Title: Merged talks - An effective field theory for non-maximal quantum chaos; 66- Effective description of sub-maximal chaos: stringy effects for SYK scrambling

Speakers: Ping Gao, Felix Haehl

Collection: It from Qubit 2023

Date: August 03, 2023 - 1:45 PM

URL: https://pirsa.org/23080025

Abstract: 66 - It has been proposed that the exponential decay and subsequent power law saturation of out-of-time-order correlation functions can be universally described by collective 'scramblon' modes. We develop this idea from a path integral perspective in several examples, thereby establishing a general formalism. After reformulating previous work on the Schwarzian theory and identity conformal blocks in two-dimensional CFTs relevant for systems in the infinite coupling limit with maximal quantum Lyapunov exponent, we focus on theories with sub-maximal chaos: we study the large-q limit of the SYK quantum dot and chain, both of which are amenable to analytical treatment at finite coupling. In both cases we identify the relevant scramblon modes, derive their effective action, and find bilocal vertex functions, thus constructing an effective description of chaos. The final results can be matched in detail to stringy corrections to the gravitational eikonal S-matrix in holographic CFTs, including a stringy Regge trajectory, bulk to boundary propagators, and multi-string effects that are unexplored holographically.

Effective description of sub-maximal chaos: stringy effects for SYK scrambling

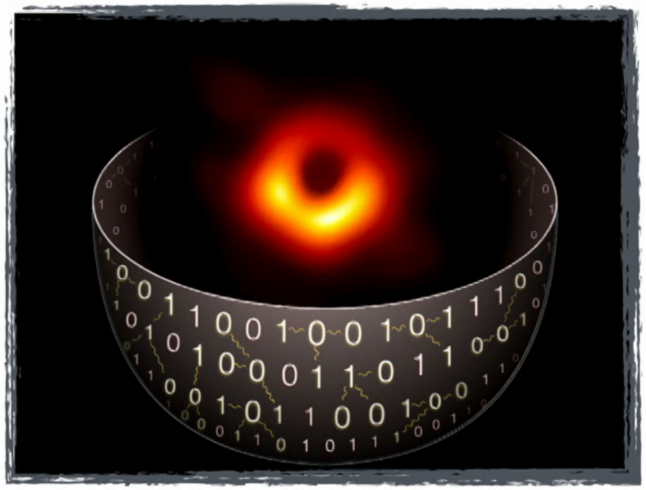
Felix Haehl
University of Southampton

It From Qubit 2023

2301.05698
with C. Choi, M. Mezei, G. Sarosi

UK Research and Innovation

erc



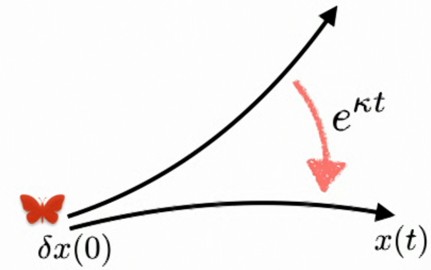
$$\frac{\ell_{AdS}}{\ell_{Planck}} \sim N \qquad \frac{\ell_{AdS}}{\ell_{string}} \sim \lambda$$

- $N \gg 1, \ \lambda \gg 1:$ classical gravity (geometry, black holes, shockwaves, …)
- Finite (or perturbative) $1/\lambda:$ stringy effects

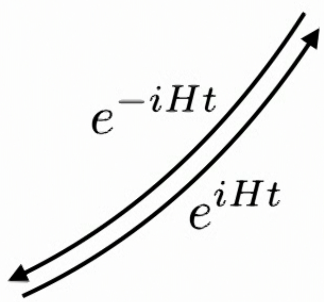
Out-of-time-ordered correlators & stringy effects

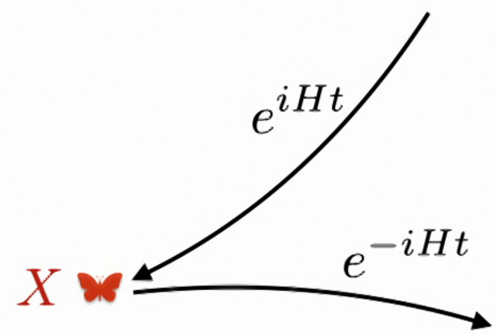
Quantum Lyapunov exponent

- **Classical** butterfly effect:
  exponential sensitivity to initial condition



- **Quantum** butterfly effect:
  $X(t) = e^{iHt} X\, e^{-iHt}$ is 'complicated' even if X was 'simple'

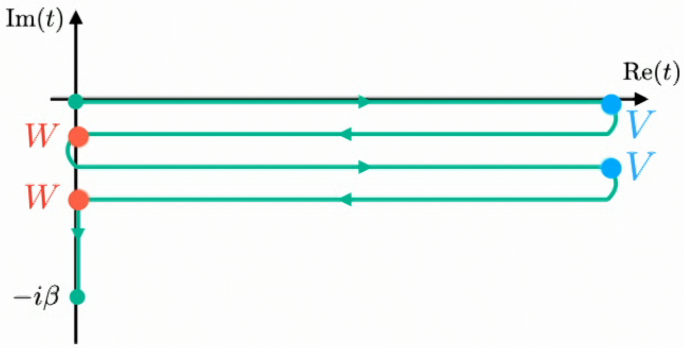




$$e^{iHt} X e^{-iHt} = X + it[H,X] - \frac{t^2}{2}[H,[H,X]] + \dots$$

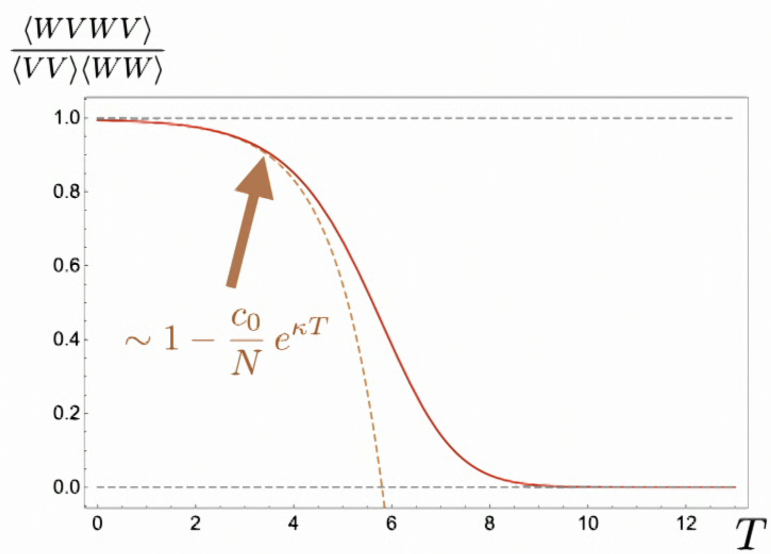
- **OTOC** = overlap of $|V(T)W(0)\rangle$ and $|W(0)V(T)\rangle$:

$$\langle W(0)V(T)|W(0)V(T)\rangle_\beta \sim \langle VV\rangle_\beta \langle WW\rangle_\beta \left(1 - \frac{c_0}{N}\, e^{\kappa T} + \dots\right)$$



