Title: Talk 125 - Partition function of a volume of space

Speakers:

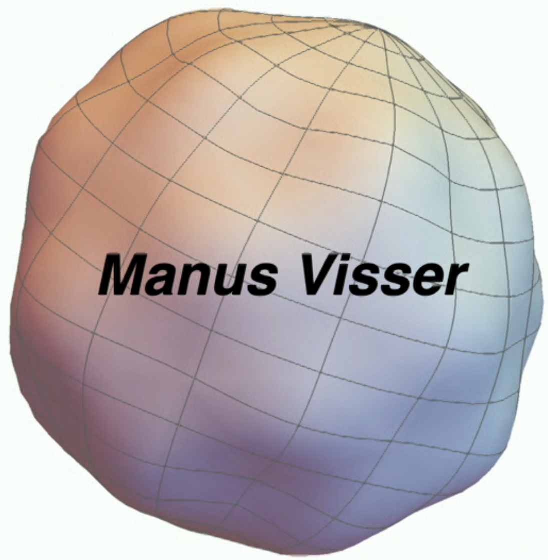
Collection: It from Qubit 2023

Date: August 03, 2023 - 10:30 AM

URL: https://pirsa.org/23080020

Abstract: We consider the quantum gravity partition function that counts the dimension of the Hilbert space of a spatial region with topology of a ball and fixed proper volume, and evaluate it in the leading order saddle point approximation. The result is the exponential of the Bekenstein-Hawking entropy associated with the area of the saddle ball boundary, and is reliable within effective field theory provided the mild curvature singularity at the ball boundary is regulated by higher curvature terms. This generalizes the classic Gibbons-Hawking computation of the de Sitter entropy for the case of positive cosmological constant and unconstrained volume, and hence exhibits the holographic nature of nonperturbative quantum gravity in generic finite volumes of space.

Partition function for a volume of space



UNIVERSITY OF CAMBRIDGE

Swiss National Science Foundation

It from Qubit 2023, Perimeter Institute

Based on work with Ted Jacobson
(2212.10607, *Physical Review Letters*)

Horizon entropy

- Bekenstein-Hawking entropy provides a low-energy window into the realm of quantum gravity

$$S = \frac{A}{4\hbar G}$$

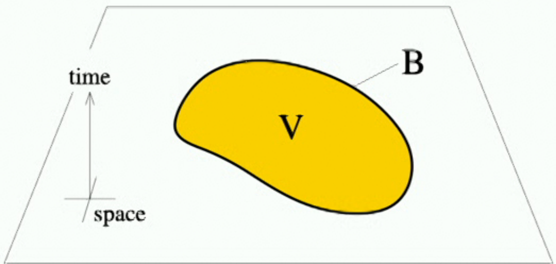
- *Universal formula:* applies to black hole horizons, cosmological horizons, acceleration horizons, and entanglement wedges.

- All horizon entropies presumably have the same statistical origin.

"Entropy = area" for any volume of space

- *Claim:* gravitational entropy is not only associated to the area of black hole or de Sitter horizon, but to the area of *any* boundary separating a region of space.

Bousso (1999), Banks-Fischler (2001), Jacobson-Parentani (2003), Bianchi-Myers (2014), …



How to justify this?

Gravitational path integral

Gibbons and Hawking (1977) derived the entropy of black hole and de Sitter horizons from a Euclidean saddle approximation of the quantum gravity partition function.

Can the entropy of any volume of space be derived from a quantum gravity partition function?

Yes! Using the method of constrained instantons

Outline

1. Gravitational partition function for de Sitter space

   - Review of Gibbons-Hawking, Action integrals and partition functions in quantum gravity, PRD, 1977.

   - Jacobson-Banihashemi, Thermodynamic ensembles with cosmological horizons, JHEP, 2022.

2. Constrained sphere partition function

   - Jacobson & MV, *Partition function of a volume of space*, PRL, 2023.

