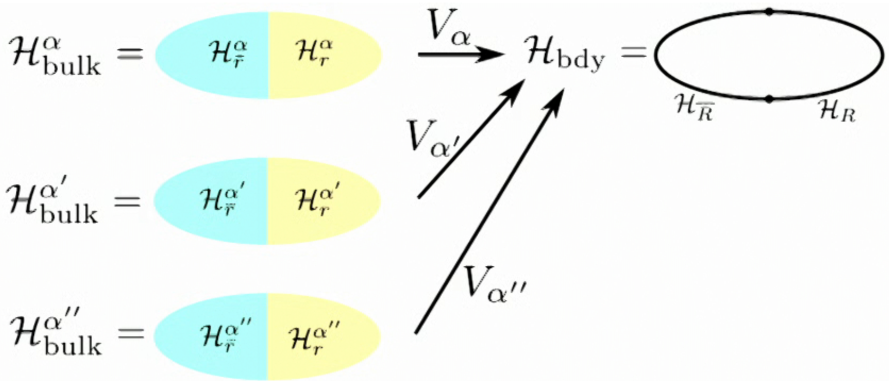
- Start with a small code consisting of fixed-area states on a given bulk geometry:

- Add more geometries:

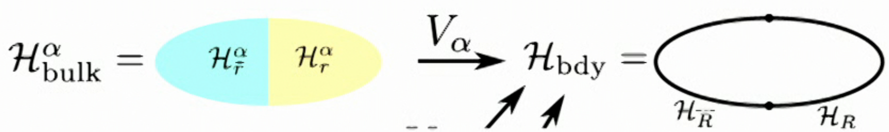


- The large code is defined by taking the direct sum of many small codes:

$$\mathcal{H}_{\mathrm{bulk}} := \bigoplus_{\alpha} \mathcal{H}_{\mathrm{bulk}}^{\alpha}, \qquad V := \bigoplus_{\alpha} V_{\alpha}$$



- We assume:
  1. The FLM formula holds approximately in each small code
  $$S(\rho_R^\alpha) = \frac{A^\alpha}{4G} + S(\rho_r^\alpha) + \mathcal{O}(\epsilon_1)$$
  2. Different small code subspaces are approximately orthogonal on R (and on $\overline{R}$) in a certain way.
- Roughly speaking, approximate orthogonality means conditions such as:
  $\text{Tr}_{\overline{R}}\left(V_\alpha|\psi^\alpha\rangle\langle\psi^\beta|V_\beta^\dagger\right)$ has a small trace norm for $\alpha \neq \beta$.
- We do not assume V is isometric, but the conditions above ensure V is an approximate isometry.

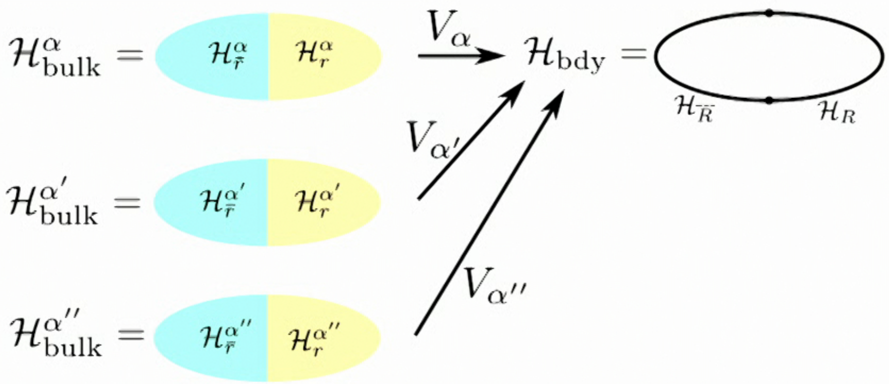


Theorem 1 (Approximate JLMS in each small code)

The JLMS formula holds approximately for any log-stable state ρ^α in $\mathcal{H}^\alpha_{bulk}$:

$$\left\| V_\alpha^\dagger K_{\rho^\alpha_R} V_\alpha - \left(\frac{A^\alpha}{4G} + K_{\rho^\alpha_r} \right) \right\|_\infty \leq \epsilon$$

- Roughly speaking, log-stable states have bounded eigenvalues: $\left\| (\rho^\alpha_r)^{-1} \right\|_\infty \leq 1/\epsilon_0$.
- The theorem guarantees $\epsilon \sim \sqrt{\epsilon_1/\epsilon_0}$.
- To prove it, vary the FLM formula under $\delta\rho$ but now bound the $\mathcal{O}(\delta\rho^2)$ terms.
- In holography, $\epsilon_1 = O(G)$, so we can comfortably choose $\epsilon_0 \gg \epsilon_1$ to ensure JLMS holds approximately.