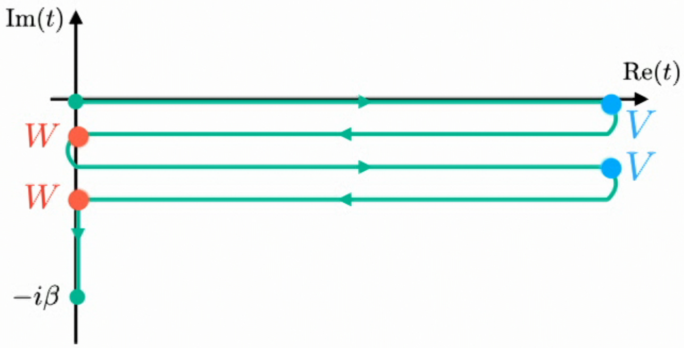
quantum Lyapunov exponent κ

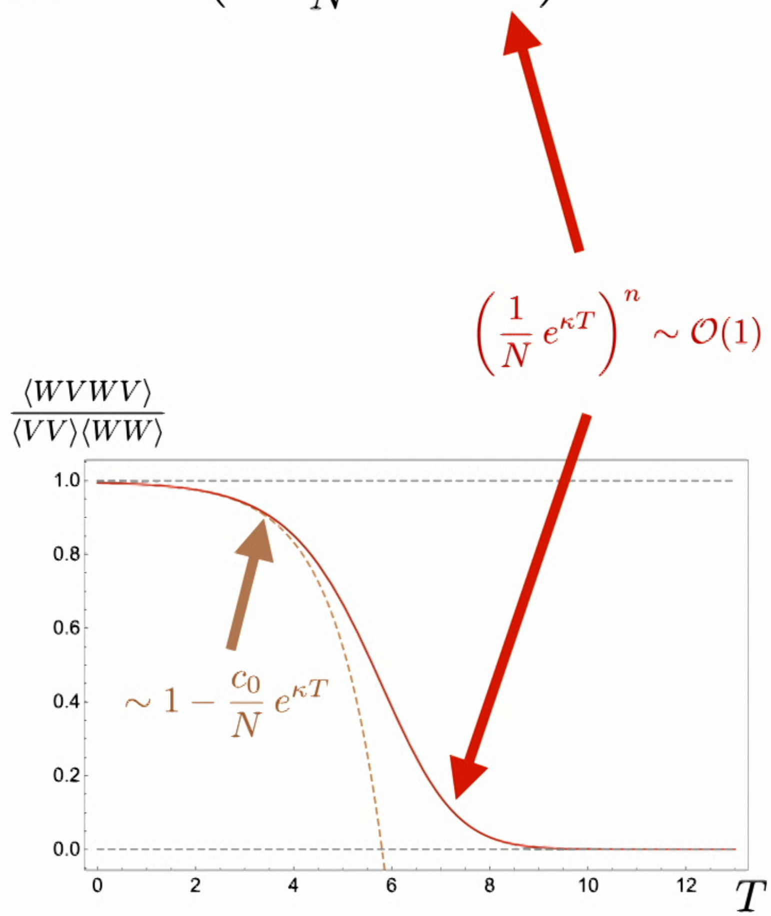
[Larkin/Ovchinnikov '68] [Kitaev '14] [Shenker/Stanford '14]



- **OTOC** = overlap of $|V(T)W(0)\rangle$ and $|W(0)V(T)\rangle$:

$$\langle W(0)V(T)|W(0)V(T)\rangle_\beta \sim \langle VV\rangle_\beta \langle WW\rangle_\beta \left(1 - \frac{c_0}{N}\, e^{\kappa T} + \dots\right)$$





- **OTOC** = overlap of $|V(T)W(0)\rangle$ and $|W(0)V(T)\rangle$:

$$\langle W(0)V(T)|W(0)V(T)\rangle_\beta \sim \langle VV\rangle_\beta \langle WW\rangle_\beta \left(1 - \frac{c_0}{N}\, e^{\kappa T} + \dots\right)$$

- **Scrambling time:**

$$t_\star \sim \frac{1}{\kappa} \log N$$

- **Chaos bound:**

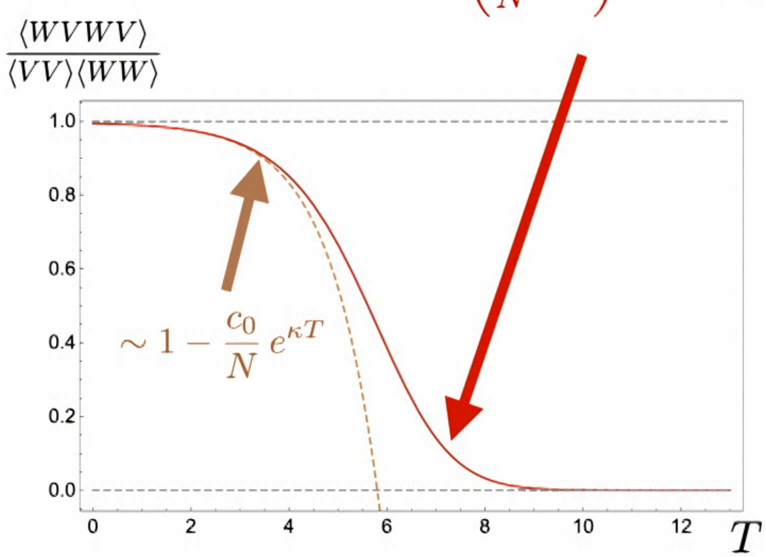
$$\kappa \leq 2\pi/\beta$$

[Maldacena/Shenker/Stanford '16]

- Gravity & SYK at $\beta J \to \infty$:

$$\kappa = 2\pi/\beta$$

$$\left(\frac{1}{N}\, e^{\kappa T}\right)^n \sim \mathcal{O}(1)$$



SYK and stringy effects

$$H = i^{q/2} \sum_{i_1,\dots,i_q} J_{i_1\cdots i_q} \chi_{i_1} \cdots \chi_{i_q}$$

collective fields

$$\overline{Z(J)} = \int D\Sigma\, DG\, e^{-N S_{\text{eff}}(\Sigma, G)}$$

$$S_{\text{eff}}(\Sigma, G) = -\log \text{Pf}(\partial_t - \Sigma) + \frac{1}{2}\int d\tau_1 d\tau_2 \left(\Sigma G - \frac{J^2}{q} G^q\right)$$

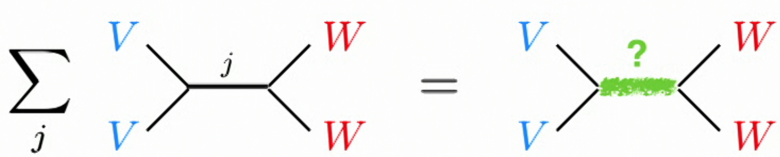
large βJ

$$\overline{Z(J)} \approx \int Df \exp\left[\frac{\alpha N}{\beta J} \int d\tau \left\{\tan \frac{f(\tau)}{2}, \tau\right\}\right]$$

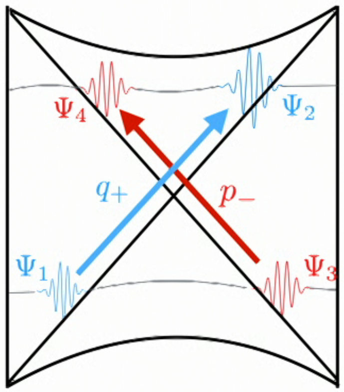
Schwarzian effective action $\longrightarrow$ $\kappa = 2\pi/\beta$

[Sachdev/Ye '93] [Sachdev '10, '15] [Kitaev '15] [Maldacena/Stanford '16]
[Almheiri/Polchinski '14] [Maldacena/Stanford/Yang '16]

- Finite βJ: is there an effective mode for sub-maximal chaos?



Bulk description



$$|\psi_{\text{out}}\rangle \equiv V(T/2)W(-T/2)|\text{TFD}\rangle$$

$$|\psi_{\text{in}}\rangle \equiv W(-T/2)V(T/2)|\text{TFD}\rangle$$

wave functions (bulk-boundary propagators)

$$\text{OTOC} = \langle\psi_{\text{out}}|\psi_{\text{in}}\rangle$$

$$= \int \frac{dq_+}{q_+}\frac{dp_-}{p_-}\,\Psi_1(q_+)\Psi_2(q_+)\Psi_3(p_-)\Psi_4(p_-)\,e^{i\delta(q_+,p_-)}$$

scattering phase

- e.g. AdS3 Einstein gravity: $\delta_{\text{grav}} \propto G_N q_+ p_- e^{T-\mu|\Delta x|}$ (eikonal approx.)

[Dray/'tHooft '85]...[Shenker/Stanford '14]...

- e.g. AdS_3 Einstein gravity:

$$\delta_{\text{grav}} \propto G_N q_+ p_- e^{T-\mu|\Delta x|}$$

- Elastic, tree-level string theory correction:

$$\delta \propto G_N \int dk \, \frac{e^{ik\cdot\Delta x}}{k^2+\mu^2}\left(-i\alpha' q_+ p_- \, e^T\right)^{J(k)-1}$$

$$J(k) = 2 - \alpha'(k^2+\mu^2)/2r_0^2$$

[BPST '07]...[Stanford/Shenker '14]...

- e.g. AdS3 Einstein gravity:

$$\delta_{\text{grav}} \propto G_N q_+ p_- e^{T-\mu|\Delta x|}$$

- Elastic, tree-level string theory correction:

$$\delta \propto G_N \int dk \, \frac{e^{ik\cdot\Delta x}}{k^2+\mu^2}\big(-i\alpha' q_+ p_- \, e^T\big)^{J(k)-1}$$

$$J(k) = 2 - \alpha'(k^2+\mu^2)/2r_0^2$$

[BPST '07]...[Stanford/Shenker '14]...

Large $|\Delta x|/T$: 'graviton' pole $k^2 = -\mu^2$ dominates

$$\delta = \delta_{\text{grav}}$$

Small $|\Delta x|/T$: saddle point dominates

$$\delta \propto G_N \big(-i\alpha' q_+ p_- e^{T-\mu|\Delta x|}\big)^{1-\alpha'\mu^2/2r_0^2} \times (\cdots)$$

Goals

- e.g. AdS3 Einstein gravity:

$$\delta_{\text{grav}} \propto G_N q_+ p_- e^{T-\mu|\Delta x|}$$

- Elastic, tree-level string theory correction:

$$\delta \propto G_N \int dk \, \frac{e^{ik\cdot\Delta x}}{k^2+\mu^2} \left(-i\alpha' q_+ p_- \, e^T \right)^{J(k)-1}$$

$$J(k) = 2 - \alpha'(k^2+\mu^2)/2r_0^2$$

[BPST '07]...[Stanford/Shenker '14]...