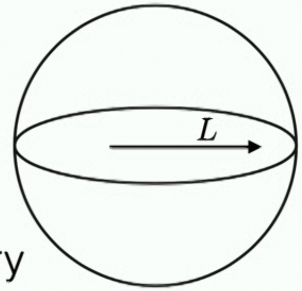
De Sitter entropy from path integral

- In a saddle approximation the path integral can be estimated as:

$$Z \sim \exp\left(-I_E^{\text{saddle}}/\hbar\right)$$

- If the saddle geometry is Euclidean de Sitter space (a round sphere with radius equal to the dS curvature scale *L*), then

$$I_E^{\text{saddle}}/\hbar = -\frac{A(L)}{4\hbar G} = -S_{dS}$$



S^D

NB:
1. Hamiltonian *H* vanishes, since there is no boundary
2. action is independent of temperature

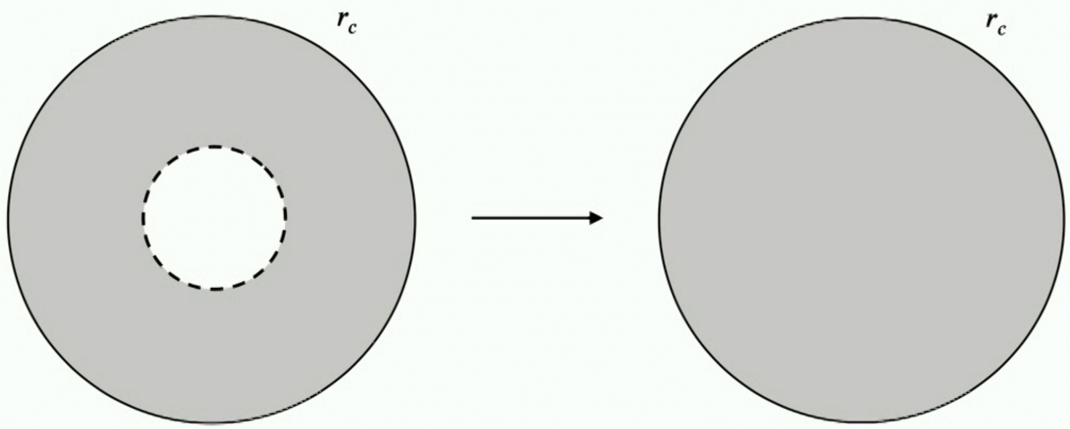
9

Sphere partition function

- What is the meaning of the sphere partition function? What is the ensemble?

Jacobson-Banihashemi (2022)

- Resolution: introduce an *artificial "York" boundary*, where H is defined, and examine the limit in which it disappears



topology: $D^2 \times S^{D-2}$ $\quad\quad S^D$

10

Dimension of the Hilbert space

- Canonical partition function with a York boundary

$$Z = \mathrm{Tr}\, e^{-\beta H_{\mathrm{BY}}}$$

- Jacobson-Banihashemi '22: In the vanishing boundary limit the Brown-York Hamiltonian vanishes and the path integral is over all metric on the sphere S^D.

→

$$Z \to \mathrm{Tr}_{\mathcal{H}}\, 1 = e^{S_{dS}}$$

= dimension of Hilbert space of states surrounded by a horizon, i.e. states of a ball

Outline

1. Gravitational partition function for de Sitter space

   - Review of Gibbons-Hawking, Action integrals and partition functions in quantum gravity, PRD, 1977.

   - Jacobson-Banihashemi, Thermodynamic ensembles with cosmological horizons, JHEP, 2022.

2. Constrained sphere partition function

   - Jacobson & MV, *Partition function of a volume of space*, PRL, 2023.

Partition function for a volume of space

- Should not "area = entropy" apply to *any* volume of space (topological ball)?
- To specify a "region of space", one must somehow fix its size.
- We fix the spatial volume, by adding a constraint in the path integral that all spatial slices have volume V

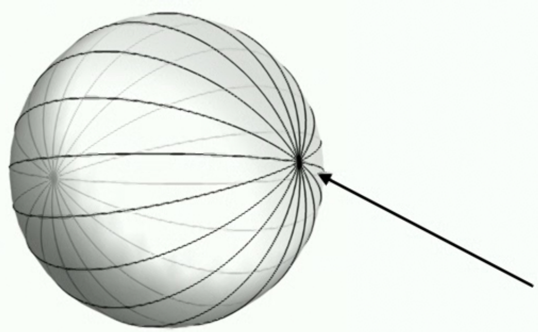
$$Z[V] = \mathrm{Tr}_{\mathcal{H}} 1$$ = dimension of Hilbert space of geometries with spatial volume V

