Title: Type I Von Neumann algebras from bulk path integrals: RT as entropy without AdS/CFT

Speakers: Donald Marolf

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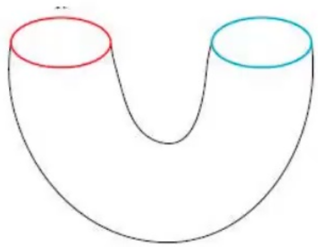
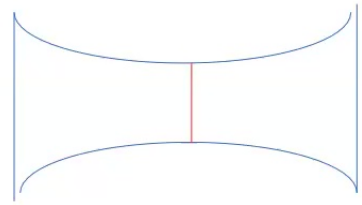
Type I von Neumann algebras from bulk Path Integrals: RT as entropy w/o AdS/CFT

Donald Marolf (UCSB)

Work with Eugenia Colafranceschi, Xi Dong, and Zhencheng Wang

I. Motivation from Ryu-Takayanagi





Consider a familiar TFD-like setup for discussing Ryu-Takayanagi Entropy

$$S_{RT} = \frac{A}{4G} + \ \dots$$

If there is a dual CFT, we understand there to be a Hilbert space $\mathcal{H}_B \otimes \mathcal{H}_{\bar{B}}$, and S_{RT} describes the entropy of the associated density matrix on either factor.

How much of this story can we derive from the bulk without invoking AdS/CFT?

Two recent developments suggest we might be able to do quite a bit:

1) Page curve story where B is a non-gravitating bath.
2) Chandrasekaran-Longo-Penington-Witten & P-W type II algebras and their entropy

Without invoking AdS/CFT, both give entropies that can be computed from RT (in an appropriate limit for #2).

II. Axioms for a (Euclidean) bulk path integral

What do we mean by a bulk theory?
We would like to work at finite couplings; i.e., *not* in the semiclassical limit.

We will assume that some clever person has given us a UV-completion of some bulk quantum gravity theory which contains something we can call a Euclidean path integral. It does not matter if this is actually anything like a sum over geometries. However, we assume it to satisfy certain axioms.

II. Axioms for a (Euclidean) bulk path integral

1) We assume there is some space X^d of smooth manifolds with smooth fields that define boundary conditions for our bulk path integral. As a result, the path integral defines a function

$$\zeta: X^d \to \mathbb{C}$$

2) (Reality) $$[\zeta(M)]^* = [\zeta(M^*)]$$

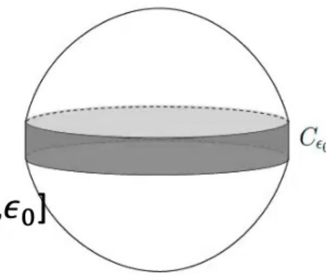
3) (Reflection positivity) $\zeta(M) > 0$ when M is appropriately reflection-symmetric.

4) (Continuity under changing the length of a cylinder)

$$lim_{\epsilon \to \epsilon_0} \zeta(M_\epsilon) = \zeta(M_{\epsilon_0})$$

$C_{\epsilon_0} \subset M_{\epsilon_0}$

$C_{\epsilon_0} = B \times [0, \epsilon_0]$



5) (Factorization) $\zeta(M_1 \sqcup M_2) = \zeta(M_1)\, \zeta(M_2)$

All of these of course hold if there is a "dual" CFT, but the above are much weaker than the full axioms for a CFT! How much of the structure of a CFT do we really need to derive something like R-T and to show it to be an entropy?

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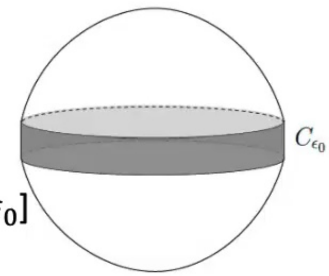


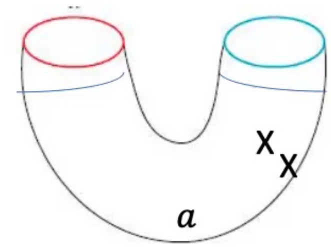
5) (Factorization) $\zeta(M_1 \sqcup M_2) = \zeta(M_1)\,\zeta(M_2)$

Comment on Factorization Axiom: It would be nicer to *derive* factorization, or to derive *"effective factorization"* (baby universe superselection sectors, aka α-sectors, so that the full path integral decomposes into a sum over ζ's that factorize). But this is hard due to issues associated with unbounded operators.
I will try to comment further on this below.

