Title: Sparse random Hamiltonians are quantumly easy

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Sparse random Hamiltonians are quantumly easy

[2302.03394]
QIP 2023 Best student paper

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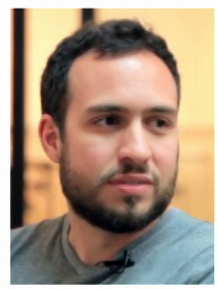
with Alex Dalzell, Mario Berta, Fernando Brandão, Joel Tropp

July Aug 2 @ Π






Prologue

Quantum computing ⟷ **Random matrix theory**

Recent interest:
quantum advantage?

New tools:
Non-asymptotic bounds

Quantum computers are provably good at finding extremal eigenvectors of certain random matrices.

Fact I: energy measurements (Born rule)

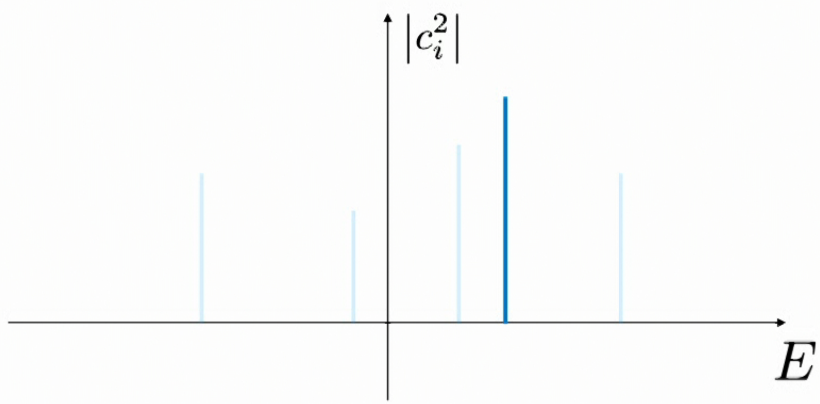
Consider a non-degenerate Hermitian matrix with eigenvectors

$$\boldsymbol{H} |\psi_i\rangle = E_i |\psi_i\rangle .$$

Given a quantum state $|\phi\rangle = \sum_i c_i |\psi_i\rangle$, quantum computer can "measure the energies of $\boldsymbol{H}$" by

$$\text{returning} \quad |\psi_i\rangle \quad \text{with probability} \quad |c_i^2| .$$

$$|\phi\rangle = \sum_i c_i |\psi_i\rangle$$



*using "quantum phase estimation" and "Hamiltonian simulation" algorithms
[Kitaev '95] [Lloyd '96]

Fact II: Semi-circular law

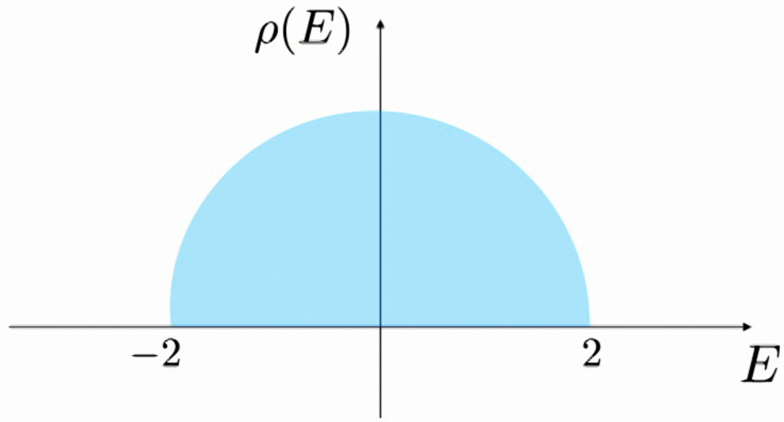
The spectral distribution of GUE approaches a semi-circle.