13

Euclidean sphere geometry

- What are the topologies that we integrate over in the path integral?
- Consider a spatial topological *(D-1)*-ball whose boundary has topology S^{D-2}.
- The Euclidean manifold generated by rotating the ball through a complete circle about the ball boundary is a topological *D*-sphere

e.g. *D=2* version:



$$S^D$$

Constrained sphere partition function

Method of constrained instantons
Affleck (1981), Stanford (2020), Cotler-Jensen (2021)

$$Z[V,\Lambda] = \int \mathcal{D}\lambda\,\mathcal{D}g\,\exp\left[\frac{1}{16\pi\hbar G}\int d^Dx\sqrt{g}(R-2\Lambda) + \frac{1}{\hbar}\int d\phi\,\lambda(\phi)\left(\int d^{D-1}x\sqrt{\gamma} - V\right)\right]$$

- Foliate S^D by *(D-1)*-balls at constant ϕ with induced metric $\gamma_{ab} = g_{ab} - N^2\phi_{,a}\phi_{,b}$

$$N \equiv (g^{ab}\phi_{,a}\phi_{,b})^{-1/2}$$

- The saddle point equations are the Einstein equations sourced by an effective perfect fluid with vanishing energy density,

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} \quad \text{with} \quad T_{ab} = \frac{\lambda}{N}\gamma_{ab} \equiv P\gamma_{ab}$$

Static, spherically symmetric saddle

- Saddle with minimum action presumably is the most symmetric one:

$$ds^2 = N^2(r)d\phi^2 + h(r)dr^2 + r^2 d\Omega_{D-2}^2$$

- *N(r)* is determined by the Tolman-Oppenheimer-Volkoff-equation, with boundary conditions:

  1) $N = 0$ at $r = R_V$, the "horizon"

  2) $N' = -\sqrt{h}$ at $r = R_V$, to prevent conical singularity

- $\Lambda = 0$ solution:

$$R_V = [(D-1)V/\Omega_{D-2}]^{1/(D-1)}$$

$$ds^2 = \frac{1}{4R_V^2}(R_V^2 - r^2)^2 d\phi^2 + dr^2 + r^2 d\Omega_{D-2}^2$$

Constrained sphere partition function

Method of constrained instantons
Affleck (1981), Stanford (2020), Cotler-Jensen (2021)

$$Z[V,\Lambda] = \int \mathcal{D}\lambda\, \mathcal{D}g\, \exp\left[\frac{1}{16\pi\hbar G}\int d^D x\sqrt{g}(R-2\Lambda) + \frac{1}{\hbar}\int d\phi\, \lambda(\phi)\left(\int d^{D-1}x\sqrt{\gamma} - V\right)\right]$$

- Foliate S^D by *(D-1)*-balls at constant ϕ with induced metric $\gamma_{ab} = g_{ab} - N^2\phi_{,a}\phi_{,b}$

$$N \equiv (g^{ab}\phi_{,a}\phi_{,b})^{-1/2}$$

- The saddle point equations are the Einstein equations sourced by an effective perfect fluid with vanishing energy density,

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} \quad \text{with} \quad T_{ab} = \frac{\lambda}{N}\gamma_{ab} \equiv P\gamma_{ab}$$

Static, spherically symmetric saddle

- Saddle with minimum action presumably is the most symmetric one:

$$ds^2 = N^2(r)d\phi^2 + h(r)dr^2 + r^2 d\Omega^2_{D-2}$$

- *N(r)* is determined by the Tolman-Oppenheimer-Volkoff-equation, with boundary conditions:

  1) $N = 0$ at $r = R_V$, the "horizon"

  2) $N' = -\sqrt{h}$ at $r = R_V$, to prevent conical singularity

- $\Lambda = 0$ solution:

$$R_V = [(D-1)V/\Omega_{D-2}]^{1/(D-1)}$$

$$ds^2 = \frac{1}{4R_V^2}(R_V^2 - r^2)^2 d\phi^2 + dr^2 + r^2 d\Omega^2_{D-2}$$

Constrained sphere partition function

Method of constrained instantons

Affleck (1981), Stanford (2020), Cotler-Jensen (2021)

$$Z[V,\Lambda] = \int \mathcal{D}\lambda\, \mathcal{D}g\, \exp\left[\frac{1}{16\pi\hbar G}\int d^D x\sqrt{g}(R-2\Lambda) + \frac{1}{\hbar}\int d\phi\, \lambda(\phi)\left(\int d^{D-1}x\sqrt{\gamma} - V\right)\right]$$

- Foliate S^D by *(D-1)*-balls at constant ϕ with induced metric $\gamma_{ab} = g_{ab} - N^2\phi_{,a}\phi_{,b}$

$$N \equiv (g^{ab}\phi_{,a}\phi_{,b})^{-1/2}$$

- The saddle point equations are the Einstein equations sourced by an effective perfect fluid with vanishing energy density,

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} \quad \text{with} \quad T_{ab} = \frac{\lambda}{N}\gamma_{ab} \equiv P\gamma_{ab}$$

Euclidean action

- Euclidean constrained instanton has topology S^D, is conformally flat, and has a $1/(r-R_V)$ curvature singularity at the horizon.