III. The Hilbert space sector $\mathcal{H}_{B \sqcup \bar{B}}$

B $\bar{B}$

$C = B \times [0,\epsilon_1]$ $\bar{C} = \bar{B} \times [0,\bar{\epsilon}_1]$



Consider *rimmed* manifolds $a, b \in Y^d_{B \sqcup B}$ with boundary $B \sqcup \bar{B}$.
Call the space of such rimmed manifolds $Y^d_{B \sqcup \bar{B}}$

B $\bar{B}$

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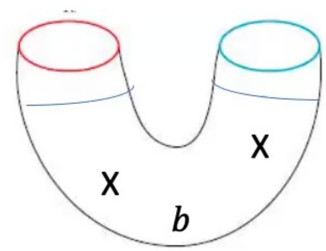
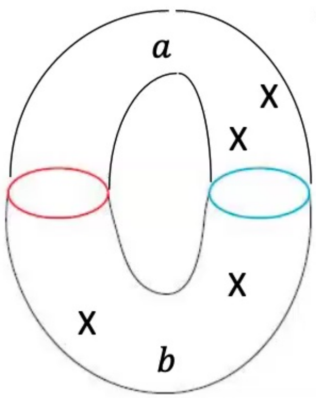


There exist states $|a>, |b> \in \mathcal{H}_{B \sqcup \bar{B}}$ and their inner product is

$$< a|b > = \zeta(M_{a^* b})$$



Remarks:

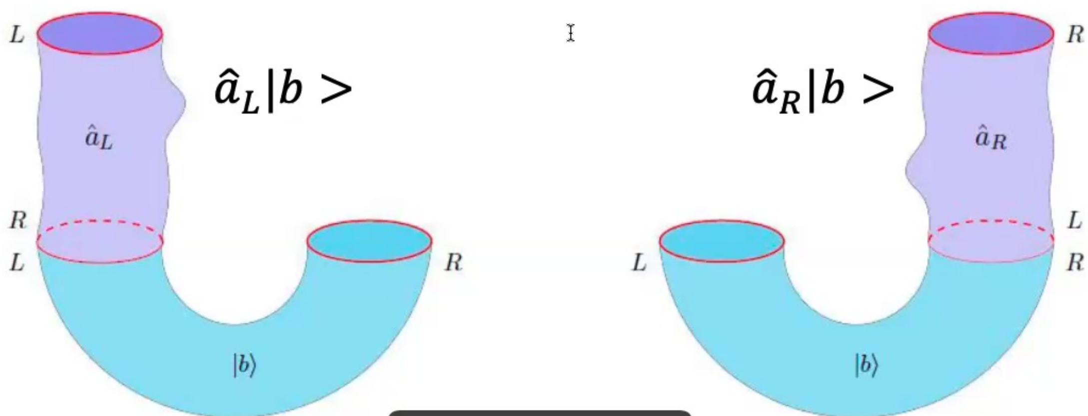
1) Points on $B \sqcup \bar{B}$ should be thought of as being labelled so that the gluing is unique.

2) We compute only inner products of states with matching rims (same B's).

IV. Operators, Algebras and Traces

Let us specialize to the diagonal case $B \sqcup B$ and consider two rimmed surfaces $a, b \in Y^d_{B \sqcup B}$. Here the left- and right- boundaries are distinguished, so we shall call them B_L, B_R (or just L,R), even though they are identical.

