Title: Complexity = (almost) anything

Speakers: Robert Myers

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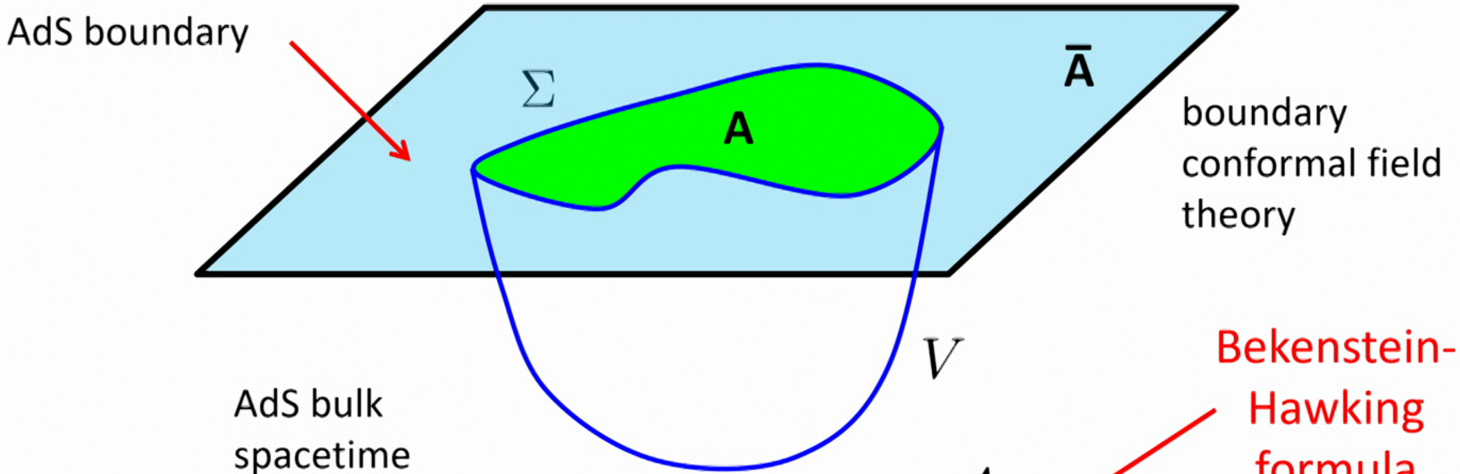
Complexity Equals
(Almost) Anything
Alexandre Belin, RCM, Shan-Ming Ruan, Gabor Sarosi & Antony Speranza
[arXiv:2111.02429; arXiv:2210.09647]
Eivind Jorstad, RCM & Shan-Ming Ruan [arXiv:2304.05453]
PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

- CFT dof within **A** described by density matrix $\rho_A = Tr_{\bar{A}}(|\psi\rangle\langle\psi|)$