Theorem 2 (Approximate JLMS in the large code)

The JLMS formula holds approximately for any log-stable state ρ in $\mathcal{H}_{bulk}$:

$$V^\dagger K_{\rho_R} V - \left(\frac{\hat{A}}{4G} + K_{\rho_r} \right) = small$$

- Here $\hat{A} := \bigoplus_\alpha A^\alpha \mathbb{1}_\alpha$.
- Roughly speaking, ρ is log-stable if
  1. upon decomposing $\rho_r = \bigoplus_\alpha p_\alpha \rho_r^\alpha$, each ρ_r^α is log-stable, and
  2. p_α is sufficiently smooth so that a diagonal approximation holds for K_{ρ_R}.
- Semiclassical states prepared by gravitational path integrals naturally satisfy these conditions.

Exponentiated JLMS

- For various reasons, we not only want the regular JLMS

$$V^\dagger K_{\rho_R} V - \left(\frac{\hat{A}}{4G} + K_{\rho_r} \right) = \text{small}$$

but also want the exponentiated version

$$\left\| V^\dagger e^{-is\, K_{\rho_R}} V - e^{-is\left(\frac{\hat{A}}{4G} + K_{\rho_r}\right)} \right\|_\infty = \text{small} \qquad (1)$$

- For a code with exact complementary recovery, the exponentiated JLMS is equivalent to a flat entanglement spectrum in fixed-area states. [Akers/Rath, XD/Harlow/Marolf]
- Here we generalize this relation to approximate codes:

Theorem 3 (Exponentiated JLMS)

Replacing the FLM formula by a Renyi version

$$S_n(\rho_R^\alpha) = \frac{A^\alpha}{4G} + S_n(\rho_r^\alpha) + \textit{small}$$

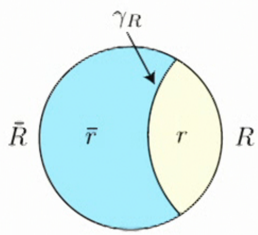
with $n = 1 +$ is upgrades Theorems 1 and 2 to exponentiated versions such as (1).

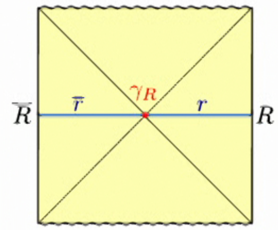
A (too) quick derivation of "modular flow ≈ area flow"

- Now we have

$$V^\dagger e^{-is K_{\rho_R}} V \approx e^{-is\left(\frac{\hat{A}}{4G} + K_{\rho_r}\right)}$$

in a large code subspace.





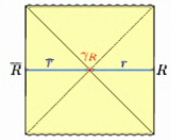
- <u>**If**</u> the bulk modular Hamiltonian K_{ρ_r} is negligible compared to $\frac{\hat{A}}{4G}$, we might expect

$$\text{modular flow } |\psi_s^{\text{mod}}\rangle \approx \text{ area flow } |\psi_s^{\text{area}}\rangle$$

- This derivation is too quick, for K_{ρ_r} is not generally negligible compared to $\frac{\hat{A}}{4G}$.

Simple example of non-negligible K_{ρ_r}

Consider pure 3d gravity, TFD state $|\mathrm{TFD}_\beta\rangle$ dual to BTZ black hole:



- We find

$$K_{\rho_R} = \beta H_R + \log Z_\beta = \frac{\hat{A}^2}{8GA_\beta} + \log Z_\beta$$

  is different from $\frac{\hat{A}}{4G}$. (Here A_β is thermal expectation value of $\hat{A}$.)

- Their difference

$$K_{\rho_r} = K_{\rho_R} - \frac{\hat{A}}{4G} = \frac{(\hat{A} - A_\beta)^2}{8GA_\beta}$$

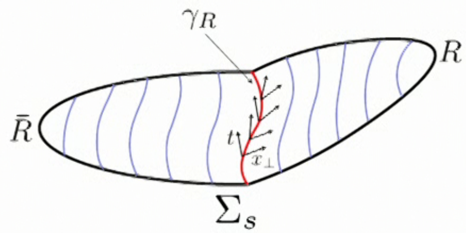
  can be as large as $\frac{\hat{A}}{4G}$ if acting on a state whose wavefunction in A has significant support far away from A_β.