Large $|\Delta x|/T$: 'graviton' pole $k^2 = -\mu^2$ dominates

$$\delta = \delta_{\text{grav}}$$

Small $|\Delta x|/T$: saddle point dominates

$$\delta \propto G_N \left(-i\alpha' q_+ p_- e^{T-\mu|\Delta x|} \right)^{1-\alpha'\mu^2/2r_0^2} \times (\cdots)$$

(1) discover this in the Schwarzian theory (SYK at $\beta J \to \infty$, JT gravity)

(3) discover this structure in an SYK chain at finite βJ

Goals

- e.g. AdS3 Einstein gravity:

$$\delta_{\text{grav}} \propto G_N q_+ p_- e^{T-\mu|\Delta x|}$$

(1) discover this in the Schwarzian theory (SYK at $\beta J \to \infty$, JT gravity)

- Elastic, tree-level string theory correction:

$$\delta \propto G_N \int dk \, \frac{e^{ik\cdot\Delta x}}{k^2+\mu^2}\big(-i\alpha' q_+ p_- \, e^T\big)^{J(k)-1}$$

$$J(k) = 2 - \alpha'(k^2+\mu^2)/2r_0^2$$

[BPST '07]...[Stanford/Shenker '14]...

Large $|\Delta x|/T$: 'graviton' pole $k^2 = -\mu^2$ dominates

$$\delta = \delta_{\text{grav}}$$

Small $|\Delta x|/T$: saddle point dominates

$$\delta \propto G_N \big(-i\alpha' q_+ p_- e^{T-\mu|\Delta x|}\big)^{1-\alpha'\mu^2/2r_0^2} \times (\cdots)$$

(2) Intermediate step: single SYK 'site' at finite βJ

$$\delta \propto \frac{\beta J}{N}\big(-i q_+ p_- e^T\big)^v$$

$$(0 < v < 1)$$

(3) discover this structure in an SYK chain at finite βJ

(1) Maximal chaos from reparametrization modes

- Simplest examples: Schwarzian theory, identity block in CFT2

  [Almheiri/Polchinski '14] [Kitaev '15] [Maldacena/Stanford '16]...[Roberts/Stanford '14] [Cotler/Jensen '18] [FH/Rozali '18]

  - Maximal chaos
  - EFT of time reparametrizations $\tau \to f(\tau)$
  - Universal coupling to microscopic operators:

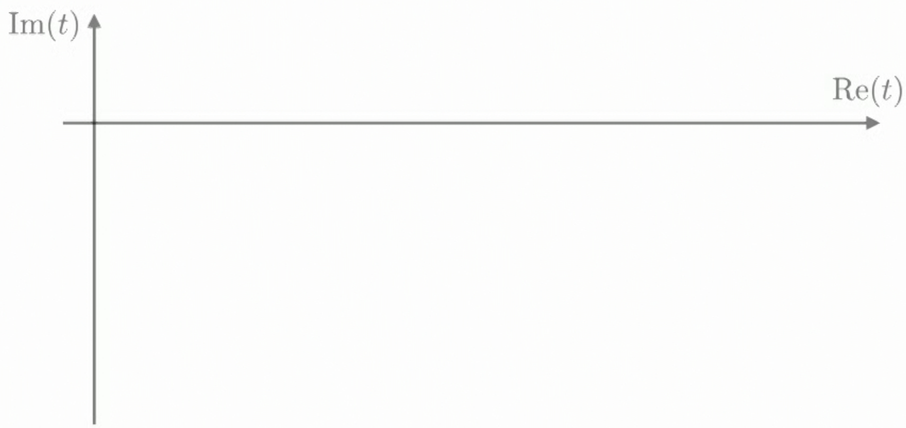
$$\langle V(\tau_1)V(\tau_2)\rangle \longrightarrow \left[\frac{f'(\tau_1)f'(\tau_2)}{(f(\tau_1)-f(\tau_1))^2}\right]^{\Delta_V} \equiv \mathcal{G}_{(f)}^{\Delta_V}(\tau_1,\tau_2)$$

  - Universal contribution to 4pt function:

$$\langle V(\tau_1)V(\tau_2)W(\tau_3)W(\tau_4)\rangle = \int \mathcal{D}f\, e^{-S_{\mathrm{eff}}[f]}\, \mathcal{G}_{(f)}^{\Delta_V}(\tau_1,\tau_2)\, \mathcal{G}_{(f)}^{\Delta_W}(\tau_3,\tau_4)$$

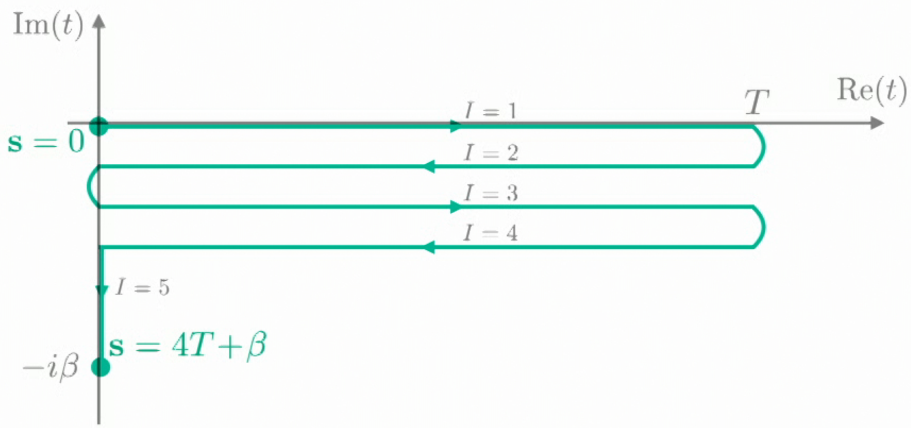
- Maximal chaos due to **interaction of reparametrization zero modes** ('scramblons') dominating the **real-time path integral**

$$\mathrm{OTOC} \equiv \langle W(0)V(T)W(0)V(T)\rangle_\beta = \int Df\, e^{-S_{\mathrm{eff}}[f]}\, \mathcal{G}_{(f)}^{\Delta_V}(T,T)\mathcal{G}_{(f)}^{\Delta_W}(0,0)$$



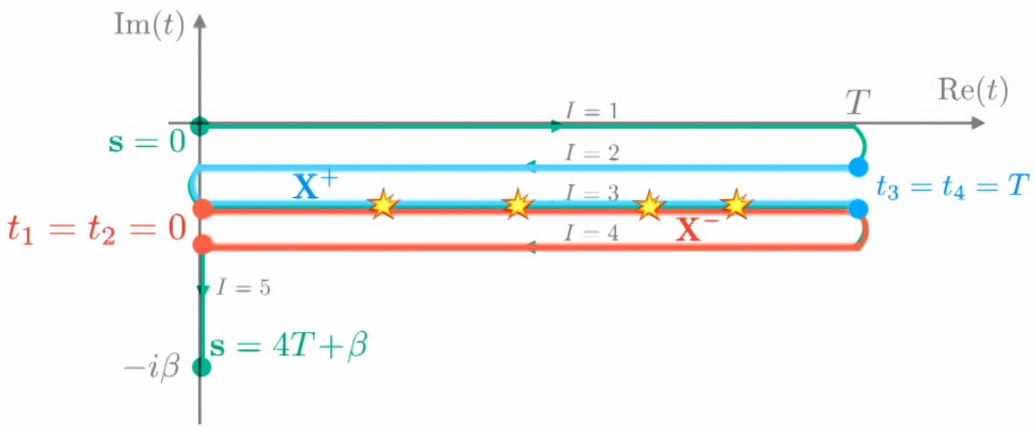
- Maximal chaos due to **interaction of reparametrization zero modes** ('scramblons') dominating the **real-time path integral**

$$\text{OTOC} \equiv \langle W(0)V(T)W(0)V(T)\rangle_\beta = \int Df\, e^{-S_{\text{eff}}[f]}\, \mathcal{G}_{(f)}^{\Delta_V}(T,T)\mathcal{G}_{(f)}^{\Delta_W}(0,0)$$



- Maximal chaos due to **interaction of reparametrization zero modes** ('scramblons') dominating the **real-time path integral**

$$\text{OTOC} \equiv \langle W(0)V(T)W(0)V(T)\rangle_\beta = \int Df\, e^{-S_{\text{eff}}[f]}\, \mathcal{G}^{\Delta_V}_{(f)}(T,T)\mathcal{G}^{\Delta_W}_{(f)}(0,0)$$



