Title: Talk 74 - The Riemann Zeta Function, Poincare Recurrence, and the Spectral Form Factor

Speakers: Michael Winer

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Abstract: The Spectral Form Factor is an important diagnostic of level repulsion Random Matrix Theory (RMT) and quantum chaos. The short-time behavior of the SFF as it approaches the RMT result acts as a diagnostic of the ergodicity of the system as it approaches the thermal state. In this work we observe that for systems without time-reversal symmetry, there is a second break from the RMT result at late times: specifically at the Heisenberg Time $T_H=2\pi \rho$. That is to say that after agreeing with the RMT result to exponential precision for an amount of time exponential in the system size, the spectral form factor of a large system will very briefly deviate in a way exactly determined by its early time thermalization properties. The conceptual reason for this is the Riemann-Siegel Lookalike formula, a resummed expression for the spectral determinant relating late time behavior to early time spectral statistics. We use the lookalike formula to derive a precise expression for the late time SFF for semiclassical systems, and then confirm our results numerically. We find that at late times, the various modes act on the SFF via repeated, which may give hints as to the analogous behavior for systems with time-reversal symmetry.

The Spectral Form Factor: From Thermalization to the Riemann Zeta Function

Mike Winer

Brian Swingle

Victor Galitski

Plan For The Talk

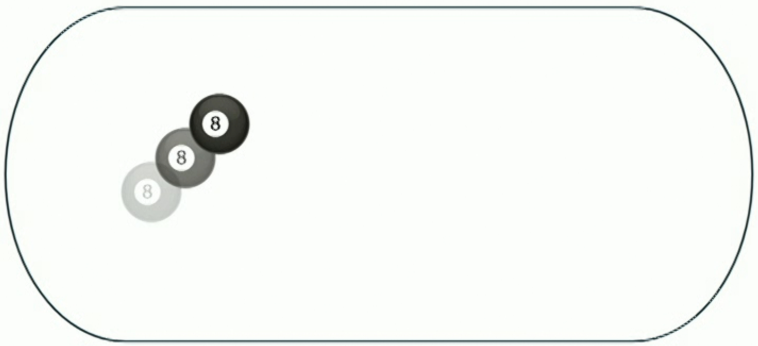
1. Two Perspectives on Quantum Chaos
2. The SFF as a diagnostic of thermalization
3. The Total Return Probability
4. The SFF is suppressed at the recurrence time
5. Riemann-Siegel and its lookalike
6. A formula for late-time SFF

The Problem of Quantum Thermalization

- System starts in particular states with particular values for operators. How does it relax to thermal average values?
- Ergodicity: Over long enough time, system samples entire state space
- The problem of timescales: time to sample all of Hilbert space= Poincare recurrence time~Heisenberg time.
- Time to equilibrate much smaller

Fast Quantum Relaxation

- Quantum ball bouncing around billiard table
- Trajectories diverge exponentially
- Localized wavepacket quickly fills entire table