(in the large dimension limit $N \to \infty$)

Gaussian Unitary Ensemble

$$\boldsymbol{H}_{ij} = \frac{g_{ij} + \mathrm{i}g'_{ij}}{\sqrt{2N}} \quad \text{if } j > i$$

$$\boldsymbol{H}_{ii} = \frac{g_{ii}}{\sqrt{N}}.$$

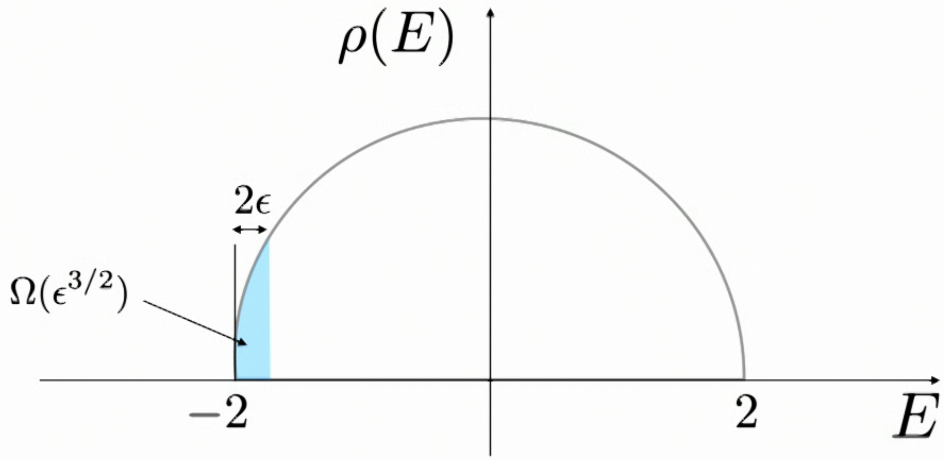


$$\rho(x) \approx \frac{\sqrt{4 - x^2}}{2\pi}$$

Fact I + Fact II = quantumly easy

Start with a uniform mixture of energies states, and then measure.

With decent chance, we find a low-energy state.



(Quantum computing) Why are low-energy states interesting?

(Non-asym. RMT) Which matrices satisfy the semi-circle law?

Quantum computing motivations

Quantum computers

(Hilbert space) $|\psi\rangle$

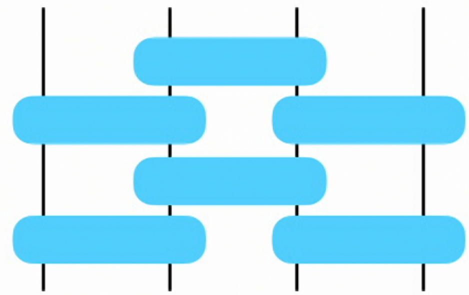
(Unitary operator) $\boldsymbol{U}$

(Measurement)

- qubit tensor products ($dim = 2^n$)

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

- circuit of local gates



- What are quantum computers good at? (“BQP”)

What are quantum computers good at?

Task	Application	Q. Adv?
Factoring [Shor '94]	Breaks RSA	Y (if RSA secure)
quantum simulation [Feynmann '82]	Q. Chemistry, Materials	?

Also: sampling [Aaronson&Arkhipov '10], search [Yamakawa&Zhandry '22]

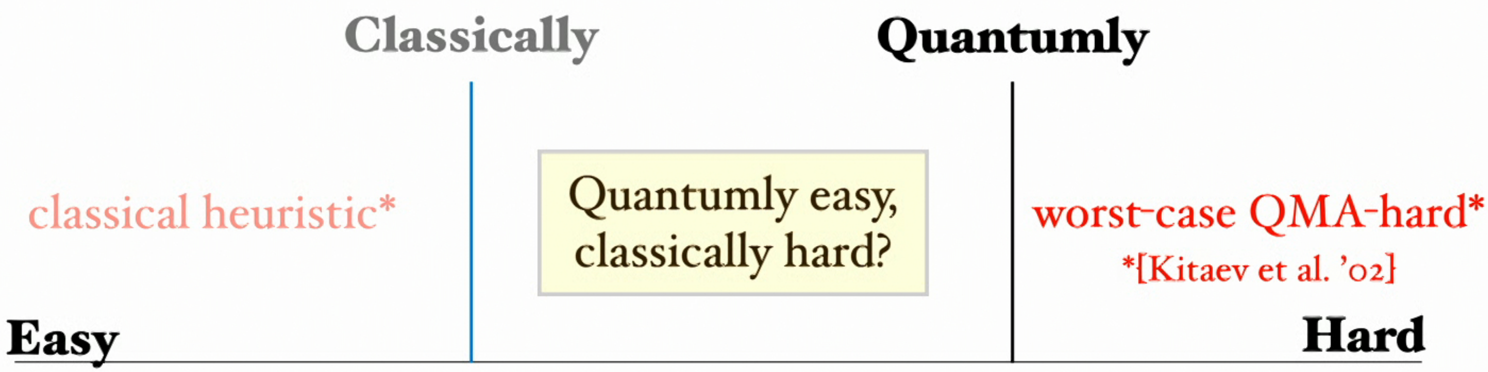
Killer app: Quantum simulation

"What's the ground energy of CO_2?"

