Title: A symmetry algebra in double-scaled SYK

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A symmetry algebra in double-scaled SYK

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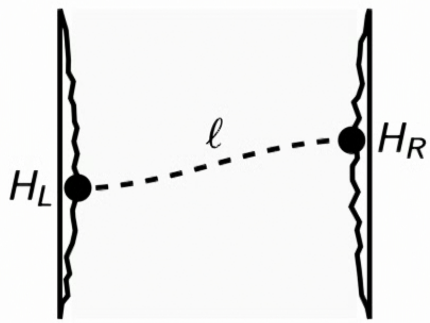
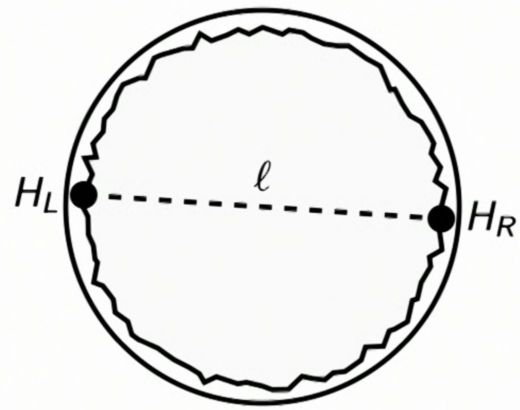
Based on a recent paper with Henry Lin and discussions with Zhenbin Yang.

JT algebra

Consider the operators H_L, H_R, ℓ in JT gravity plus matter on an interval:



Lorentzian



Euclidean

These operators generate an algebra [Harlow/Wu, Lin]

$$[H_L, H_R] = 0, \qquad [H_L, \ell] = \mathrm{i}k_L, \qquad [k_L, \ell] = \mathrm{i}, \qquad [k_L, k_R] = 0$$

$$[k_L, H_L] = H_L - k_L^2 \qquad [k_L, H_R] = e^{-\ell}$$

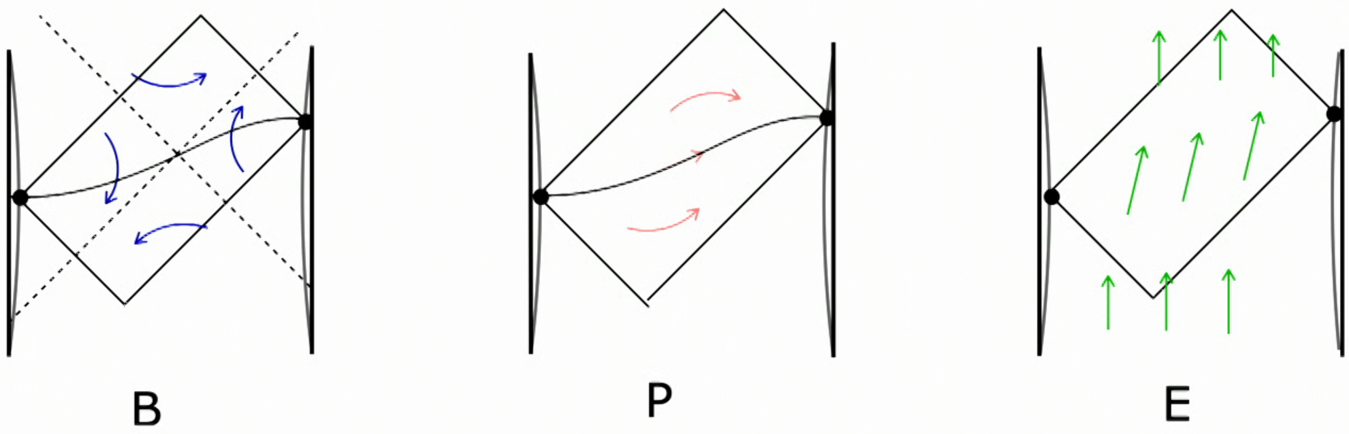
This is actually a very simple algebra in disguise. If we define

$$B = e^{\ell/2}[H_L - H_R, [H_L + H_R, \ell]]$$
$$E = -e^{\ell/2}[H_L - H_R, [H_L - H_R, \ell]]$$
$$P = \mathrm{i}[B, E]$$

then these form an $\mathfrak{sl}_2$ subalgebra and the entire algebra is simply the universal enveloping algebra of

$$\text{Heisenberg} \times \mathfrak{sl}_2.$$

The first factor is generated by ℓ and $[H_L + H_R, \ell]$ and the second by [Lin/Maldacena/Zhao]



This algebra determines lots of important stuff in JT gravity [Lin/Maldacena/Zhao, Harlow/Wu], like the chaos properties (maximal Lyapunov exponent), the traversable wormhole experiment, . . .