- E.g., Schwarzian theory:

$$f_I(t) = \tanh\left(\frac{t+\delta\epsilon_I(t)}{2}\right)$$

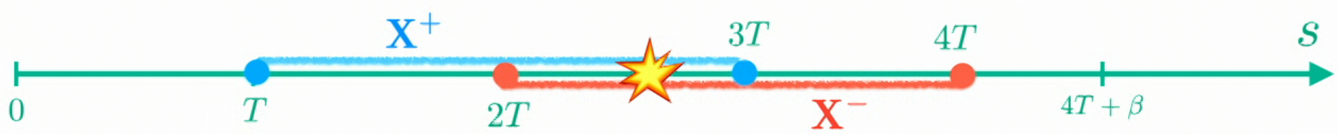
zero modes:
(generated by SL(2,R))

$$\begin{cases} \delta_+\epsilon_{I=2,3} = X^+\, e^{-t} \\ \delta_-\epsilon_{I=3,4} = X^-\, e^{t-T} \end{cases}$$

c.f. [Maldacena/Stanford/Yang '16] [Stanford/Yang/Yao '21] [Gu/Kitaev/Zhang '21]

- Maximal chaos due to **interaction of reparametrization zero modes** ('scramblons') dominating the **real-time path integral**

$$\mathrm{OTOC} \equiv \langle W(0)V(T)W(0)V(T)\rangle_\beta = \int Df\, e^{-S_{\mathrm{eff}}[f]}\, \mathcal{G}^{\Delta_V}_{(f)}(T,T)\mathcal{G}^{\Delta_W}_{(f)}(0,0)$$



- E.g., Schwarzian theory:

$$f_I(t) = \tanh\left(\frac{t+\delta\epsilon_I(t)}{2}\right)$$

zero modes:
(generated by SL(2,R))

$$\begin{cases} \delta_+\epsilon_{I=2,3} = X^+\, e^{-t} \\ \delta_-\epsilon_{I=3,4} = X^-\, e^{t-T} \end{cases}$$

c.f. [Maldacena/Stanford/Yang '16] [Stanford/Yang/Yao '21] [Gu/Kitaev/Zhang '21]

- Maximal chaos due to **interaction of reparametrization zero modes** ('scramblons') dominating the **real-time path integral**

$$\mathrm{OTOC} \equiv \langle W(0)V(T)W(0)V(T)\rangle_\beta = \int Df\, e^{-S_{\mathrm{eff}}[f]}\, \mathcal{G}^{\Delta_V}_{(f)}(T,T)\mathcal{G}^{\Delta_W}_{(f)}(0,0)$$

  - E.g., Schwarzian theory:

$$f_I(t) = \tanh\left(\frac{t+\delta\epsilon_I(t)}{2}\right)$$

zero modes:
(generated by SL(2,R))

$$\begin{cases} \delta_+\epsilon_{I=2,3} = X^+\, e^{-t} \\ \delta_-\epsilon_{I=3,4} = X^-\, e^{t-T} \end{cases}$$

- Interaction of zero modes → saddle point breaks down for $T \sim \log N$

$$S_{\mathrm{eff}}[f] \longrightarrow S_{\mathrm{eik}}[X^+, X^-] = \frac{2\alpha N}{\beta J} e^{-T}\, X^+ X^- + \ldots$$

eikonal approximation

- Exact path integral over $X^\pm$ gives:

$$\mathrm{OTOC} \sim \langle VV\rangle_\beta \langle WW\rangle_\beta \times z^{-2\Delta} U(2\Delta, 1, z^{-1}) \qquad z \propto \frac{e^T}{N}$$

c.f. [Maldacena/Stanford/Yang '16] [Stanford/Yang/Yao '21] [Gu/Kitaev/Zhang '21]

(2)
Large q SYK:
“fake” sub-maximal chaos

(2) Large q SYK: "fake" sub-maximal chaos

- G-Σ action in the **large q limit:** $N \gg q^2 \gg 1\,,\quad \mathcal{J}^2 \equiv \frac{qJ^2}{2^{q-1}} = \text{fix}$

$$G(\tau_1,\tau_2) = G_{\text{free}}(\tau_{12})\left[1 + \frac{1}{q}\,g(\tau_1,\tau_2) + \mathcal{O}(q^{-2})\right]$$

$$S[g] = \frac{N}{4q^2}\int d\tau_1 d\tau_2 \left[\frac{1}{4}\partial_{\tau_1} g\,\partial_{\tau_2} g - \mathcal{J}^2\,e^g\right]$$

[Maldacena/Stanford '16]
[Cotler et al. '17]

- OTOC at $\mathcal{O}(1/N)$ is known for all $\beta\mathcal{J}$ [Streicher '19] [Choi/Mezei/Sarosi '19]

$$\text{OTOC} \sim 1 - \frac{1}{N}\,e^{-\frac{i\pi v}{2}}\,\sec\left(\frac{\pi v}{2}\right)^3 e^{\kappa T} + \dots$$

phase factor
c.f. [Kitaev/Suh '17]

Lyapunov exponent: $\kappa = \frac{2\pi v}{\beta}$

$0 \le v \le 1 \qquad \beta\mathcal{J} = \pi v\,\sec\left(\frac{\pi v}{2}\right)$

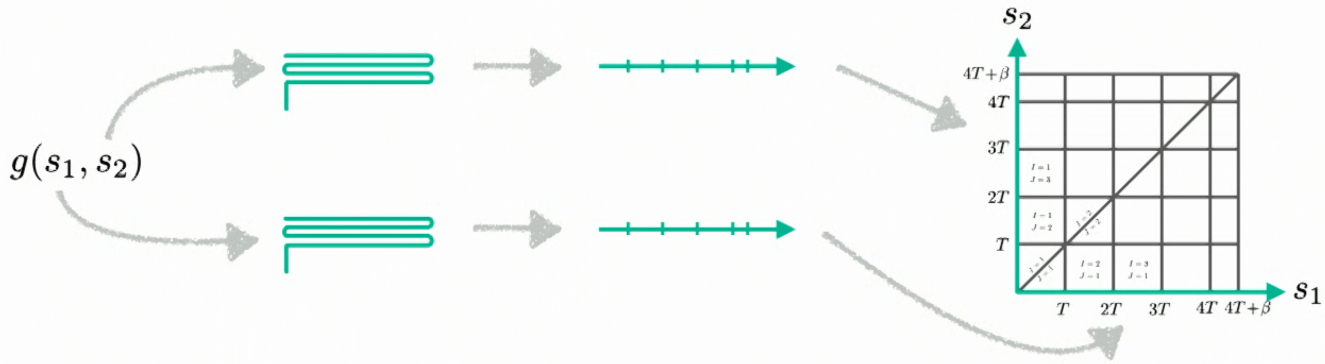
- Eikonal result for OTOC in large q SYK:

$$\text{OTOC} = z^{-2\Delta}\, U(2\Delta, 1, z^{-1}) \qquad z = \frac{1}{4\Delta^2 N}\, \sec\left(\frac{\pi v}{2}\right)^3 e^{-\frac{i\pi v}{2} + vT}$$

[Gu/Kitaev/Zhang '21]
[Choi/FH/Mezei/Sarosi '23]
[Gao/Liu '23]

- Again: comes from interaction of zero modes $\delta_+ g_{IJ}(s_1, s_2)$ & $\delta_- g_{IJ}(s_1, s_2)$ (generated by SL(2,R))

$$S_{\text{eff}}[\delta_+ g, \delta_- g] \longrightarrow S_{\text{eik}}[X^+, X^-] = \frac{N}{2q^2}\, \cos\left(\frac{\pi v}{2}\right) e^{\frac{i\pi v}{2} - vT} X^+ X^-$$



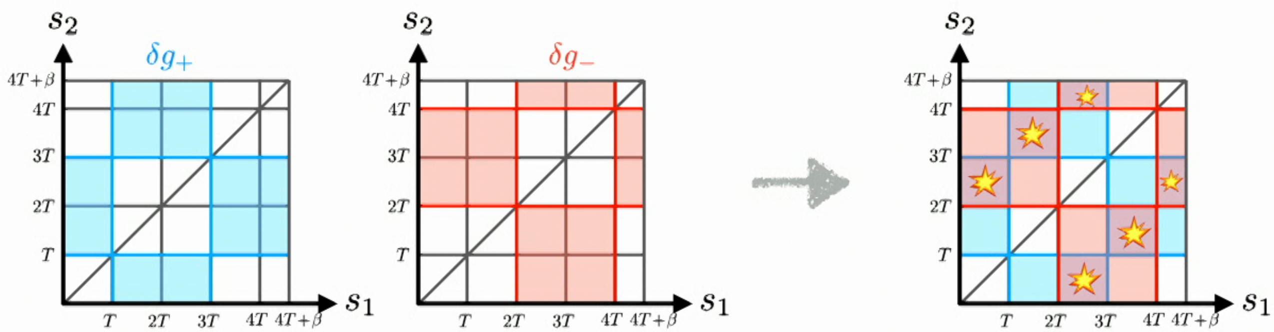
- Eikonal result for OTOC in large q SYK:

$$\mathrm{OTOC} = z^{-2\Delta}\, U(2\Delta, 1, z^{-1}) \qquad z = \frac{1}{4\Delta^2 N}\, \sec\left(\frac{\pi v}{2}\right)^3 e^{-\frac{i\pi v}{2} + vT}$$