?	state preparation	$\lvert\psi\rangle$
O	time evolution	e^{iHt}
O	measurement	O

Amazon[20' Chamberland,...,B], Google[21' Lee], Microsoft[21' von Burg],..

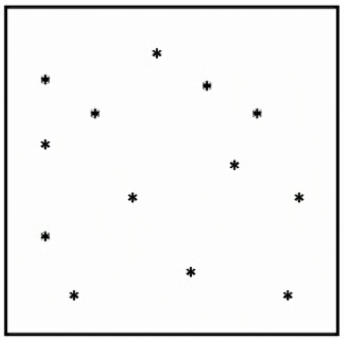
The low-energy state problem



- Which systems?
  - sparse random Hamiltonians [today]
- Which algorithms?
  - "measurements"(quantum phase estimation) [today]

Why sparse random?

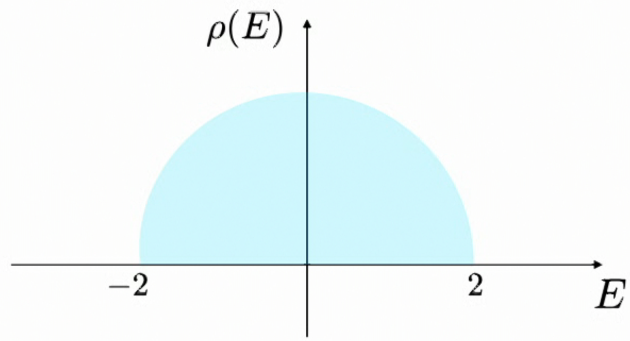
sparse



=> easy measurement*

[Berry et al '14]

random matrix



=> measurement gives low-energy state

(dense: measurements become difficult*)

*under suitable access model

The Pauli string ensemble

$$\boldsymbol{H}_{PS} := \sum_{j=1}^{m} \frac{r_j}{\sqrt{m}} \boldsymbol{\sigma}_j \quad \text{where} \quad \boldsymbol{\sigma}_j \overset{\text{i.i.d.}}{\sim} \{\boldsymbol{I}, \boldsymbol{\sigma}^x, \boldsymbol{\sigma}^y, \boldsymbol{\sigma}^z\}^{\otimes n} \quad \text{and} \quad r_j \overset{\text{i.i.d.}}{\sim} \{1, -1\}.$$

- non-local, non-commuting Hamiltonians
- (m)-sparse matrices*

e.g., m = n = 3 $$\frac{\boldsymbol{\sigma}_1^x \boldsymbol{\sigma}_2^y \boldsymbol{I}_3 - \boldsymbol{\sigma}_1^z \boldsymbol{I}_2 \boldsymbol{\sigma}_3^y - \boldsymbol{\sigma}_1^y \boldsymbol{\sigma}_2^z \boldsymbol{\sigma}_3^x}{\sqrt{3}}$$

Main results.

- (**quantumly easy.**) Quantum algorithms efficiently find low-enengy states
- (**classically non-trivial.**) Small-sized circuits give bad energy

Quantumly easy

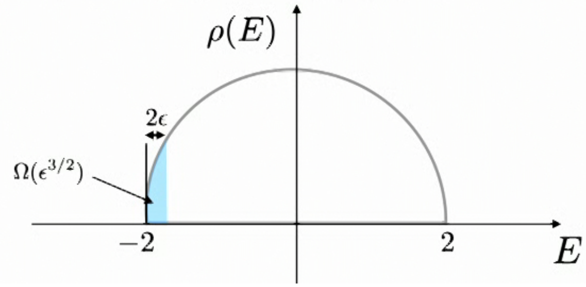
$$\boldsymbol{H}_{PS} := \sum_{j=1}^{m} \frac{r_j}{\sqrt{m}} \boldsymbol{\sigma}_j \quad \text{where} \quad \boldsymbol{\sigma}_j \overset{\text{i.i.d.}}{\sim} \{\boldsymbol{I}, \boldsymbol{\sigma}^x, \boldsymbol{\sigma}^y, \boldsymbol{\sigma}^z\}^{\otimes n} \quad \text{and} \quad r_j \overset{\text{i.i.d.}}{\sim} \{1, -1\}.$$