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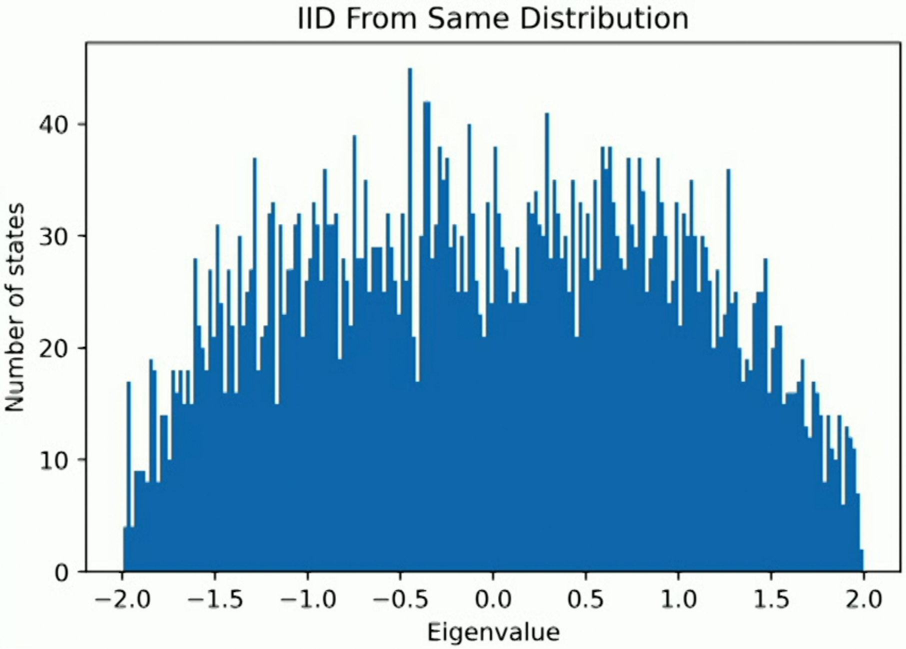
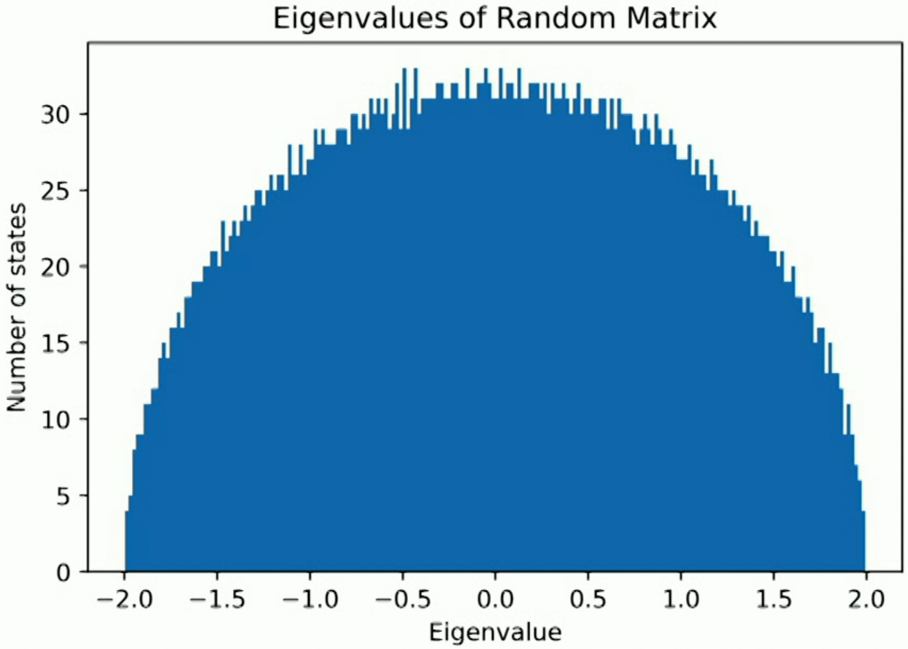
Level Repulsion Diagnoses Quantum Chaos