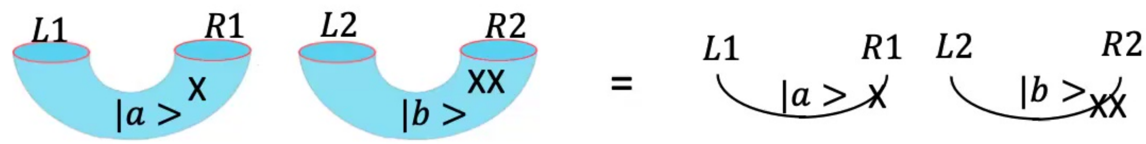
Then we can define operators $\hat{a}_L$, $\hat{a}_R$ on $\mathcal{H}_{B \sqcup B}$ via



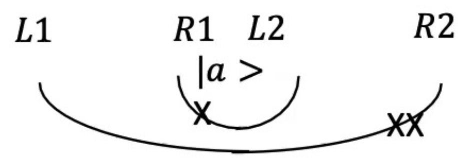
IV. These operators are bounded!

Let us use $a, b \in Y^{d}_{B\sqcup B}$ to define states in $\mathcal{H}_{B\sqcup B\sqcup B\sqcup B}$.
Call the boundaries $B_{L1}, B_{R1}, B_{L2}, B_{R2}$, and consider the following states

$|a_{L1R1}, b_{L2R2}>$ =

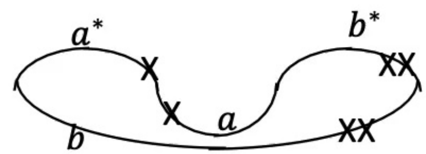


$|a_{L2R1}, b_{L1R2}>$ =



$$Norms: < a_{L2R1}, b_{L1R2} | a_{L2R1}, b_{L1R2} > = < a_{L1R1}, b_{L2R2} | a_{L1R1}, b_{L2R2} > = < a|a >< b|b >$$

$Inner\ Product: < a_{L1R1}, b_{L2R2} | a_{L2R1}, b_{L1R2} > =$



$$= < b^* \left| \hat{a}_L^{\dagger} \hat{a}_L \right| b > = \ real\ \&\ positive =: < a_{L2R1}, b_{L1R2} | a_{L1R1}, b_{L2R2} >$$

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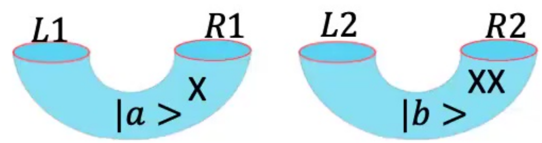

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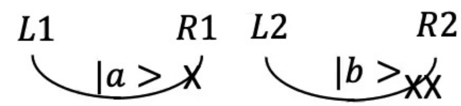

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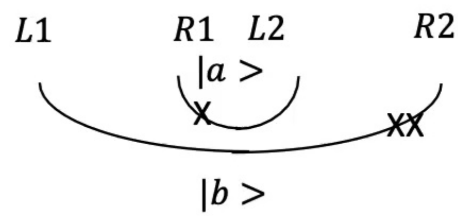

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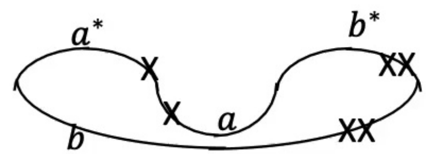

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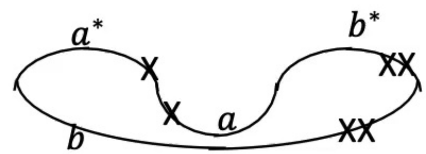
IV. These operators are bounded!

Apply Cauchy-Schwarz: $|<c|d>| \leq \sqrt{<c|c><d|d>}$

$$<b^* \left| \hat{a}_L^\dagger \hat{a}_L \right| b> \leq \sqrt{(<a|a><b|b>)^2} = <a|a><b|b>$$

i.e., $\widehat{||a_L^\dagger} \hat{a}_L|| \leq$ < a|a >, so our operators are bounded!

$$Norms: <a_{L2R1}, b_{L1R2}|a_{L2R1}, b_{L1R2}> = <a_{L1R1}, b_{L2R2}|a_{L1R1}, b_{L2R2}> = <a|a><b|b>$$

$Inner\ Product: <a_{L1R1}, b_{L2R2}|a_{L2R1}, b_{L1R2}> =$ 

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V. Von Neumann Algebras and traces

The left- and right- representations on $\mathcal{H}_{B\sqcup B}$ can now be completed to define von Neumann algebras $\mathcal{A}_L, \mathcal{A}_R$.