We will discuss a **deformation of this algebra** that arises in double-scaled SYK, and that reduces to the JT algebra in a low-temperature limit.

The algebra has practical applications simplifying some SYK computations.

Conceptually, it represents a deformation of something that appeared to be quite rigid: the near-horizon Poincare symmetry. Also

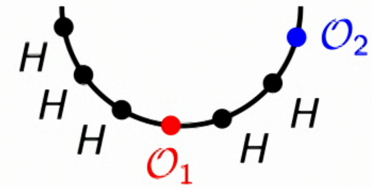
1. The deformed algebra is compatible with a type of discrete geometry.
2. In a semiclassical limit the discreteness goes away, and the algebra reduces to $\mathfrak{sl}_2$ with a nonstandard action that determines the sub-maximal Lyapunov exponent.

Let's start by explaining qualitatively where the algebra comes from.

Consider a general quantum mechanical system and label two-sided states in the way David Kolchmeyer did in his talk on Monday:

$$|H^3\mathcal{O}_1 H^2 \mathcal{O}_2\rangle = H^3\mathcal{O}_1 H^2 \mathcal{O}_2 \otimes 1|\text{MAX}\rangle$$

$$=$$



Labeling states this way makes it easy to act with H_L and H_R:

$$H_L|H^3\mathcal{O}_1 H^2 \mathcal{O}_2\rangle = |H^4\mathcal{O}_1 H^2 \mathcal{O}_2\rangle$$
$$H_R|H^3\mathcal{O}_1 H^2 \mathcal{O}_2\rangle = |H^3\mathcal{O}_1 H^2 \mathcal{O}_2 H\rangle$$

To get the algebra we are interested in, we also need an analog of the third operator ℓ in JT gravity.

For the analog of ℓ, we choose a multiple of the *size operator* [Roberts/DS/Streicher, Streicher/Qi]:

$$s|\mathbb{I}\rangle = 0 \qquad s|\psi_1\rangle = |\psi_1\rangle \qquad s|\psi_{17}\psi_2\rangle = 2|\psi_{17}\psi_2\rangle \qquad s|\psi_1\psi_{17}\psi_1\rangle = |\psi_1\psi_{17}\psi_1\rangle$$

The special feature of double-scaled SYK is that H_L, H_R and the size operator generate a fairly simple algebra.

To explain this further, we need to define double-scaled SYK and a more subtle labeling of the Hilbert space than the one on the last slide.

Double-scaled SYK

We take the limit $p \to \infty$ and N to infinity in the SYK model

$$\{\psi_i, \psi_j\} = 2\delta_{ij},$$

$$H = \mathrm{i}^{p/2} \sum_{1 \le i_1 < \cdots < i_p \le N} J_{i_1 \ldots i_q} \psi_{i_1} \cdots \psi_{i_p}, \quad \left\langle J^2_{i_1 \ldots i_p} \right\rangle = \frac{\mathcal{J}^2}{\lambda \binom{N}{p}}$$

holding fixed

$$\lambda \equiv 2p^2/N, \quad q \equiv \exp(-\lambda).$$

The model has two dimensionless parameters:

- λ controls the semiclassical-ness. $\lambda \ll 1 \implies$ small fluctuations
- $\beta\mathcal{J}$ acts as a sort of coupling, where high temperature is more weakly coupled and low temperature is more strongly coupled.