Theorem (low energy states are easy)

For any accuracy $\epsilon \geq 2^{-n/c_1}$, let $m = \left\lfloor c_2 \frac{n^5}{\epsilon^4} \right\rfloor$ for some constants c_1, c_2. Then, with high probability drawing $\boldsymbol{H}_{PS}$ from the Pauli string ensemble, we can prepare a low-energy state $\boldsymbol{\rho}$ such that

$$\mathrm{Tr}[\boldsymbol{\rho}\boldsymbol{H}_{PS}] \leq (1-\epsilon)\lambda_{\min}(\boldsymbol{H}_{PS}) \quad \text{using} \quad G = \mathrm{Poly}(m, \epsilon^{-1}) \quad \text{gates.}$$

- arbitrarily low energies
- Algorithm: measure the energy of the maximally mixed state
- average-case



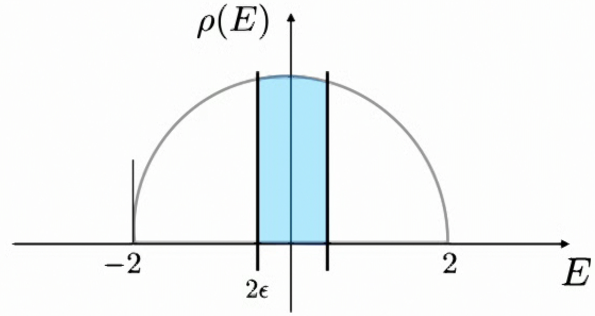
Classically non-trivial

Theorem (small circuits give bad energy)

Consider a family of states $Circ(G)$ parameterized by a circuit architecture with G gates. For any $\epsilon \geq 0$, suppose $m \leq \epsilon^2 \cdot 2^n$. Then, with high probability,

$$G = \tilde{o}(\epsilon\sqrt{m}) \quad \text{implies} \quad \inf_{|\phi\rangle \in Circ(G)} \langle\phi| \boldsymbol{H}_{PS} |\phi\rangle \geq \epsilon \cdot \mathbb{E}\lambda_{\min}(\boldsymbol{H}_{PS}).$$

Namely, none of the states $|\psi\rangle \in Circ(G)$ produce any low-energy state.



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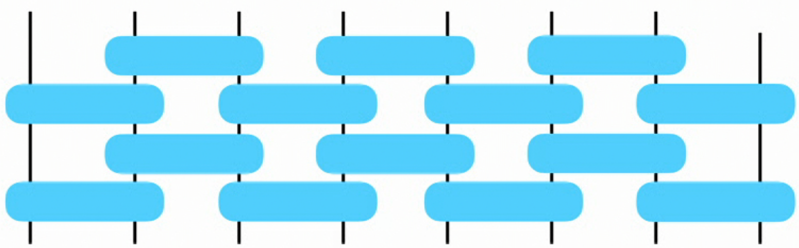
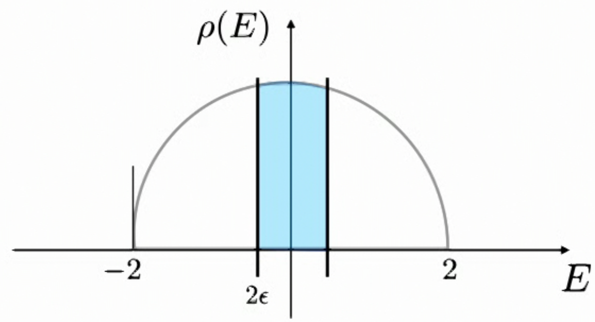
- Complexity grows with the number of Pauli strings m