- BUT the on-shell Euclidean action is finite:

for $\Lambda = 0$: $I_{\text{saddle}} = -\frac{1}{16\pi G}\int d^D x\sqrt{g}R = -\frac{A_V}{4G}$

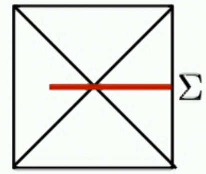
- Hence, in the zero-loop saddle-point approximation:

$$Z[V] \approx \exp(A_V/4\hbar G)$$

This shows that finite volumes of space have a BH “entropy” (log dim H).

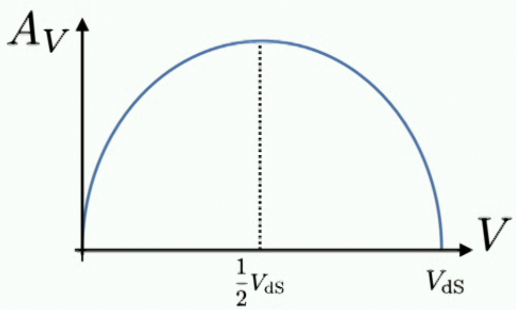
$\Lambda \neq 0$ case

For nonzero Λ a similar saddle exists, where spatial ball is embedded in H^{D-1} for $\Lambda < 0$ or S^{D-1} for $\Lambda > 0$



BUT for $\Lambda > 0$ there are significant differences:

- If V is larger than the dS spatial hemisphere, entropy *decreases* as V increases.
- There is no saddle if V is larger than the full spatial dS sphere.
- The integral over all V is indeed dominated by the de Sitter saddle:



$$Z = \int_0^{V_{\rm dS}} dV \exp(A_V/4\hbar G)$$
$$\approx \exp(A_{\rm dS}/4\hbar G)$$

recovers Gibbons-Hawking entropy!

20

Regulation of the saddle singularity

- Bekenstein-Hawking result for action seems reasonable, but a curvature singularity exists at the horizon.

- Despite the singularity, the result could be reliable provided the corrections are small, EITHER because

1. Curvature corrections might regularise the saddle without significantly changing the entropy, while remaining consistent with EFT

$$S_{\text{Wald}} \sim \frac{A}{\ell_P^2}(1 + \ell^2 R + \ell^4 R^2 + \ell^4 \Box R + \cdots)$$

OR

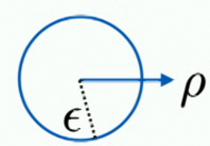
$$\sim \frac{A}{\ell_P^2}\left(1 + \frac{\ell^2}{\ell_s R_V} + \frac{\ell^4}{\ell_s^2 R_V^2} + \frac{\ell^4}{\ell_s^3 R_V} \cdots\right)$$

Suppose curvature scalar saturates at $\rho = \ell_s$

$$R \sim \frac{1}{R_V \ell_s}$$

2. non-EFT UV completion exists but does not significantly change the entropy, because It is only relevant in a small region surrounding the horizon.

$$I \sim \int d^D x \sqrt{g} R \sim \int_0^{\epsilon} \frac{1}{\rho R_V} \rho d\rho \sim \epsilon / R_V \ll 1$$



Conclusions

- Partition function of a volume of space = dimension of the quantum gravity Hilbert space of a topological ball with fixed proper volume.

- The Hilbert space dimension is given by the semiclassical Bekenstein-Hawking entropy corresponding to the boundary of the saddle ball

$$Z[V] = \dim \mathcal{H} \approx \exp(A_V/4\hbar G)$$

- This reflects the holographic nature of nonperturbative quantum gravity in a generic finite volume of space.

24

Future directions

- *Higher curvature corrections* deserve further work: do they regularise the curvature singularity at the horizon?

- Constrained partition function maybe gives a nonperturbative rationale for Jacobson's maximal vacuum entanglement hypothesis.

- Different constraint: fixing spacetime volume gives a smooth saddle.
  Is there a Hilbert space interpretation of this constraint?

25