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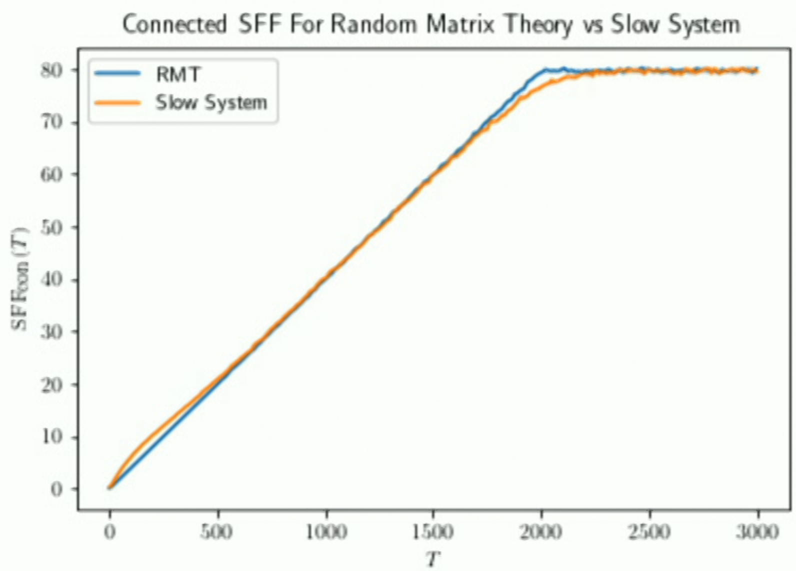
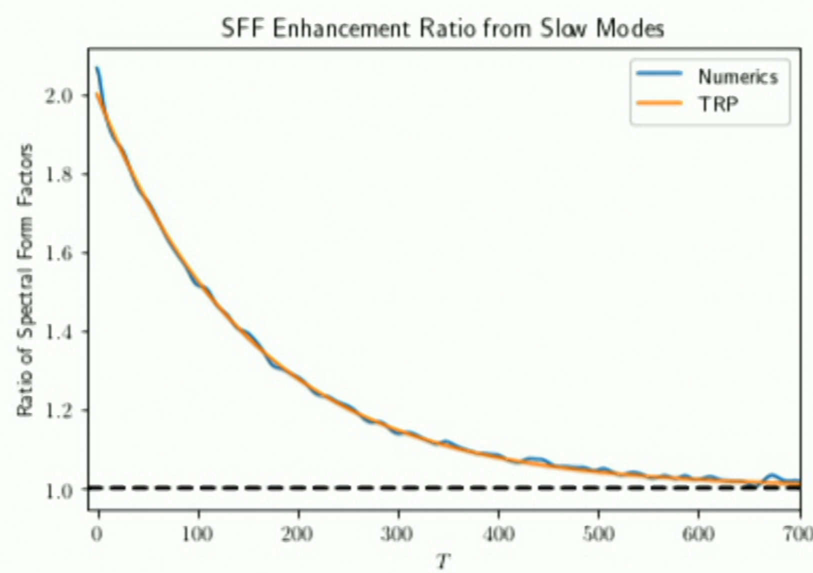
The SFF Is Enhanced by the Total Return Probability





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The SFF Sum Rule

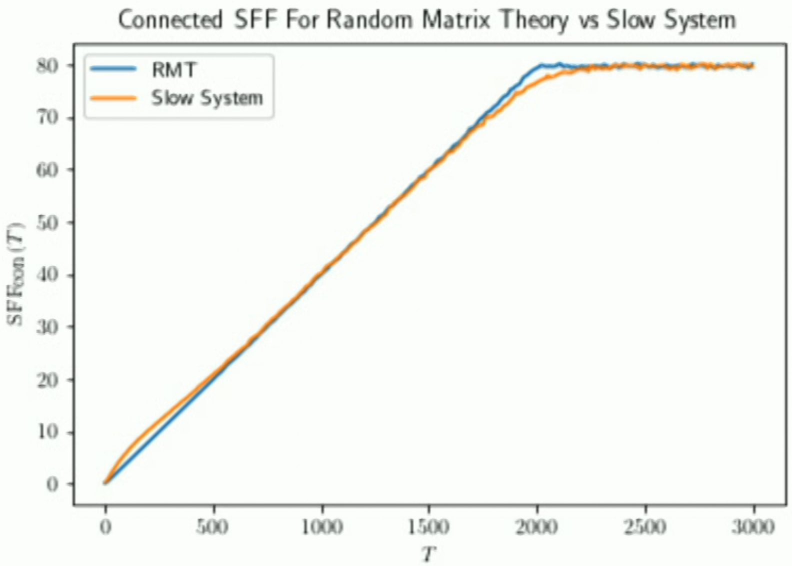
$$\int dT \, \mathrm{SFF}(T) = \int dT \, N$$

Average value doesn't depend on detailed dynamics!