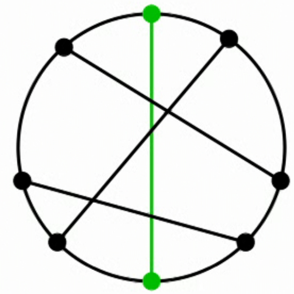
Define matter operators by a similar formula

$$\mathcal{O}_s = \mathrm{i}^{s/2} \sum_{1 \le i_1 < \cdots < i_s \le N} K_{i_1 \ldots i_s} \psi_{i_1} \ldots \psi_{i_s}, \qquad \langle K^2_{i_1 \ldots i_s} \rangle = \frac{1}{\binom{N}{s}}$$

[Erdos/Schroder (spins), CGHPSSSST, Berkooz/Isachenkov/Narovlansky/Torrents]

Correlation functions are given by a sum over all possible "chord diagrams" that describe the Wick contractions of the $J_{i_1 \ldots i_p}$ and $K_{i_1 \ldots i_s}$ couplings:

$$\mathrm{tr}\big(H^3 \mathcal{O} H^3 \mathcal{O}\big) \supset$$



We get a factor of q for each intersection between Hamiltonian chords, and a factor of q^{Δ} for intersections between Hamiltonian chords and matter chords ($\Delta = s/p$), so the above evaluates to

$$q^2 (q^{\Delta})^3$$

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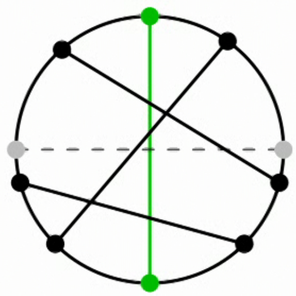
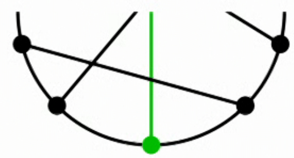
[Erdos/Schroder (spins), CGHPSSSST, Berkooz/Isachenkov/Narovlansky/Torrents]

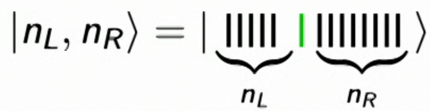
One can cut open these chord diagrams to get a Hilbert space
[Berkooz/Isachenkov/Narovlansky/Torrents, Lin]



$\longrightarrow$



$= \; |\,|||\,\rangle \; = \; |1, 1\rangle$



Different from previous labeling scheme, because some of the operators might have contracted:

$$|H^2\mathcal{O}H^2\rangle = \text{ superposition of } |0,0\rangle, |0,2\rangle, |2,0\rangle, |1,1\rangle, |2,2\rangle.$$

States $|n_L, n_R\rangle$ have definite size: $\frac{s}{p} = \bar{n}$ where the "chord number" is:

$$\bar{n}|n_L, n_R\rangle = (n_L + \Delta + n_R)|n_L, n_R\rangle.$$

We define ℓ as the rescaled chord number = rescaled size operator:

$$\ell = \lambda \bar{n} \qquad \text{discrete!}$$

So we now have a collection of states where the ℓ operator acts simply. The H_L and H_R operators also act fairly simply on these states, so we can compute their algebra. It is convenient to write

$$H_L = \mathfrak{a}_L^\dagger + \mathfrak{a}_L$$

where the two terms correspond to creating or annihilating a chord. Then

$$[\bar{n}, \mathfrak{a}_i^\dagger] = \mathfrak{a}_i^\dagger, \quad [\bar{n}, \mathfrak{a}_i] = -\mathfrak{a}_i$$
$$\mathfrak{a}_i \mathfrak{a}_i^\dagger - q \mathfrak{a}_i^\dagger \mathfrak{a}_i = 1 \quad \text{(no sum)}$$

So far this is the algebra of two "q-deformed Harmonic oscillators." But the two oscillators are not quite independent:

$$[\mathfrak{a}_L, \mathfrak{a}_R] = [\mathfrak{a}_L^\dagger, \mathfrak{a}_R^\dagger] = 0$$
$$[\mathfrak{a}_L, \mathfrak{a}_R^\dagger] = [\mathfrak{a}_R, \mathfrak{a}_L^\dagger] = q^{\bar{n}}.$$

This is the most "beautiful" way we found to write the algebra, but one can also write it out in terms of H_L, H_R, k_L, k_R, ℓ and it looks similar to the JT algebra with a couple of extra terms on the RHS. At low temperatures these extra terms vanish.

In JT gravity, the most useful part of the algebra is the $\mathfrak{sl}_2$ subalgebra. Does that have an analog?