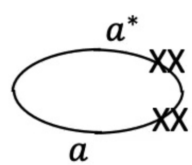
These algebras have useful traces!

For a *surface* $a, Y^d_{B\sqcup B}$, define $tr(a^*a) \;:= < a|a > = sup_\epsilon < C_\epsilon \left|\hat{a}_L^\dagger \hat{a}_L\right| C_\epsilon > /||C_\epsilon||^2$

Traces on vN algebras need only be defined on positive elements, of the form a^*a.

$$< a|a >< b|b > \;\geq\; < b^* \left|\hat{a}_L^\dagger \hat{a}_L\right| b > = < b^*\, a^*|ab >$$

$tr(a^*a)\; tr(b^*b) \geq tr(b^*a^*ab)$ (*) "The Trace Inequality"

$tr(a^*a)$ = 

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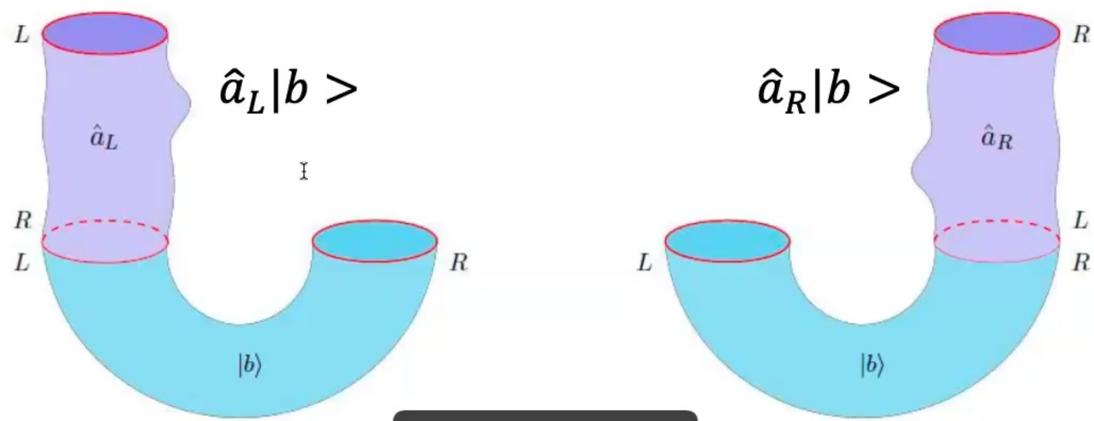

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And now the magic….. The final form $sup_\epsilon < C_\epsilon \left|\hat{a}^\dagger_L \hat{a}_L\right| C_\epsilon > /||C_\epsilon||^2$ is well-defined on the vN algebra (and gives a faithful, normal, semi-finite trace).

Also, when $tr(a^*a)$ is finite, there is still a state $|a>$ of norm $tr(a^*a)$. So C-S argument still works, and (*) in fact still holds on $\mathcal{A}_L, \mathcal{A}_R$. (Thanks, Xi!)

V. Von Neumann Algebras and traces

Theorem Review (familiar from other talks…):

- Every vN algebra is a direct sum of "vN factors" (vN algebras with trivial centers)
- Every vN factor is of type I, II, or III.
- There is no (faithful, normal, semifinite) trace on a type III factor.
  So our factors are not type III!!!!

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  So our factors are not type III!!!!
- For any such trace on a type II factor, there are non-trivial projections P with arbitrarily small trace. But this is not allowed by our trace inequality:

$$tr(a^*a)\, tr(b^*b) \geq tr(b^*a^*ab) \qquad (*) \qquad \text{“The Trace Inequality”}$$

For $a = b = P$ w/ $P^2 = P$ we find

$$tr(a^*a)\, tr(b^*b) = [tr(P)]^2 \geq tr(b^*a^*ab) = tr(P)$$

$$tr(P) = 0 \quad \Longrightarrow \quad P = 0$$

Otherwise, $tr(P) \geq 1$

Our algebras are not type II and so must be type I!!!!

VI. Quick summary of further results

The (von Neumann or Rieffel) bicommutant theorem then tells us that $\mathcal{A}_L, \mathcal{A}_R$ are commutants.