[Gu/Kitaev/Zhang '21]
[Choi/FH/Mezei/Sarosi '23]
[Gao/Liu '23]

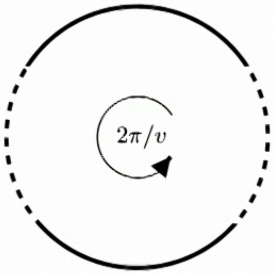
- Again: comes from interaction of zero modes $\delta_+ g_{IJ}(s_1, s_2)$ & $\delta_- g_{IJ}(s_1, s_2)$ (generated by SL(2,R))

$$S_{\text{eff}}[\delta_+ g, \delta_- g] \longrightarrow S_{\text{eik}}[X^+, X^-] = \frac{N}{2q^2}\cos\left(\frac{\pi v}{2}\right) e^{\frac{i\pi v}{2} - vT} X^+ X^-$$



$$\text{OTOC} = z^{-2\Delta}\, U(2\Delta, 1, z^{-1}) \qquad z = \frac{1}{4\Delta^2 N} \sec\left(\frac{\pi v}{2}\right)^3 e^{-\frac{i\pi v}{2} + vT}$$

- A lot like maximal chaos with 'twisted' temperature
  - Extended range of analyticity:
    "fake disk" / "twisted thermal circle"



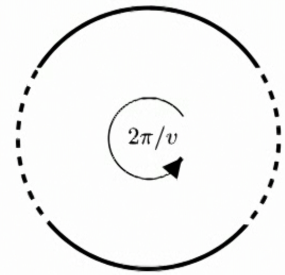
$$\langle e^{g(t_1,t_2)}\rangle = \left[\frac{\cos\left(\frac{\pi v}{2}\right)}{\cosh\left(\frac{v}{2}(t_{12} + i\pi)\right)}\right]^2$$

[Streicher '18]
[Choi/FH/Mezei/Sarosi '23][Gao/Liu '23]
[Lin/Stanford '23]

$$\langle e^{g(t_1,t_2)} e^{g(t_3,t_4)}\rangle = \langle e^{g(t_1,t_2)}\rangle\langle e^{g(t_3,t_4)}\rangle \left[1 - \frac{2}{\cos\left(\frac{\pi v}{2}\right)} \cosh\left(\frac{v}{2}(t_1 + t_2 - t_3 - t_4 + i\pi)\right) + \dots\right]$$

$$\mathrm{OTOC} = z^{-2\Delta}\, U(2\Delta, 1, z^{-1}) \qquad z = \frac{1}{4\Delta^2 N} \sec\left(\frac{\pi v}{2}\right)^3 e^{-\frac{i\pi v}{2} + vT}$$

- A lot like maximal chaos with ‘twisted’ temperature
  - Extended range of analyticity: “fake disk” / “twisted thermal circle”
  - Solutions generated by reparametrizations; preserve ‘twisted’ SL(2,R) symmetry



[Streicher ’18]
[Choi/FH/Mezei/Sarosi ’23][Gao/Liu ’23]
[Lin/Stanford ’23]

general solution to e.o.m.: $$e^{g_{IJ}(s_1,s_2)} = \pm\frac{1}{\mathcal{J}^2}\frac{F'_{IJ}(s_1)G'_{IJ}(s_2)}{(F_{IJ}(s_1) - G_{IJ}(s_2))^2}$$

SL(2,R): $$F_{IJ}(s) \longrightarrow \frac{a\,F_{IJ}(s)+b}{c\,F_{IJ}(s)+d}, \qquad G_{IJ}(s) \longrightarrow \frac{a\,G_{IJ}(s)+b}{c\,G_{IJ}(s)+d}$$

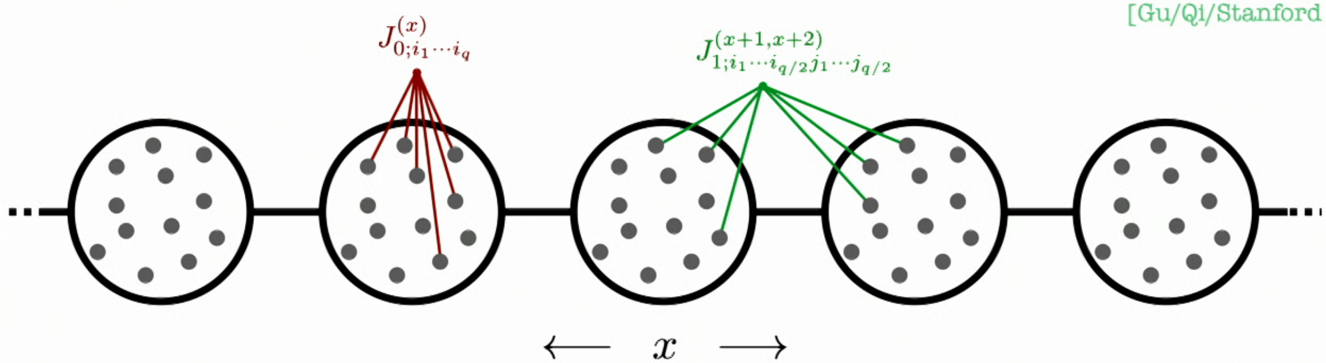
$$\tilde{L}_n^{\mathrm{diag}} = \tilde{L}_n^{(1)} + \tilde{L}_n^{(2)} \qquad \tilde{L}_n^{(a)} = -ie^{-in\tilde{s}_a}\left(\partial_{\tilde{s}_a} - i\Delta n\right) \qquad \tilde{s}_{1,2} \equiv v s_{1,2} \pm \frac{1-v}{2}\pi$$

(3)
Large q SYK chain: stringy effects

Large q SYK chain

$$H = i^{q/2} \sum_{x=0}^{M-1} \left[\sum_{i_1,\dots,i_q} J^{(x)}_{0;i_1\cdots i_q} \chi_{i_1} \cdots \chi_{i_q} + \sum_{\substack{i_1,\dots,i_{q/2} \\ j_1,\dots,j_{q/2}}} J^{(x,x+1)}_{1;i_1\cdots i_{q/2} j_1\cdots j_{q/2}} \chi^{(x)}_{i_1} \cdots \chi^{(x)}_{i_{q/2}} \chi^{(x+1)}_{j_1} \cdots \chi^{(x+1)}_{j_{q/2}} \right]$$

[Gu/Qi/Stanford '16]



- Large q limit:

$$S = \frac{N}{4q^2} \sum_{x=0}^{M-1} \int ds_1 ds_2 \left[\frac{1}{4} \partial_1 g_x \, \partial_2 g_x \pm \mathcal{J}_0^2 \, e^{g_x} \pm \mathcal{J}_1^2 \, e^{\frac{1}{2}(g_x + g_{x+1})} \right]$$

[Choi/Mezei/Sarosi '20]

Large q SYK chain

- Saddle point: *x*-independent, same as for single SYK
- Fluctuations: momentum dependent
  - Not reparametrizations; no SL(2,R)
  - But soft modes still exist and still dominate OTOC path integral:

$$\delta_+ g_{IJ,k}(t_1,t_2) = A_{+,IJ}(k)\, X^+(k) \left[\frac{e^{-v(t_1+t_2)/2}}{\cos\left(\frac{v}{2}(\pi - it_{12})\right)}\right]^{h(k)-1}$$

$$\delta_- g_{IJ,k}(t_1,t_2) = A_{-,IJ}(k)\, X^-(k) \left[\frac{e^{v(t_1+t_2-2T)/2}}{\cos\left(\frac{v}{2}(\pi - it_{12})\right)}\right]^{h(k)-1}$$

$1 \leq h \leq 2$:

$$\frac{h(h-1)}{2} = 1 + \frac{\gamma}{2}\left[\cos k - 1\right]$$

$$\gamma = \mathcal{J}_1^2/(\mathcal{J}_0^2 + \mathcal{J}_1^2)$$

- **Soft mode action:**

$$S_{\rm eik} = \int_{-\pi}^{\pi} dk\, C_{\rm eik}(k) X^+(k) X^-(k)$$

$$C_{\rm eik}(k) = \frac{iN}{2q^2} \frac{e^{\left(\frac{i\pi}{2} - T\right)\kappa}}{\sqrt{\pi}} \cos\left(\frac{\pi\kappa}{2}\right) \cos\left(\frac{\pi\kappa}{v}\right) \Gamma\left(\frac{\kappa}{v}+1\right) \Gamma\left(\frac{1}{2} - \frac{\kappa}{v}\right)$$

$$\kappa(k) \equiv v\big(h(k)-1\big) = v(1 - \tfrac{\gamma}{6}\, k^2 + \ldots)$$

- OTOC:

"stringy" wave functions
(v-deformed bulk-boundary propagators)

$$\mathrm{OTOC} \approx \int \frac{dq_+}{q_+} \frac{dp_-}{p_-}\, \Psi_1^{(v)}(t_1; q_+) \Psi_2^{(v)}(t_2; q_+)\, \Psi_3^{(v)}(t_3; p_-) \Psi_4^{(v)}(t_4; p_-)\, e^{i\delta^{(v)}(q_+, p_-)}$$

$$\delta^{(v)} \propto \int dk\, \frac{e^{ikx}}{iC_{\rm eik}(k)} \big(-iq_+ p_-\big)^{v(h(k)-1)} + (\text{multi-string?})$$

pole at $\kappa = 1$

c.f. twisted thermal circle: **"fake" contribution**

momentum-dependent Regge shift: **"stringy" contribution**

c.f. stringy effects: $\delta \propto G_N \int d^{d-1}k\, \frac{e^{ikx}}{k^2+\mu^2} \left(-i\alpha'\, q_+ p_- e^T\right)^{J(k)-1}$ $\quad J(k) = 2 - \alpha'(k^2+\mu^2)/2r_0^2$

[Shenker/Stanford '14]

Conclusion

Summary

- Path integral computation of OTOC is dominated at times $T \sim \log N$ by an $\mathcal{O}(1)$ 'eikonal' effective action

$$S_{\text{eik}} \sim N \int dk\, e^{\left(\frac{i\pi}{2} - \kappa T\right)} (\cdots) X^{+}(k) X^{-}(-k)$$

  - Origin: interaction of exponentially growing/decaying soft modes

- Special cases: zero modes associated with reparametrizations & symmetry
  - maximal chaos (gravity)
  - sub-maximal chaos due to 'extended thermal circle' (large q SYK)

- Large q SYK chain:
  - velocity-dependent Lyapunov exponent
  - mechanism works as in string theory
  - result suggests 'multi-string' exchanges

Questions

- Use formalism for QFTs in higher dimensions
  - e.g. CFT2: effective description of lightray operators & sub-maximal chaos?
- How to handle ‘multi-string’ terms in SYK?
  - Stringy mechanism?

AN EFFECTIVE FIELD THEORY FOR NON-MAXIMAL CHAOS

Ping Gao

MIT

(a work with Hong Liu, 2301.05256)

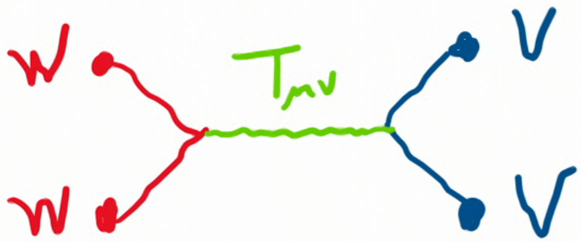
@Perimeter Institute

It from Qubit 2023

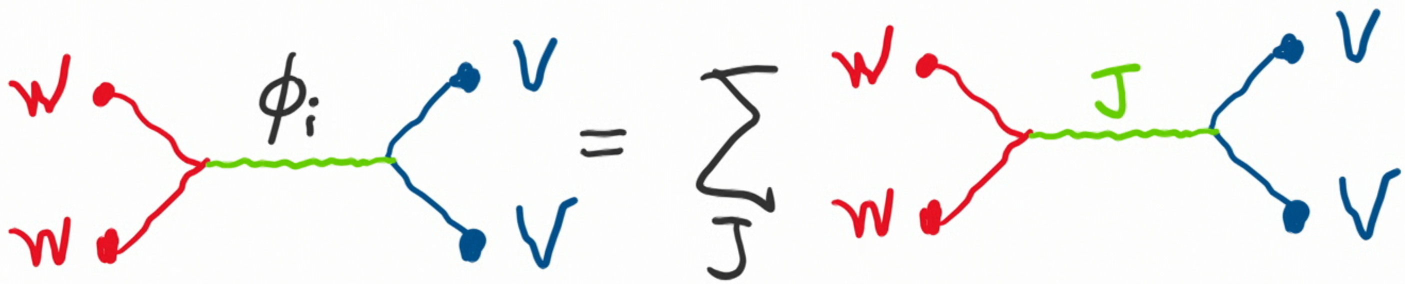
Motivation

Lyapunov exponent: Felix's κ = My λ

- For maximal chaos



- For non-maximal chaos, we would like to construct an EFT in similar structure

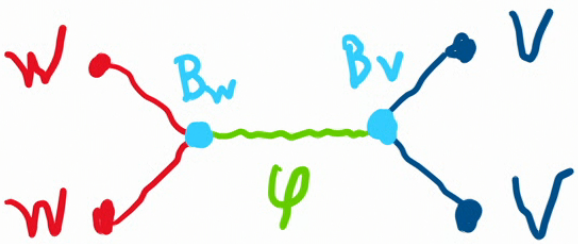


[Costa-Goncalves-Penedones '12]
[Cornalba '07]

An EFT for maximal chaos

A quick review

- Quantum chaos is related to exchange some intermediate modes in the scattering. It is very natural to consider each operator couples to this mode φ.
- Each operator W consists of a bare operator W_0 and a "cloud" of φ in expansion of $W(t) = W_0(t) + L_t[W_0\varphi] + O(\varphi^2)$, L_t is a differential operator.
- Two W fuses into one vertex B: $\langle W(t)W(t')\rangle = g_W(t-t') + B_W(t,t') + O(\varphi^2)$, $B_W(t,t')= L_t[g_W\varphi(t)]+ L_{t'}[g_W\varphi(t')]$
- W_0 and V_0 have no correlation.

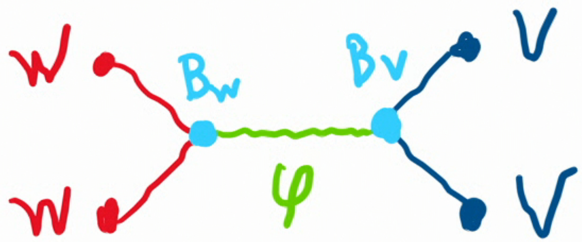


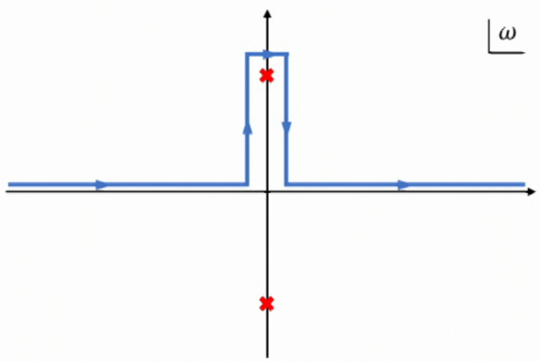
[Blake-Lee-Liu '18]

A quick review

- To have exponential growth, we need to impose shift symmetry $\varphi \to \varphi + ae^{\pm\lambda t}$ for any a for the effective action and also vertex B_W. The action has kernel $\partial_t^2 - \lambda^2$
- One can show that this shift symmetry leads to exponential growth of OTOC and no exponential growth for TOC.
- However, the construction is too restrictive and only compatible with **maximal chaos** $\lambda = 2\pi/\beta$ due to KMS symmetry.

[Blake-Liu '21]



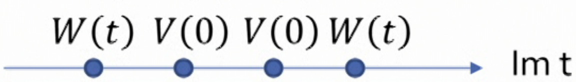


The goals

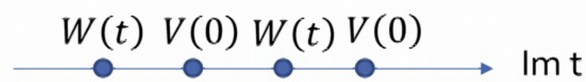
- Four-point function reduces to two-point function of effective modes via vertex. (we do not have a natural candidate for the effective mode, the treatment is phenomenogical)
- Exponential growth like $e^{\lambda t}$ for OTOC with non-maximal λ.
- No exponential growth for TOC
- Compatible with KMS symmetry.

KMS symmetry of 4-pt function

- For a general thermal 4-pt function
$$F_{abcd}(t_1, t_2, t_3, t_4) = Tre^{-2\pi H} O_a(t_1) O_b(t_2) O_c(t_3) O_d(t_4)$$
we have KMS symmetry
$$F_{abcd}(t_1, t_2, t_3, t_4) = F_{bcda}(t_2, t_3, t_4, t_1 + 2\pi i)$$
and it is analytic in the domain $\mathcal{D}$: Im $t_4 - 2\pi <$Im $t_1 <$Im $t_2 <$Im $t_3 <$Im t_4.
- Define $\hat{F}_{WWVV}(t_1, t_2; t_3, t_4) = \langle \mathcal{T} W(t_1) W(t_2) V(t_3) V(t_4) \rangle$
- $\mathcal{T}$ is ascending ordering of imaginary part of time.
- Symmetric under t_1, t_2 swap and t_3, t_4 swap
- For different imaginary part of t_i, it can be TOC or OTOC



TOC



OTOC

Ingredient I: Nonlocality

- In the EFT for maximal chaos, the coupling between bare operator and effective mode is local. The expansion $W(t) = W_0(t) + L_t[W_0\varphi] + O(\varphi^2)$ means that $\varphi(t)$ only knows $W_0(t)$ but ignorant about the locations of other bare operators.
- Locality always leads to maximal chaos even if we have multiple modes.
- Nonlocality might be reasonable since non-maximal chaos is related to stringy effect, which is essentially nonlocal (in the target space).
- Consider the expansion of a pair of operators:

$$W(t)W(t') = W_0(t)W_0(t') + D(t,t')\phi(t,t') + O(\phi^2)$$

- As a consistency, there should be no nonlocality in 4-pt function $\hat{F}_{WWVV}$ (constraints)