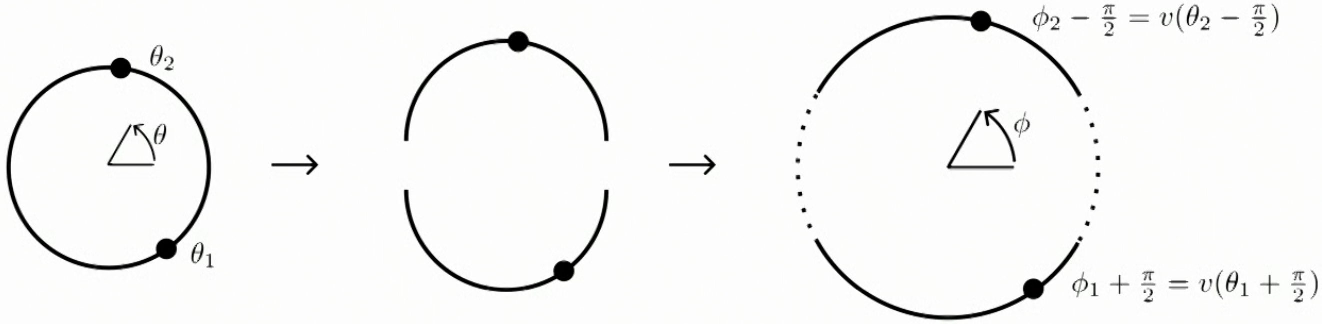
One can use exactly the same formulas as in the JT case and define

$$B = e^{\ell/2}[H_L - H_R, [H_L + H_R, \ell]]$$
$$E = -e^{\ell/2}[H_L - H_R, [H_L - H_R, \ell]]$$
$$P = \mathrm{i}[B, E]$$

These operators generate a subalgebra that commutes with $\ell = \lambda \bar{n}$. This generalizes the $\mathfrak{sl}_2$ algebra from the JT gravity case. Proper subalgebra of the "quantum algebra" $U_{\sqrt{q}}(\mathfrak{sl}_2)$.

In the semiclassical limit $\lambda \to 0$, the algebra of B, E, P simplifies to $\mathfrak{sl}_2$, but with a strange action on the boundary that can be explained by introducing the "fake circle."

The fake circle



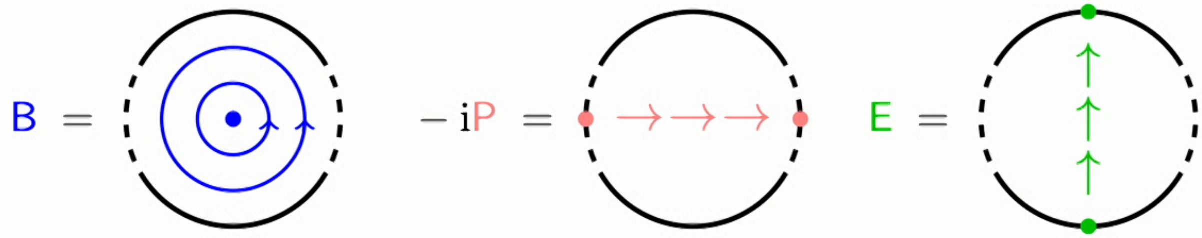
Here $0 < v < 1$ is a determined by the temperature:

$$\frac{\pi v}{\beta} = \cos \frac{\pi v}{2}.$$

which is the only parameter remaining in the limit $\lambda \to 0$.