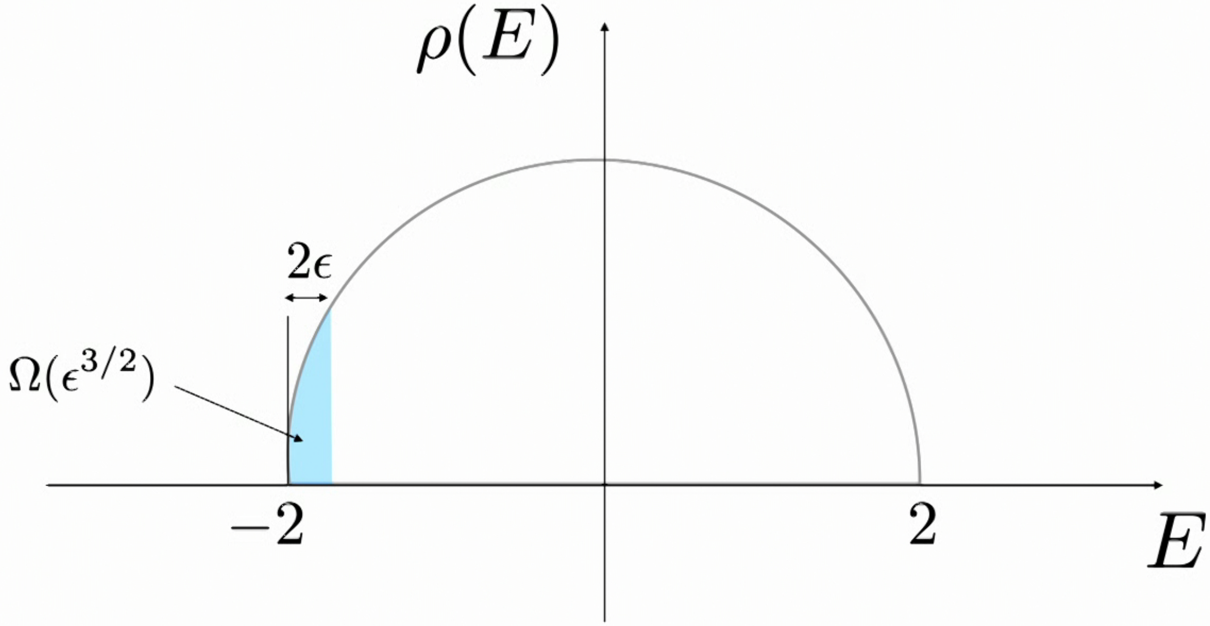




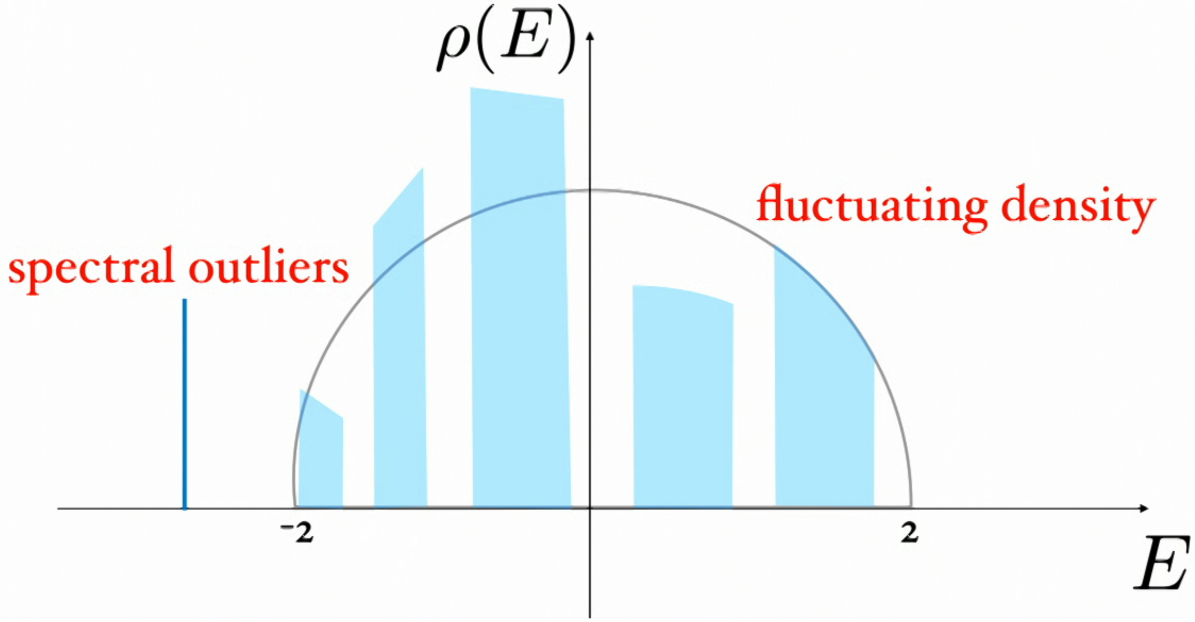
Proof idea:
Quantum easiness from non-asymptotic RMT

Semi-circular spectrum

The Pauli string ensemble typically has a semi-circular spectrum, which have lots of low-energy states.