⟶ calculate von Neumann entropy: $S(A) = -Tr\left[\rho_A \log \rho_A\right]$



$$S(A) = \min_{\partial V \sim \Sigma} \frac{A_V}{4G_N}$$

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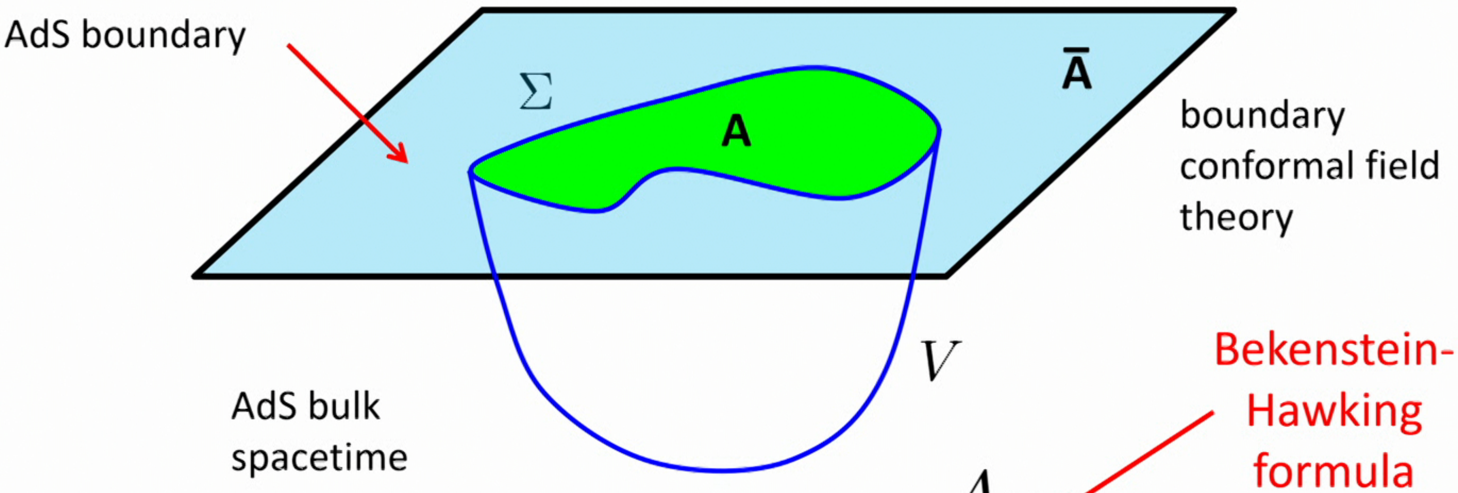
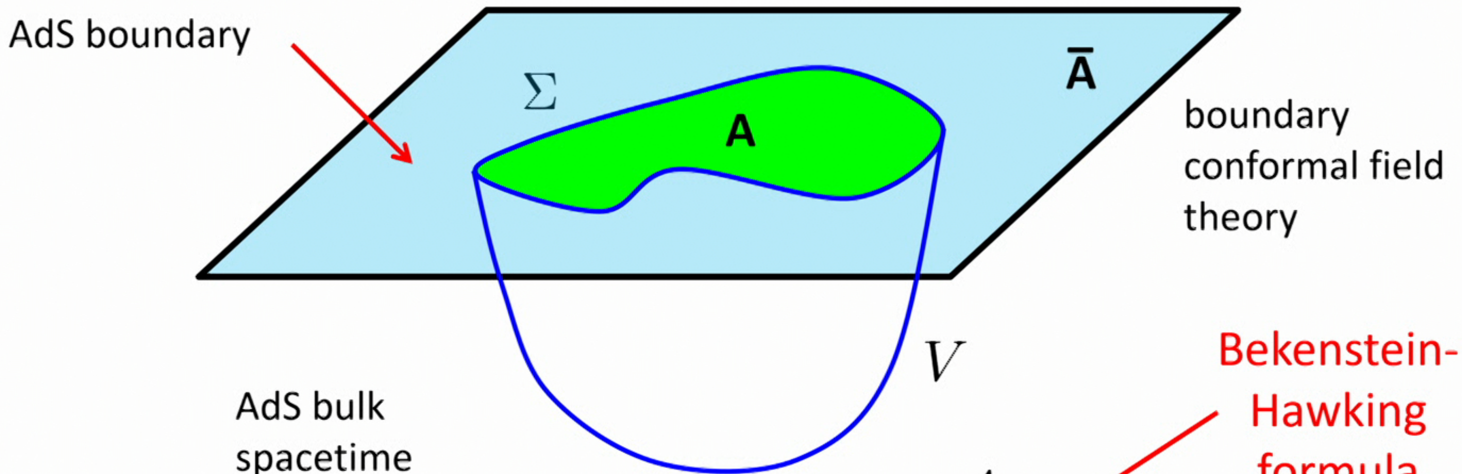


$$S(A) = \min_{\partial V \sim \Sigma} \frac{A_V}{4G_N}$$

- holographic EE is a fruitful forum for bulk-boundary dialogue:
  - → new lessons about quantum field theories
  - → new lessons about quantum gravity

Holographic Entanglement Entropy:

(Ryu & Takayanagi `06)