Ingredient II: Two modes

- We will keep a minimal amount of nonlocality.
- One way is to consider two modes $\phi_{1,2}$. Each "mainly" couples to one bare operator W_0 but "slightly" couples to the other W_0.
- The "main" coupling can be understood as the local piece and the "slight" coupling can be understood as the nonlocal piece.

$$e^{\sigma} = \frac{f'(\tau_1)g'(\tau_2)}{(f(\tau_1) - g(\tau_2))^2}$$

Large q SYK

Ingredient II: Two modes

- We consider the coupling

$$W(t)W(t') = W_0(t)W_0(t') + W_0(t_L)L_{t_S}[W_0(t_S)\phi_1(\bar{t}\,;t_S)] + W_0(t_S)L_{t_L}[W_0(t_L)\phi_2(\bar{t}\,;t_L)] + O(\phi^2)$$

- t_L=max Im of $\{t, t'\}$, τ_S=min Im of $\{t, t'\}$, $\bar{t} = (t+t')/2$, $L_t[A(t)f(t)] = \sum c_{mn}\partial^m A \partial^n f$
- This coupling guarantees the symmetry of switching two fields W.
- The dependence of ϕ_i on $\bar{t}$ is nonlocal. But we want this dependence weak, which can be formulated in derivative expansion of $\bar{t}$ in both L_t and effective action.
- The differential operator L_t at leading order has no $\bar{t}$ derivative, which is local.
- Each W_0 mainly and locally couples to one ϕ_i, which slightly knows nonlocal information of the other W_0.

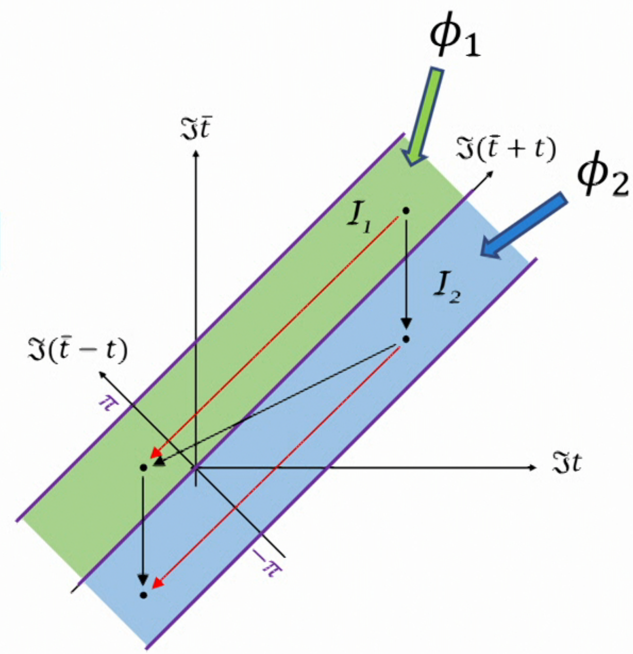
KMS symmetry of 2-pt function

- By definition, $\phi_1(\bar{t}\,; t\,)$ is defined for Im $(\bar{t} - t\,) \in (0, \pi)$; $\phi_2(\bar{t}\,; t\,)$ is defined for Im $(\bar{t} - t\,) \in (-\pi, 0)$.
- Four-point function at quadratic order of ϕ_i

$$\hat{\mathcal{F}}_{WWVV}(t_1, t_2; t_3, t_4) = g_W g_V + \sum_{i,j=1,2} L_{t_i} \tilde{L}_{t_{j+2}} \left[g_W g_V \left\langle \hat{\mathcal{T}} \phi_i(\bar{t}_W; t_i) \phi_j(\bar{t}_V; t_{j+2}) \right\rangle \right]$$

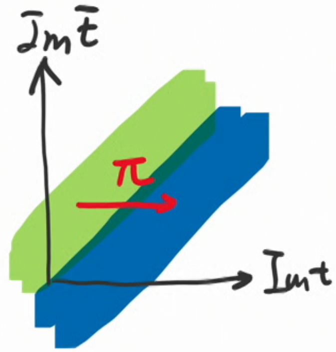
- $\hat{\mathcal{T}}$ is an ordering to be specified later
- KMS symmetry couples two fields

$$\left\langle \hat{\mathcal{T}} \phi_1(\bar{t}; t) \phi_i(\bar{t}'; t') \right\rangle \simeq \left\langle \hat{\mathcal{T}} \phi_2(\bar{t} + \pi i; t + 2\pi i) \phi_i(\bar{t}'; t') \right\rangle,$$
$$\left\langle \hat{\mathcal{T}} \phi_2(\bar{t}; t) \phi_i(\bar{t}'; t') \right\rangle \simeq \left\langle \hat{\mathcal{T}} \phi_1(\bar{t} + \pi i; t) \phi_i(\bar{t}'; t') \right\rangle,$$



Diagonalize KMS

- KMS symmetry relates $\phi_{1,2}$ on the strip $I_1 \cup I_2$. However, the domains on which $\phi_{1,2}$ are defined are different. To construct an EFT, we need to have a domain on which all fields are commonly living.
- Consider two new fields $\eta_{\pm}(\bar{t}\,;t) = \frac{1}{\sqrt{2}}\big(\phi_1(\bar{t}\,;t - i\pi) \pm \phi_2(\bar{t}\,;t)\big)$
- $\eta_{\pm}$ now are both defined on the strip I_2
- $\hat{\mathcal{T}}$ is defined as the imaginary part time ordering of t for $\eta_{\pm}$



$$\left\langle \hat{\mathcal{T}} \eta_s(\bar{t};t)\eta_{s'}(\bar{t}';t')\right\rangle \equiv \begin{cases} \langle \eta_s(\bar{t};t)\eta_{s'}(\bar{t}';t')\rangle\,, & \Im t < \Im t' \\ \langle \eta_{s'}(\bar{t}';t')\eta_s(\bar{t};t)\rangle\,, & \Im t > \Im t' \end{cases} \quad , \quad s, s' = \pm$$

Reformulate KMS

- KMS is diagonal $\left\langle \hat{\mathcal{T}}\eta_s(\bar{t};t)\eta_{s'}(\bar{t}';t')\right\rangle = s\left\langle \hat{\mathcal{T}}\eta_s(\bar{t}+i\pi;t+i\pi)\eta_{s'}(\bar{t}';t')\right\rangle$
- $\eta_\pm$ decouples $\left\langle \hat{\mathcal{T}}\eta_+(\bar{t};t)\eta_-(\bar{t}';t')\right\rangle = -\left\langle \hat{\mathcal{T}}\eta_+(\bar{t};t)\eta_-(\bar{t}';t')\right\rangle = 0$
- The KMS is **unconventional** because it shifts both t and $\bar{t}$. We need to rewrite it as standard KMS.
- Decompose $\eta_s(\bar{\tau};t) = \eta_{s,0}(\bar{\tau};t) + \eta_{s,+}(\bar{t};t) + \eta_{s,-}(\bar{\tau};t)$ with boundary condition

$$\eta_{s,p}(\bar{\tau}+\pi;t) = e^{2\pi i p/3}\eta_{s,p}(\bar{\tau};t), \quad p = 0, \pm$$

- Regarding p as charge. Charge conservation leads to

$$\left\langle \hat{\mathcal{T}}\eta_{s,0}(\bar{\tau};t)\eta_{s,\pm}(0;0)\right\rangle = \left\langle \hat{\mathcal{T}}\eta_{s,+}(\bar{\tau};t)\eta_{s,+}(0;0)\right\rangle = \left\langle \hat{\mathcal{T}}\eta_{s,-}(\bar{\tau};t)\eta_{s,-}(0;0)\right\rangle = 0$$

- Correlations obey ordinary KMS up to a phase

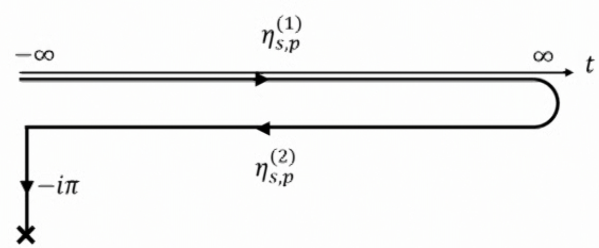
$$\left\langle \hat{\mathcal{T}}\eta_{s,p}(\bar{\tau};t-i\pi)\eta_{s,-p}(0;0)\right\rangle = se^{-2\pi i p/3}\left\langle \hat{\mathcal{T}}\eta_{s,p}(\bar{\tau};t)\eta_{s,-p}(0;0)\right\rangle$$