What could have gone wrong

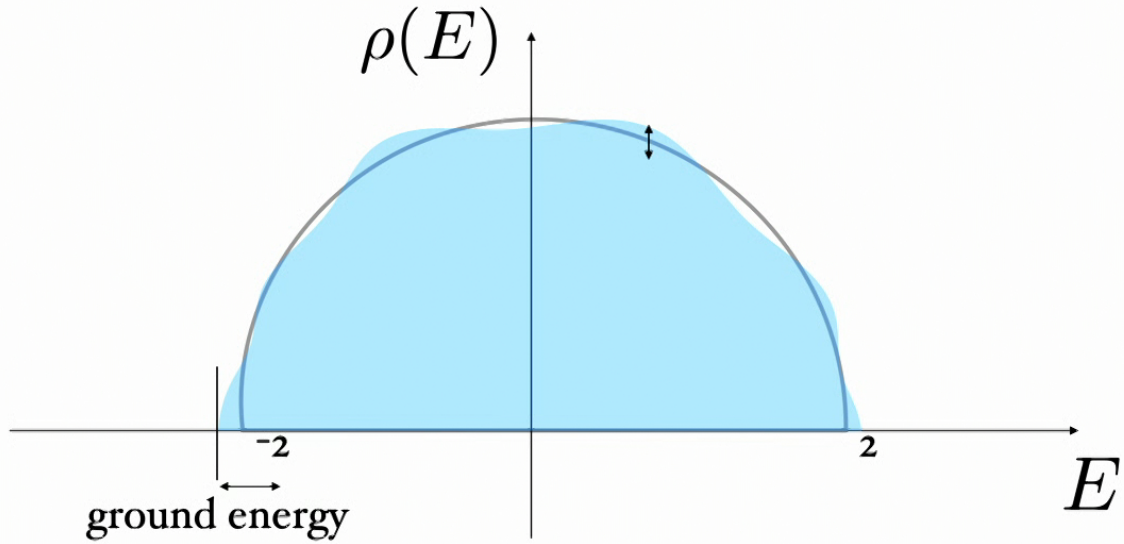


Non-asymptotic random matrix theory

The Pauli string ensemble mimics "smooth" properties of the GUE, such as the minimal eigenvalue and the coarse-grained spectral density.

~~spectral outliers~~ ~~fluctuating density~~

- The exponential complexity of GUE is not necessary for the semi-circle!
- Hint: second moment matches $\mathbb{E}_{\boldsymbol{\sigma}}[\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}] = \mathbb{E}_{\boldsymbol{G}}[\boldsymbol{G} \otimes \boldsymbol{G}]$



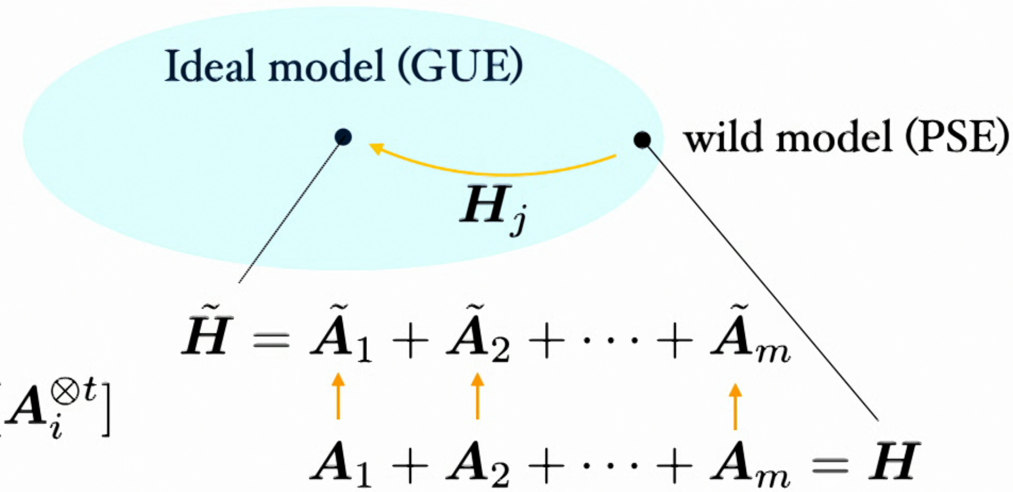
What was known in RMT?