Bekenstein-
Hawking
formula

$$S(A) = \min_{\partial V \sim \Sigma} \frac{A_V}{4G_N}$$

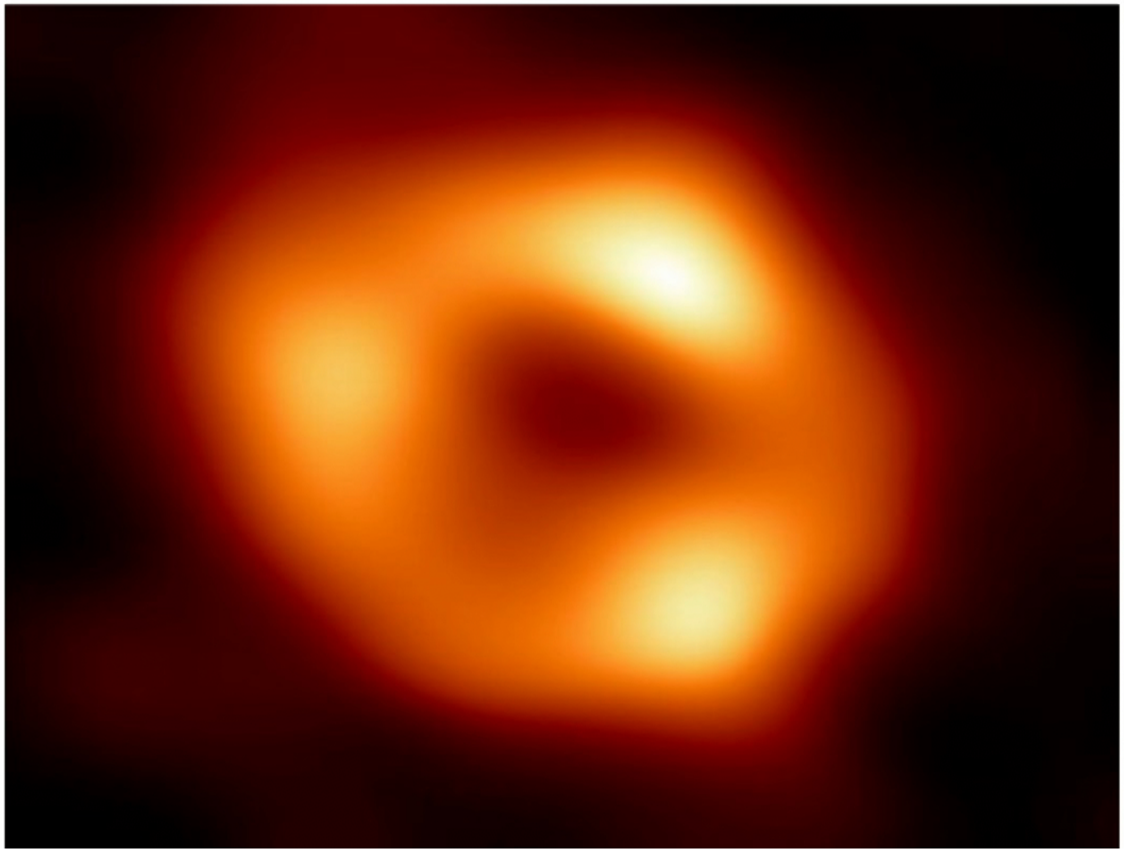
- holographic EE is a fruitful forum for bulk-boundary dialogue:

Spacetime Geometry = Entanglement

(van Raamsdonk `10)

Susskind: Entanglement Entropy is not enough!

- "to understand the rich geometric structures that exist behind black hole horizons and which are predicted by general relativity."