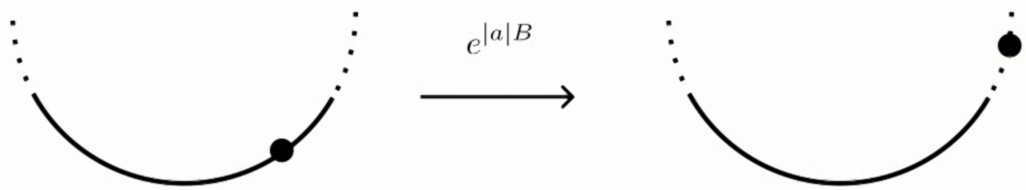
The generators act as

$$\langle\mathcal{O}(\theta_2)\mathsf{B}\mathcal{O}(\theta_1)\rangle = \partial_{\phi_1}\langle\mathcal{O}(\theta_2)\mathcal{O}(\theta_1)\rangle$$
$$\langle\mathcal{O}(\theta_2)\mathsf{E}\mathcal{O}(\theta_1)\rangle = (\cos(\phi_1)\partial_{\phi_1} - \Delta\sin\phi_1)\langle\mathcal{O}(\theta_2)\mathcal{O}(\theta_1)\rangle$$
$$\langle\mathcal{O}(\theta_2)\mathsf{P}\mathcal{O}(\theta_1)\rangle = \mathrm{i}(\sin(\phi_1)\partial_{\phi_1} + \Delta\cos\phi_1)\langle\mathcal{O}(\theta_2)\mathcal{O}(\theta_1)\rangle$$

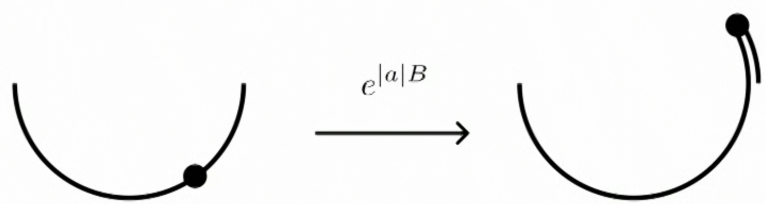


What do the fake portions of the fake circle mean?

If ϕ starts out in the physical region, can act with a symmetry generator to leave the physical region:



If we don't introduce the fake circle, we can alternatively describe this using a Euclidean timefold:



What does this state look like in the chord Hilbert space?

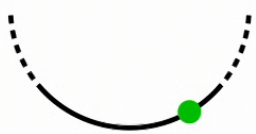
We can start out at finite λ with the exact formulas for the B, E, P generators, and with a state that is a superposition

$$\sum_{n_L} \psi(n_L)|n_L, n - n_L\rangle$$

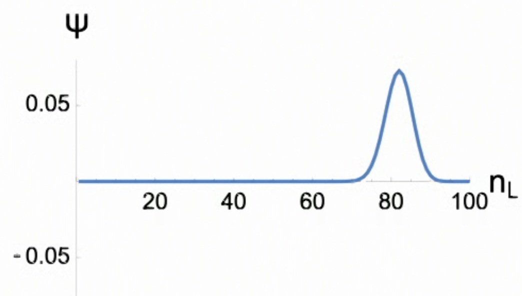
Start with the wave function where the particle is all the way to the right:

$$\leftrightarrow \qquad |\text{III}\ldots\text{IIII}\rangle \qquad = \qquad |n, 0\rangle$$

Evolving back into the physical region we get a smooth wave function:

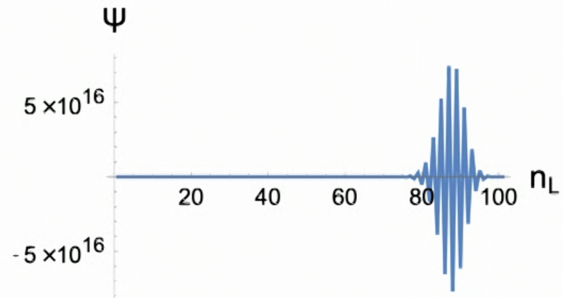


$\leftrightarrow$



Evolving into the fake region, we get a crazy, numerically large, oscillating wave function

$\leftrightarrow$



The $\lambda \to 0$ limit is a continuum limit and one might have thought we could ignore such states. But we need to keep them in order to have representations of the symmetry algebra. The fake portion of the boundary gives a smooth way to parametrize them.

Application: chaos from order

By expanding the exact formulas for B, E to first order around the thermofield double state, one finds the following

$$H_R - H_L \approx \frac{2\pi v}{\beta} B$$
$$\ell - \langle \ell \rangle \approx c_1 E + c_2 (H_L + H_R - 2\langle H \rangle)$$