Holds when $\overline{\lim_{\omega \to 0^+} \rho(E)\rho(E+\omega)} = 0$

A (Nearly) Universal Constraint on the SFF

- Sum Rule means total area in the early-time gap equals the total area in the late-time gap
- Applies in wide variety of systems
  - Nearly-Block RMT
  - Rosenzweig-Porter
  - Spin Glass



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The Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

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Introduction to the Zeta Function

Riemann ζ function is important in complex analysis and number theory

Most mathematicians think all the interesting zeroes of ζ have real part ½- important implications for people who care about math

The Riemann-Siegel Formula

$$Z(t) = e^{i\theta(t)}\zeta(\frac{1}{2}+it) = \sum_{n=1}^{\infty} \frac{e^{i(\theta - t\log n)}}{\sqrt{n}}$$

θ is defined so that Z is a real function

$$\theta(t) = \arg\left(\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\right) - \frac{\log\pi}{2}t$$

Quantum Chaos and the Riemann Hypothesis

In 80s and 90s Michael Berry and Jon Keating argued that the zeroes of the ζ function are like energy levels of chaotic system

Used ζ function techniques to study quantum chaos

The Secular Determinant

Unitary U captures time evolution. Function A captures spectral statistics

$$A(\varphi) = \det(1 - e^{i\varphi}U) = \sum_{k=0}^{N} A_k e^{ik\varphi}$$

$$U = e^{-iHt}$$

Braun Haake 1206.4222

What Are the A Coefficients?

$$A_k = (-1)^k \sum_{\text{subsets of size } k} \text{product of } k \text{ eigenvalues}$$

A_k related to spectral statistics at time kt

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The Key Ingredient

$$A_k = A_N A_{N-k}^*$$

Both sides are the sum of $\binom{N}{k}$ terms.

The Quantum Z Function

$$Z(\varphi) = \sqrt{A_N^*} e^{-iN\varphi/2} A(\varphi) = \sum_{n=0}^{N} \sqrt{A_N^*} A_n e^{i(n-N/2)\varphi}$$

The Quantum Z Function... Is Real!

$$Z(\varphi) = \sqrt{A_N^*} \mathrm{e}^{-i\frac{N}{2}\varphi} \sum_{n=0}^{\frac{N}{2}-1} A_n \mathrm{e}^{in\varphi} +$$

$$\sqrt{A_N^* A_{\frac{N}{2}}} + \sqrt{A_N} \mathrm{e}^{i\frac{N}{2}\varphi} \sum_{n=0}^{\frac{N}{2}-1} A_n^* \mathrm{e}^{-in\varphi}$$

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Upshots from the Lookalike Formula

Lookalike+diagonal approximation= Plateau

Lookalike+diagonal approximation+slow modes= Convolution

$$K_{\text{long time}}(T) = \frac{\lambda_1 e^{-\lambda_1|T|}}{2} * \frac{\lambda_2 e^{-\lambda_2|T|}}{2} * \frac{\lambda_3 e^{-\lambda_3|T|}}{2} ... * K^0_{\text{long time}}(T)$$

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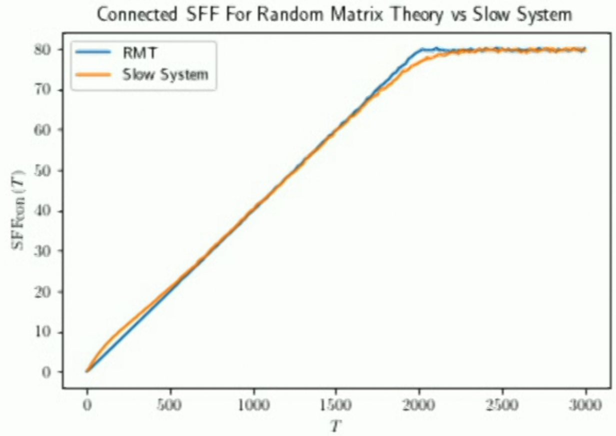