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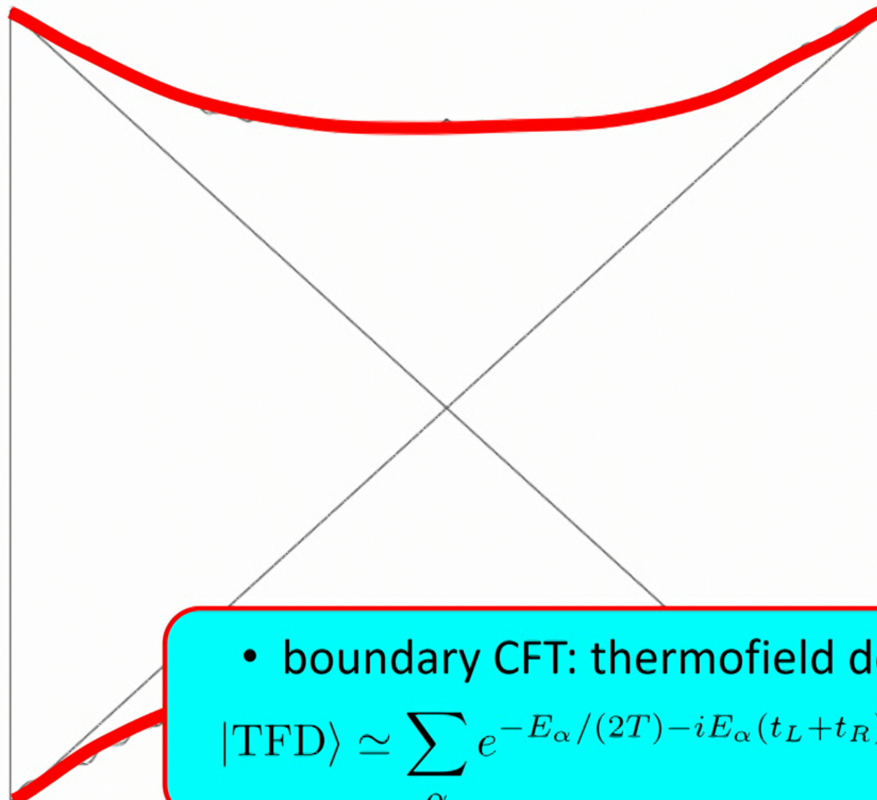

- boundary CFT: thermofield double state

$$|\mathrm{TFD}\rangle \simeq \sum_{\alpha} e^{-E_\alpha/(2T)-iE_\alpha(t_L+t_R)}\,|E_\alpha\rangle_L |E_\alpha\rangle_R$$

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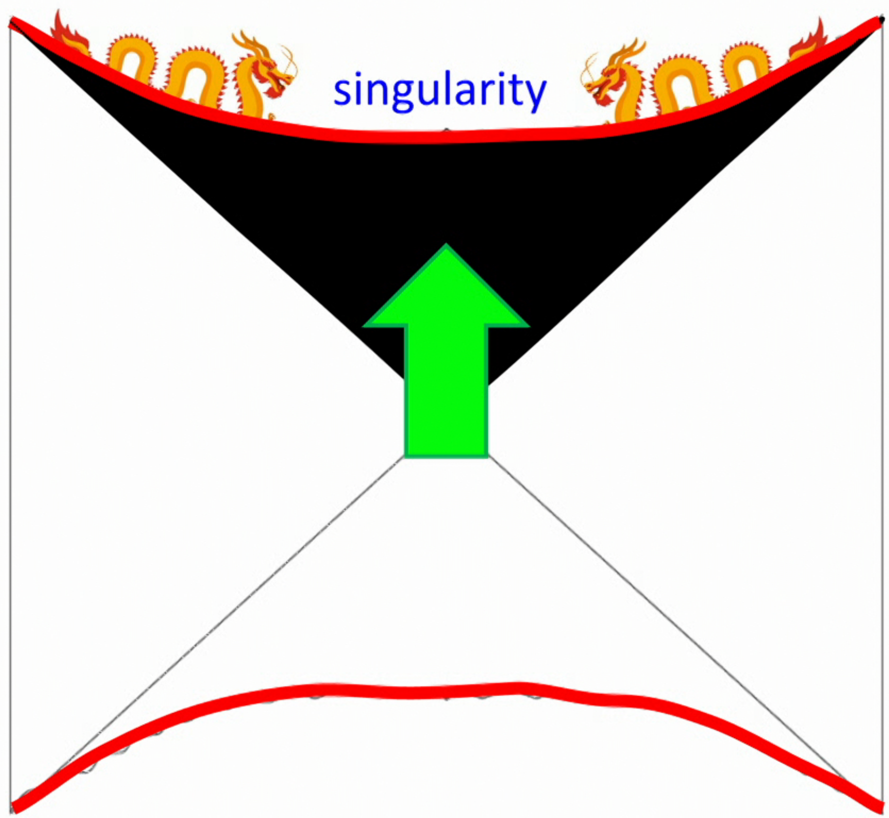