Consider the central subalgebras $\mathcal{Z}_L \subset \mathcal{A}_L$, where $\mathcal{Z}_L = \{z\colon\ az = za\ \forall a \in \mathcal{A}_L\}$, and similarly $\mathcal{Z}_R \subset \mathcal{A}_R$. Note that $\mathcal{Z}_L = \mathcal{Z}_R = \mathcal{Z}$. Diagonalizing $\mathcal{Z}$ yields eigenspaces $\mathcal{H}^z_{B\sqcup B}$ preserved by $\mathcal{A}_L, \mathcal{A}_R$ such that

$$\mathcal{H}_{B\sqcup B} = \int_\oplus dz\ \mathcal{H}^z_{B\sqcup B}\ .$$

Thus $\mathcal{A}_L = \int_\oplus dz\ \mathcal{A}^z_L$

where $\mathcal{A}^z_L$ annihilates $\mathcal{H}^{z'}_{B\sqcup B}$ for $z \neq z'$ and $\mathcal{A}^z_{B\sqcup B}$ has trivial center.
It is thus a type I von Neumann *factor* with commutant $\mathcal{A}^z_R$ on $\mathcal{H}^z_{B\sqcup B}$.
So,

$$\mathcal{H}^z_{B\sqcup B} = \mathcal{H}^z_{B\sqcup B,L} \otimes \mathcal{H}^z_{B\sqcup B,R}$$

Quantization of $tr(P)$ then requires that for each z there is an $n_z \in \mathbb{Z}^+$ such that for

$$\widetilde{\mathcal{H}}^z_{B\sqcup B,L} = \mathbb{C}^{n_z} \otimes \widetilde{\mathcal{H}}^z_{B\sqcup B,L}$$

And $a \in \mathcal{A}^z_L$ we have $tr(a^*a) = \widetilde{Tr}_z\big(1_{\mathbb{C}^{n_z}} \otimes a^*a\ \big)$, where is the usual Hilbert space trace on $\widetilde{\mathcal{H}}^z_{B\sqcup B,L}$ summing diagonal matrix elements over an O.N. basis: $\widetilde{Tr}_z(A) \coloneqq \sum_i\ < i|a|i >$

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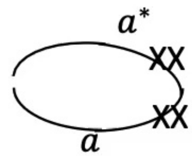
Thus for a density matrix ρ_z on $\widetilde{\mathcal{H}}^{z}_{B\sqcup B,L}$ we have

$$S(\rho_z) \coloneqq -tr(\rho_z \ln \rho_z) = -\widetilde{Tr}_z(\rho_z \ln \rho_z) =: \tilde{S}_z(\rho_z)$$

Furthermore, for a *surface* $a, Y^d_{B\sqcup B}$ that defines a state $|a>$, we can define $\rho = (a^* a)$,

$\rho =$ 

so that $tr(\rho^n)$ is computed via "the gravitational replica trick."

Thus by LM, if there is a limit described by semiclassical bulk gravity, then

$S(\rho\quad) = \frac{A}{4G} + \ \ldots$ (RT!)

But from above, if p_z give the probabilities for the state $|a>$ to be in the sector $\mathcal{H}^{Z}_{B\sqcup B}$, we have

$$S(\rho) = \int dz \ \ p_z \widetilde{S}_z(\widetilde{\rho}_z) \qquad \text{for } \widetilde{\rho}_z = 1_{\mathbb{C}^{n_z}} \otimes \rho_z$$

VII. Summary

Assume finite on smooth boundaries, reality, reflection positivity, continuity, and factorization.
Then

- There is a bulk Hilbert space associated with boundary $B \sqcup B$
- There are right and left von Neumann algebras that act on this Hilbert space.
- They are type I! This means that they contain only type I factors.
- As a result, the Hilbert space is a sum of of terms that factorize

$$\mathcal{H}_{B \sqcup B} = \int_{\oplus} dz \ \mathcal{H}^{z}_{B \sqcup B}$$

$$\mathcal{H}^{z}_{B \sqcup B} = \mathcal{H}^{z}_{B \sqcup B, L} \otimes \mathcal{H}^{z}_{B \sqcup B, R}$$

- Augmenting this construction with finite-dimensional maximally-entangled "hidden sectors" then gives a standard "state counting" interpretation of the RT entropy.