Schwinger-Keldysh formalism

- Fields are defined on a contour C. On the upper contour the field is $\eta_{s,p}^{(1)}$, and on the lower contour the field is $\eta_{s,p}^{(2)}$.
- To write down the action, we choose a section on $(\bar{t}, t) = (-i\bar{\tau}, \mathbb{R})$
- We can define the a,r variable $\eta_{s,p}^{a} = \eta_{s,p}^{(1)} - \eta_{s,p}^{(2)}$, $\eta_{s,p}^{r} = (\eta_{s,p}^{(1)} + \eta_{s,p}^{(2)})/2$
- Quadratic effective action

$$S_{\mathrm{EFT}} = \sum_{s,p} \int_0^{\pi} d\bar{\tau} \int_{-\infty}^{\infty} dt \left[\eta^a_{s,-p} K^{ar}_{s,p}(\partial_{\bar{\tau}}, \partial_t) \eta^r_{s,p} + \frac{1}{2} \eta^a_{s,-p} K^{aa}_{s,p}(\partial_{\bar{\tau}}, \partial_t) \eta^a_{s,p} \right]$$

$$\left\langle \hat{\mathcal{T}} \eta_{s,p}(\bar{\tau}; t - i\pi) \eta_{s,-p}(0;0) \right\rangle = s e^{-2\pi i p/3} \left\langle \hat{\mathcal{T}} \eta_{s,p}(\bar{\tau}; t) \eta_{s,-p}(0;0) \right\rangle$$



Schwinger-Keldysh formalism

- For simplicity, consider non-dissipation $K^{aa} = 0$ and

$$K^{ar}_{s,p}(\partial_{\bar{\tau}}, \partial_t) = K^{ar}_{s,-p}(-\partial_{\bar{\tau}}, -\partial_t)$$

- Derivative expansion in $\bar{\tau}$, leading order

$$K^{ar}_{s,\pm}(\partial_{\bar{\tau}}, \partial_t) = \partial_{\bar{\tau}} K_{s,\pm}(i\partial_t) + O(\partial^2_{\bar{\tau}})$$

$$K^{ar}_{s,0}(\partial_{\bar{\tau}}, \partial_t) = K_{s,0}(i\partial_t) + O(\partial_{\bar{\tau}})$$

$$K_{s,p}(x) = (-)^p K_{s,-p}(-x)$$

- Leading order action

$$S_{\text{EFT}} = \sum_{s=\pm} \left[\int_{-\infty}^{\infty} dt \eta^a_{s,0}(t) K_{s,0}(i\partial_t) \eta^r_{s,0}(t) + \sum_{p=\pm} \int_0^{\pi} d\bar{\tau} \int_{-\infty}^{\infty} dt \eta^a_{s,-p}(\bar{\tau};t) \partial_{\bar{\tau}} K_{s,p}(i\partial_t) \eta^r_{s,p}(\bar{\tau};t) \right]$$

- K are not completely independent by locality of four-point function F (singularity only at coincidence points).

Shift symmetry

- We will assume shift symmetry of the action

$$\eta_-^r \to \eta_-^r + \alpha_+ e^{\lambda t} + \alpha_- e^{-\lambda t}$$

- This means that at least one of $K_{-,p}$ has a factor of $\partial_t^2 - \lambda^2$
- By smoothness condition, generally we also require factor of $\partial_t^2 - \lambda^2$ in $K_{+,p}$
- Take ansatz $K_{s,p}(i\partial_t) = (\partial_t^2 - \lambda^2) k_{s,p}(i\partial_t)$
- Shift symmetry in terms of ϕ_i: $(\phi_1, \phi_2) \to (\phi_1, \phi_2) + (e^{\pm\lambda(t+i\pi)}, -e^{\pm\lambda t})$
- Vertex invariance: $L_{t_1}[g_W(t_{12}) e^{\pm\lambda(t_1+i\pi)}] = L_{t_2}[g_W(t_{12}) e^{\pm\lambda t_2}]$
- Additional condition: $G_+^{rr}(\bar{\tau}; t) = \langle \eta_+^r(\bar{\tau}; t) \eta_+^r(0; 0) \rangle = 0$ up to non-exponential terms

OTOC and TOC

- Define $F_4 = \langle W(t_1)V(t_3)W(t_2)V(t_4)\rangle\,, \quad G_4 = \langle V(t_3)W(t_1)W(t_2)V(t_4)\rangle$
- F4 is OTOC and G4 is TOC
- Using the quadratic correlation of effective modes, we have

$$
\begin{aligned}
F_4 - G_4 &= \frac{1}{2}\sum_{s,p} L_{t_1}\tilde{L}_{t_3}\left[g_W g_V \langle[\eta_{s,p}(i\bar{t}_W;t_1),\eta_{s,-p}(i\bar{t}_V;t_3)]\rangle\right] \\
&= L_{t_1}\tilde{L}_{t_3}\left[g_W g_V \Delta(t_{13})\right]
\end{aligned}
$$

Retarded function

- It has exponential growth!
- For G4, by the shift symmetry of vertex, we can show

$$G_4 \propto L_{t_1}\tilde{L}_{t_3}G_+^{rr}(t_{13}) = 0e^{\pm\lambda t_{13}}$$ (by the condition we impose for G_+^{rr})

A few remarks

- The OTOC F4 can be written in a symmetric way

[Kitaev-Suh '17]
[Gu-Kitaev-Zhang '21]

$$F_4 = \alpha e^{\lambda(t_1+t_2-t_3-t_4+i\pi)/2} \frac{G^W_{even}(\lambda, t_{12}) G^V_{even}(-\lambda, t_{34})}{\cosh\frac{\lambda(t_{12}+i\pi)}{2}\cosh\frac{\lambda(t_{34}+i\pi)}{2}} + (\lambda \leftrightarrow -\lambda)$$

- The phase $\lambda\pi/2$ is crucial for non-maximal chaos because it makes the following four-point function real

$$\mathrm{Tr}\left[e^{-\pi H/2} W(t_1) e^{-\pi H/2} V(0) e^{-\pi H/2} W(t_2) e^{-\pi H/2} V(0)\right]$$

- Taking maximal chaos limit $\lambda \to 1$, we find $\eta_{s,\pm}$ decouples. We only have p=0 components. Then one can show that two effective fields $\phi_{1,2}$ become identical and reduce to just one mode.

- The last term includes λ flipped term predicts exponential growth in the past.

A few remarks

- Unlike the EFT for maximal chaos, our construction is consistent with the KMS symmetry of vertex (and avoids reducing to maximal chaos)
- In large q SYK model, one can identify two "double reparameterization" modes as the two effective modes.

$$e^{\sigma} = \frac{f'(\tau_1)g'(\tau_2)}{(f(\tau_1) - g(\tau_2))^2}$$

(from Felix's talk)

- Our EFT is also consistent with conformal regge theory and stringy scattering in the bulk.

[Shenker-Stanford '14]
[Mezei-Sarosi '19]
[Costa-Goncalves-Penedones '12]
[Cornalba '07]

Exponentiation

- Here we only considered the diagonal shift symmetry for the vertex at linear order of ϕ_i.
- One can consider the same shift symmetry for all orders. This assumption leads to the full OTOC with an exponentiation form of the leading order OTOC by summing over all $(e^{\lambda t}/N)^k$ order terms.
- Full OTOC $F(t_1, t_2; t_3, t_4) = \langle W(t_1)V(t_3)W(t_2)V(t_4)\rangle$ with $t_{1,2} \gg t_{3,4}$ has the following form

$$F(t_1, t_2; t_3, t_4) = \int_0^\infty dy \int_0^\infty dy'\, e^{Xyy'} h_W(t_{12}, y) h_V(t_{34}, y')$$

$$X = C/N e^{\lambda(t_1+t_2-t_3-t_4+i\pi)/2}$$

[Shenker-Stanford '14]
[Gu-Kitaev-Zhang '21]
[Choi-Haehl-Mezei-Sarosi '23]

where X is leading order OTOC, C is a constant, h is "wave function".
This has a form of stringy scattering amplitude near AdS black hole horizon.

Summary

- We proposed an EFT for non-maximal chaos. It has a few nontrivial ingredients:
  1, a minimal nonlocality and derivative expansion,
  2, two effective modes that are related by KMS symmetry,
  3, constraints to the action from the locality of four-point function
  4, shift symmetry for both action and vertex, G_{+}^{rr} has no exponential growth
- By this EFT, we can show that OTOC has exponential growth and TOC does not.
- The OTOC has a string-scattering-like exponentiation formula

Questions

- How to find these two modes from a generic chaotic CFT?
- How to find these two modes from the computation of pomeron? [Brower-Polchinski-Strassler-Tan '06]
- Spatial dependence? SYK chain? Butterfly effect? Pole-skipping? [Blake-Lee-Liu '18] [Mezei-Sarosi '19] [Gu-Qi-Stanford '16]
- Relation to light-ray operator? [Kravchuk-Simmons-Duffin '18]
- What is the bulk meaning of shift symmetry?
- Relation to the bulk dual of double-scaled SYK? [Lin-Stanford '23]