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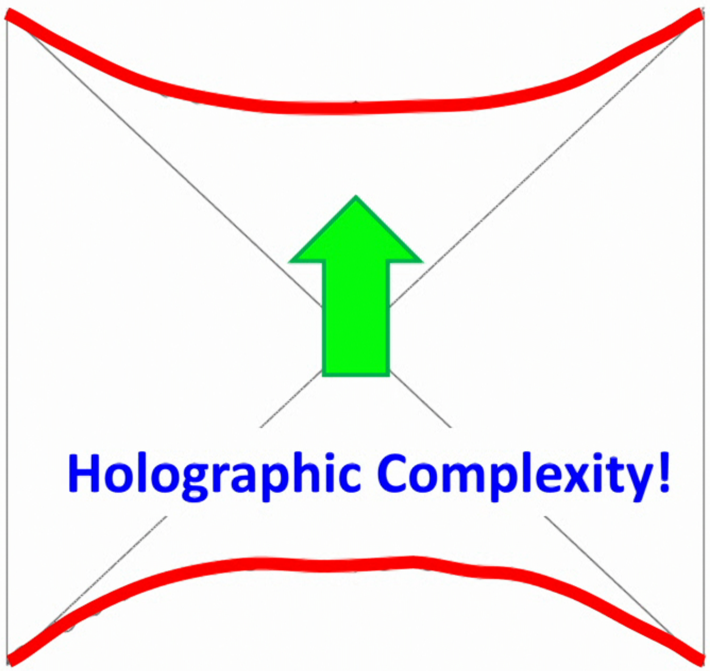


- S_{EE} only probes the eigenvalues of reduced density matrix

$$S_{EE} = -Tr\left[\rho_A \log \rho_A\right] = -\sum \lambda_i \log \lambda_i$$

- would like a new probe which is sensitive to full structure

$$|\mathrm{TFD}\rangle \simeq \sum_\alpha e^{-E_\alpha/(2T) - iE_\alpha(t_L+t_R)} \times |E_\alpha\rangle_L |E_\alpha\rangle_R$$

Complexity:

- computational complexity: how difficult is it to implement a task? eg, how difficult is it to prepare a particular quantum state?

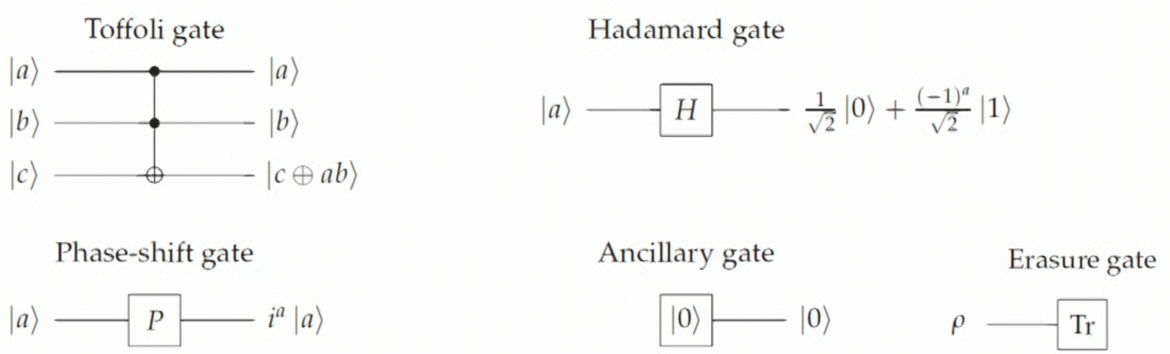
- quantum circuit model:

$$|\psi\rangle = U\,|\psi_0\rangle$$

unitary operator built from set of simple gates

unentangled

simple reference state eg, $|00000\cdots 0\rangle$



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tolerance: $\big|\,|\psi\rangle - |\psi\rangle_{\rm Target}\,\big|^2 \leq \varepsilon$

- **complexity** = minimum number of gates required to prepare the desired target state (ie, need to find optimal circuit)