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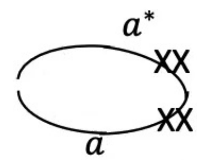

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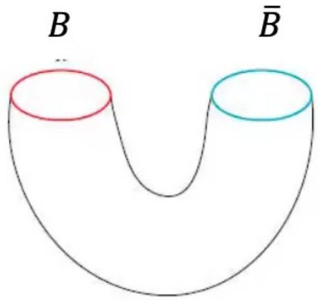
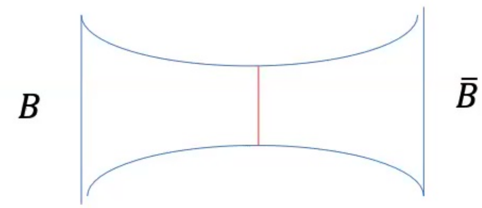
I. Motivation from Ryu-Takayanagi





Consider a familiar TFD-like setup for discussing Ryu-Takayanagi Entropy

$$S_{RT} = \frac{A}{4G} + \ \dots$$

If there is a dual CFT, we understand there to be a Hilbert space $\mathcal{H}_B \otimes \mathcal{H}_{\bar{B}}$, and S_{RT} describes the entropy of the associated density matrix on either factor.

How much of this story can we derive from the bulk without invoking AdS/CFT?
Two recent developments suggest we might be able to do quite a bit:

1) Page curve story where B is a non-gravitating bath.
2) Chandrasekaran-Longo-Penington-Witten & P-W type II algebras and their entropy

Without invoking AdS/CFT, both give entropies that can be computed from RT (in an appropriate limit for #2).

VI. Quick summary of further results

The (von Neumann or Rieffel) bicommutant theorem then tells us that $\mathcal{A}_L, \mathcal{A}_R$ are commutants.

Consider the central subalgebras $\mathcal{Z}_L \subset \mathcal{A}_L$, where $\mathcal{Z}_L = \{z: \ az = za \ \forall a \in \mathcal{A}_L\}$, and similarly $\mathcal{Z}_R \subset \mathcal{A}_R$. Note that $\mathcal{Z}_L = \mathcal{Z}_R = \mathcal{Z}$. Diagonalizing $\mathcal{Z}$ yields eigenspaces $\mathcal{H}^z_{B\sqcup B}$ preserved by $\mathcal{A}_L, \mathcal{A}_R$ such that

$$\mathcal{H}_{B\sqcup B} = \int_{\oplus} dz \ \mathcal{H}^z_{B\sqcup B} \ .$$

Thus $$\mathcal{A}_L = \int_{\oplus} dz \ \mathcal{A}^z_L$$

where $\mathcal{A}^z_L$ annihilates $\mathcal{H}^{z'}_{B\sqcup B}$ for $z \neq z'$ and $\mathcal{A}^z_{B\sqcup B}$ has trivial center.
It is thus a type I von Neumann *factor* with commutant $\mathcal{A}^z_R$ on $\mathcal{H}^z_{B\sqcup B}$.
So,

$$\mathcal{H}^z_{B\sqcup B} = \mathcal{H}^z_{B\sqcup B,L} \otimes \mathcal{H}^z_{B\sqcup B,R}$$

Quantization of $tr(P)$ then requires that for each z there is an $n_z \in \mathbb{Z}^+$ such that for

$$\mathcal{H}^z_{B\sqcup B,L} = \mathbb{C}^{n_z} \otimes \widetilde{\mathcal{H}}^z_{B\sqcup B,L}$$

And $a \in \mathcal{A}^z_L$ we have $tr(a^*a) = \widetilde{Tr}_z\big(1_{\mathbb{C}^{n_z}} \otimes a^*a\big)$, where is the usual Hilbert space trace on $\widetilde{\mathcal{H}}^z_{B\sqcup B,L}$ summing diagonal matrix elements over an O.N. basis: $\widetilde{Tr}_z(A) := \sum_i \ < i|a|i >$