	Classic models (GUE, GOE, Haar random,..)	**Wild models** (sparse, non-gaussian,..)
asym. $N \to \infty$		
Non-asym. $N = 1000$		[Today]

[Wigner, Dyson, Voiculescu, Tao,Vu, Chatterjee, Bai & Silverstein,...]

Universality principle

[Lindeberg, Chatterjee '05, Korada&Montanari '11, Tropp '18, Brailovskaya&Handel '22, Bandeira, Boedihardjo&Handel '21]



$$\mathbb{E}[\tilde{\boldsymbol{A}}_i^{\otimes t}] = \mathbb{E}[\boldsymbol{A}_i^{\otimes t}]$$

Technical contributions:

- A general framework capturing low-matching moments via Lindeberg

$$\boldsymbol{H}_j := \sum_{i=1}^{j} \tilde{\boldsymbol{A}}_i + \sum_{i=j+1}^{m} \boldsymbol{A}_i$$

- Concentration of spectral density (needed for quantum easiness)

Moments control outliers

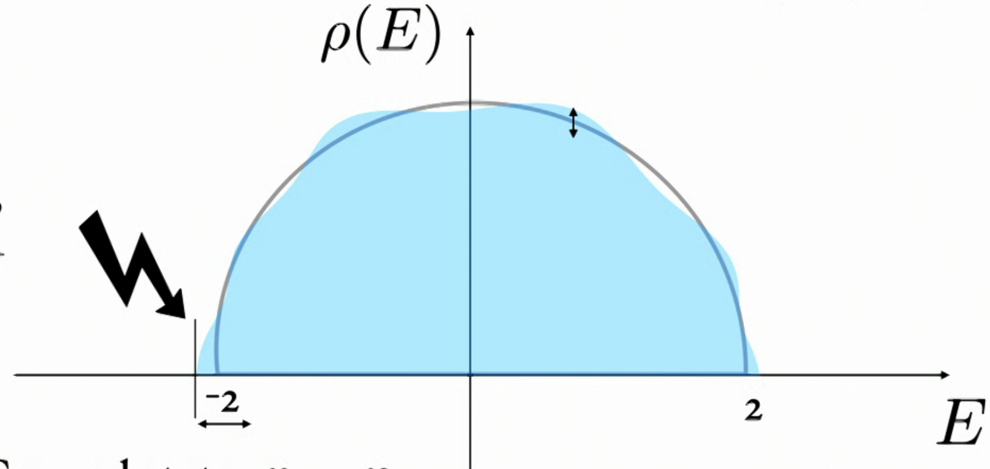
Let $p \in 2\mathbb{N}$ be an even natural number. The random Pauli string ensemble satisfies the norm bound

$$\left| |||\boldsymbol{H}_{PS}|||_p - |||\boldsymbol{G}|||_p \right| \lesssim \left(\frac{p^{3/4}}{m^{1/4}} + \frac{p}{\sqrt{m}} \right).$$

$$|||\boldsymbol{O}|||_p := \left(\frac{\mathbb{E}\mathrm{Tr}\,|\boldsymbol{O}|^p}{\mathrm{Tr}[\boldsymbol{I}]} \right)^{1/p}$$

Markov's inequality:

$$\Pr(\|\boldsymbol{O}\| \geq \epsilon) \leq \frac{\mathbb{E}\mathrm{Tr}\,|\boldsymbol{O}|^p}{\epsilon^p}$$