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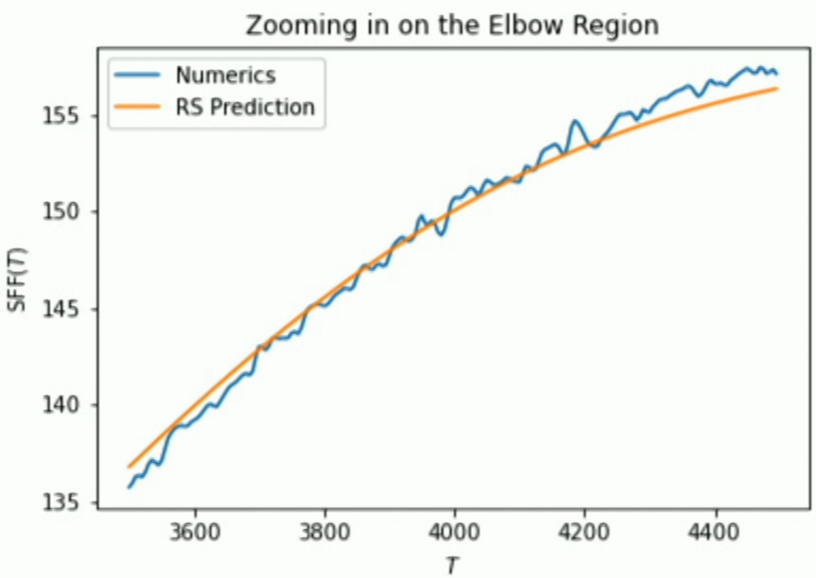
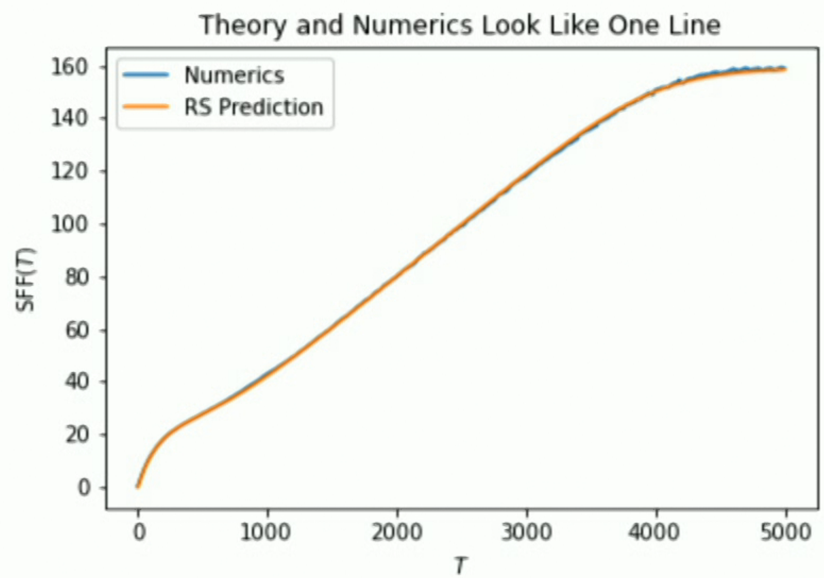
The Moment of Truth

Using Riemann-Siegel Duality and some elbow grease, we can work out the late-time formula for two-sector toy model

$$\frac{\min(T, T_H)}{2\pi} - \frac{e^{-\lambda|T-T_H|}}{4\pi\lambda}$$

The Moment of Truth





A Word on Large Systems

Exponentially large early-time enhancement

Exponentially many sectors in hydro theory

Deviation from RMT at Heisenberg time is exponentially large

Summary

- Spectral Form Factors measure thermalization of quantum system
- Early time behavior proportional to total return probability
- SFFs follow a sum rule
- Late time behavior is related to early time behavior by Riemann-Siegel lookalike