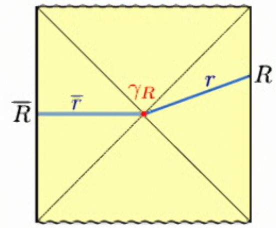
- A useful way of understanding this potentially large K_{ρ_r}: although ρ_r is peaked at A_β, it contains long tails describing small probabilities p of large area fluctuations. The corresponding $\log p$ contribution to K_{ρ_r} can be large.

General claim

The modular flow $e^{-iK_{\rho_R} s}|\psi\rangle$ is well approximated by the area flow $e^{-i\hat{A}\, s/4G}|\psi\rangle$ (in the sense the two states have the same semiclassical geometry) if

1. $|\psi\rangle$ has approximately the same semiclassical geometry as ρ, and
2. ρ is log-stable.





- This is our main result coming from applying JLMS in the large holographic code.

- In particular, condition 1 is satisfied if $\rho = |\psi\rangle\langle\psi|$. This is the scenario best studied previously. [Balakrishnan-Faulkner-Khandker-Wang; Chen-XD-Lewkowycz-Qi; Faulkner-Li-Wang; Bousso/Chandrasekaran/Rath/Shahbazi-Moghaddam]

- This result also explains and generalizes a previous proposal relating modular-flowed entropies to constrained HRT areas. [Chen-XD-Lewkowycz-Qi]

Geometric flow in higher derivative gravity

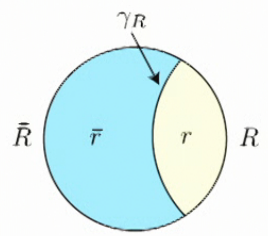
So far we have focused on Einstein gravity. We can generalize the results to higher derivative gravity.

- $\frac{A}{4G}$ is replaced by a "geometric entropy" σ of the form

$$\sigma = S_{\text{Wald}} + S_{\text{extrinsic curvature}}$$

For example, for $L = \frac{1}{16\pi G} R + \lambda R_{\mu\nu} R^{\mu\nu}$,

$$\sigma = \frac{A}{4G} + 4\pi\lambda \int_{\text{HRT}} \sqrt{\gamma} \left(R^{a}{}_{a} - \frac{1}{2} K_a K^a \right)$$



- Area flow is replaced by a "geometric flow" generated by $\hat{\sigma}$:

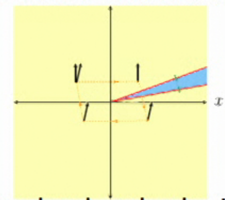
$$|\psi_s^{\text{geom}}\rangle = e^{-i\hat{\sigma}s}|\psi\rangle$$

- Again, modular flow $\approx$ geometric flow when acting on a state with the same semiclassical geometry as ρ (and if ρ is log-stable).

New Lorentzian method for σ

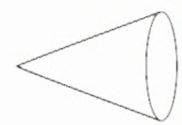
- We propose to derive the geometric entropy σ in higher derivative gravity by calculating the variation of the Lorentzian action with a small Lorentzian conical defect:

$$\sigma = -\left.\frac{dI_s}{ds}\right|_{s=0}$$



- This is very different from the Lewkowycz-Maldacena method calculating the variation of the Euclidean action with respect to a Euclidean conical angle:

$$\sigma = \left.\frac{dI_\epsilon^{\text{Euclidean}}}{d\epsilon}\right|_{\epsilon=0}$$



- In particular, the last term in

$$\sigma = S_{\text{Wald}} + S_{\text{extrinsic curvature}}$$

requires a careful analysis in the Euclidean method because it comes from second-order (in ϵ) contributions to the Lagrangian that enhance to first order after integrating near the conical defect.

- The new Lorentzian method seems much more straightforward: first-order contributions are sufficient, as no second-order terms enhance.
- We verified this proposal explicitly in 2d dilaton gravity examples coupled to scalars with higher derivative interaction. It is a nontrivial check of our conjecture.

Summary:

- We built a large holographic code that accommodates states of different geometries (and their superpositions).
- The JLMS formula (and its exponentiated version) holds approximately in the large code.
- Modular-flowed states have the same semiclassical geometry as area-flowed states (under the right condition).
- We proposed a new Lorentzian way of deriving the geometric entropy, which implies that the geometric flow is always given by a boundary-condition-preserving kink transform.

Questions:

1. We focused on scenarios where FLM holds and the holographic code is approximately isometric. Can we generalize to quantum extremal surfaces and understand features of holographic codes in the non-isometric regime?
2. We essentially used a bulk UV cutoff to make the bulk algebra Type I. Can we generalize to Type II bulk algebras?
3. In what other ways can enjoy these modular flows?