Let's focus on the first equation for the moment. The $\mathfrak{sl}_2$ algebra implies that under the adjoint action, the B operator has eigenvalues $\pm\mathrm{i}$, with eigenvectors given by

$$\mathrm{i}[B, E \pm P] = \mp (E \pm P).$$

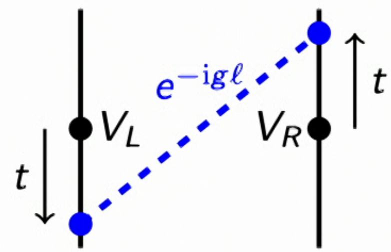
This means that under time evolution by $H_R - H_L$, the $E + P$ operator grows exponentially

$$e^{\mathrm{i}t(H_R - H_L)}(E + P)e^{-\mathrm{i}t(H_R - H_L)} = e^{\frac{2\pi v}{\beta} t}(E + P).$$

This reproduces that $\frac{2\pi v}{\beta}$ is the Lyapunov exponent [Maldacena/DS]. It also identifies $E + P$ as the "Pomeron" or "scramblon" operator.

Application: traversable wormhole

One can use these formulas to analyze the traversable wormhole experiment of [Gao/Jafferis/Wall]. We want to compute



$$= \quad \langle V_L | e^{\mathrm{i}t(H_R - H_L)} e^{-\mathrm{i}g\ell} e^{-\mathrm{i}t(H_R - H_L)} | V_R \rangle$$

The key thing is to evaluate the time evolution of the ℓ operator:

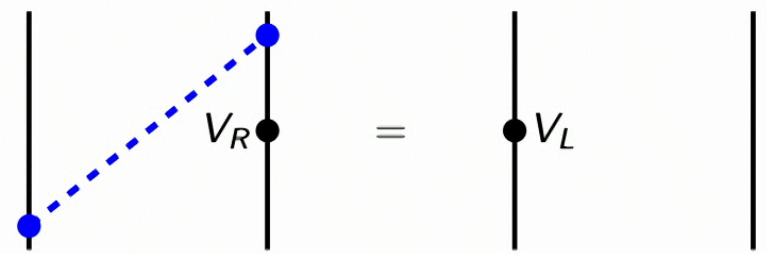
$$e^{\mathrm{i}t(H_R - H_L)} \ell e^{-\mathrm{i}t(H_R - H_L)} = \frac{c_1}{2}(E + P) e^{\frac{2\pi v}{\beta} t} + \text{non-growing}$$

Now substitute this in...

$$\langle V_L | e^{\mathrm{i}t(H_R - H_L)} e^{-\mathrm{i}g\ell} e^{-\mathrm{i}t(H_R-H_L)} | V_R \rangle = \langle V_L | e^{-\mathrm{i}a(E+P)} | V_R \rangle$$
$$= \left[\frac{\cos \frac{\pi v}{2}}{1 + \mathrm{i}\frac{a}{2} e^{\mathrm{i}\frac{\pi}{2} v}} \right]^{2\Delta_V}$$

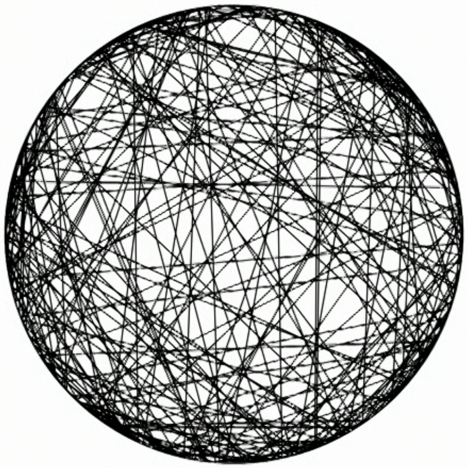
This reproduces a formula from [Gao/Jafferis] in the large p SYK model. In going to the second line we used the formula for E, P as $\mathfrak{sl}_2$ generators acting on boundary operator positions.

It turns out that the maximum of the magnitude of this, over t, is exactly one. So for this one special value of t



So the traversable wormhole works as well as possible, at any temperature. Becuase the scramblon is a symmetry generator!

Chords vs. geometry



Summary

1. JT gravity has a symmetry algebra Heisenberg $\times\mathfrak{sl}_2$ that includes the near-horizon Poincare symmetries
2. In double-scaled SYK there is a deformation of that algebra that acts on a discrete Hilbert space
3. In the semiclassical limit it contains an $\mathfrak{sl}_2$ subalgebra that acts on an extended "fake" thermal circle
4. The algebra can be used to simplify various computations