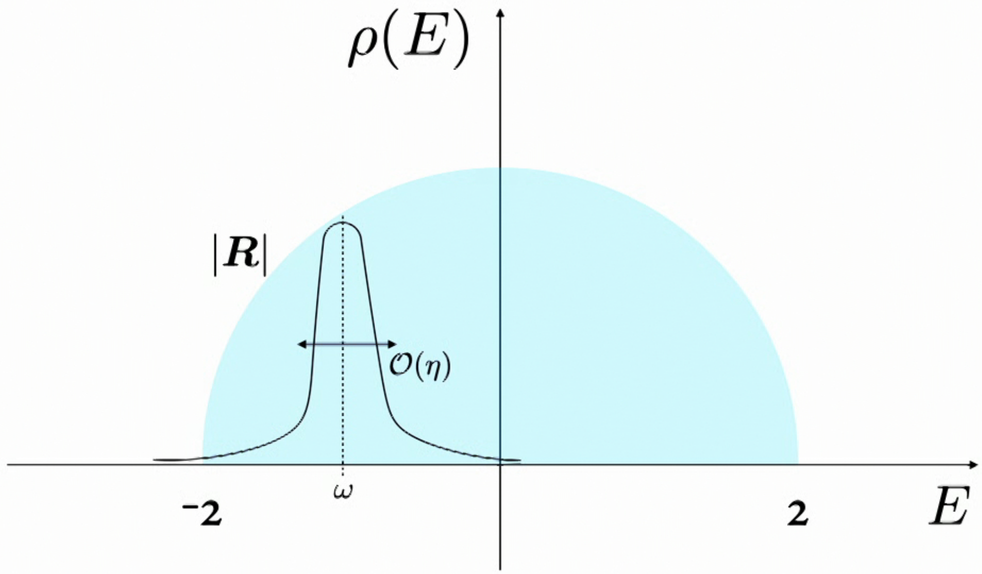
Ground state: $p \sim n$

Resolvents control spectral density

- Resolvent filters out far apart energies

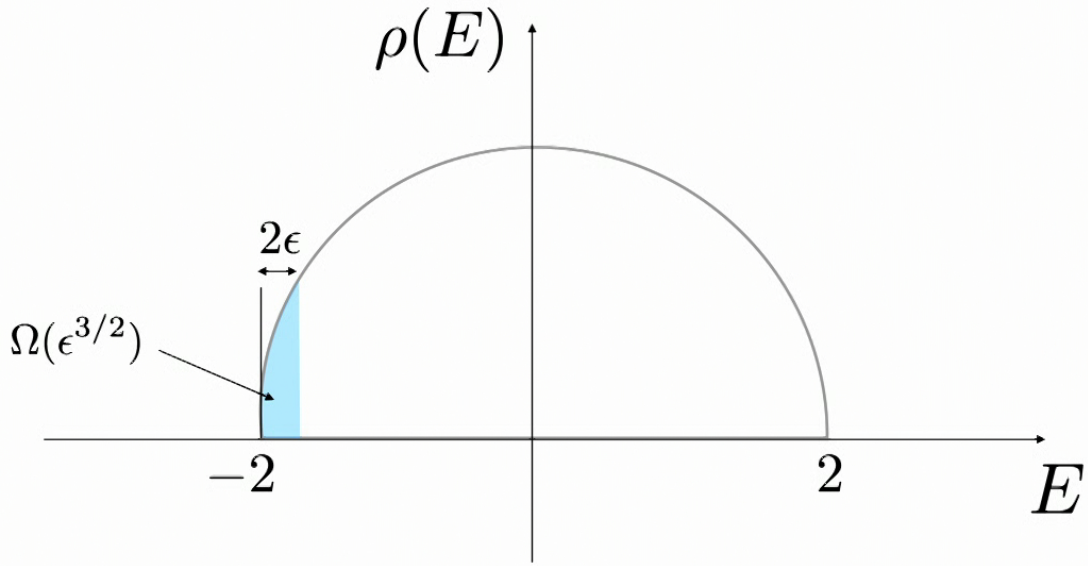
$$\boldsymbol{R}_{\omega,\eta}(\boldsymbol{H}) := \frac{1}{\boldsymbol{H} - \omega + \mathrm{i}\eta} = \mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}(\boldsymbol{H}-\omega)t - \eta t} dt$$

- Taking trace gives "smoothened" spectral density



Finale

Today: Sparse random matrices are quantumly easy



Tomorrow: Classical hardness? local Hamiltonians?