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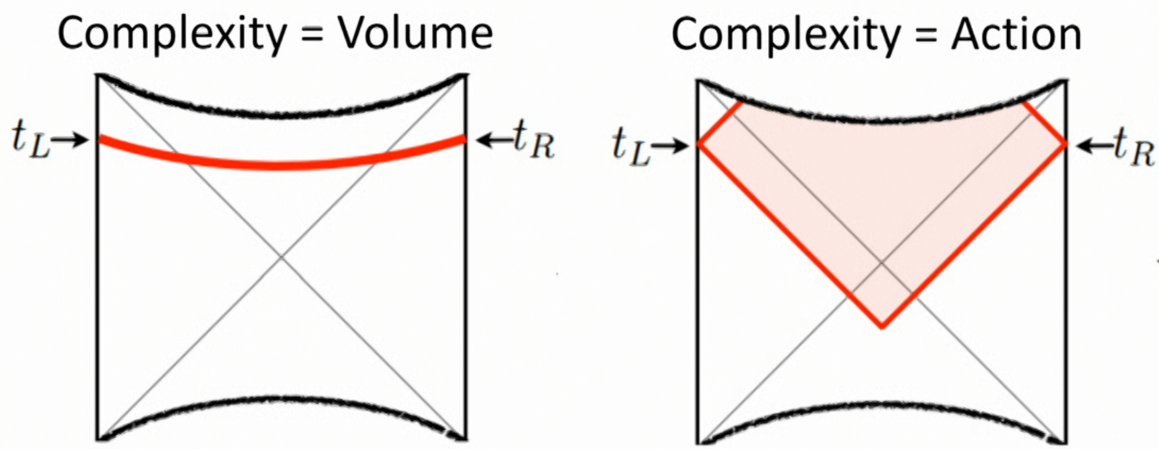
- does the answer depend on the choices?? **YES!!**

Holographic Complexity:

- complexity=volume: evaluate proper volume of extremal codim-one surface connecting Cauchy surfaces in boundary theory (cf holo EE)

(Stanford & Susskind)



- complexity=action: evaluate gravitational action for Wheeler-DeWitt patch = domain of dependence of bulk time slice connecting boundary Cauchy slices in CFT (Brown, Roberts, Swingle, Susskind & Zhao)
- both of these gravitational "observables" probe the black hole interior (at arbitrarily late times on boundary)

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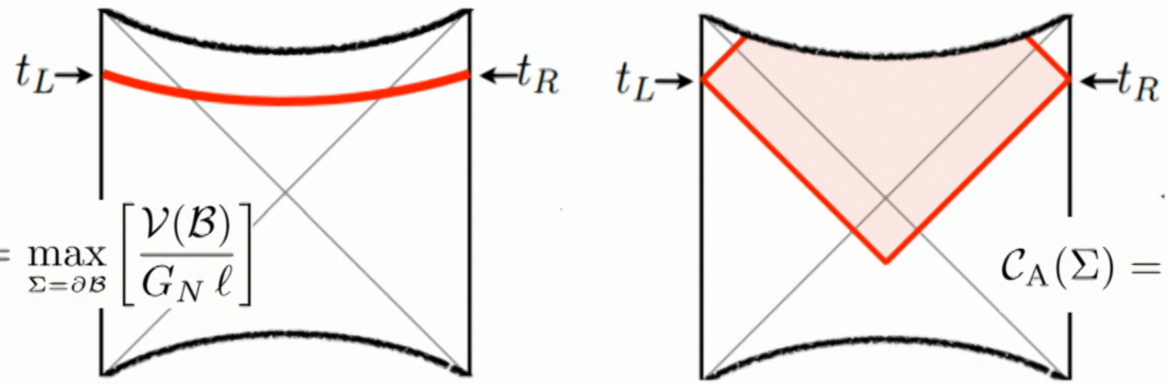
Complexity = Volume | Complexity = Action



$$\mathcal{C}_{\rm V}(\Sigma) = \max_{\Sigma=\partial\mathcal{B}} \left[\frac{\mathcal{V}(\mathcal{B})}{G_N\,\ell}\right]$$

$$\mathcal{C}_{\rm A}(\Sigma) = \frac{I_{\rm WDW}}{\pi\,\hbar}$$

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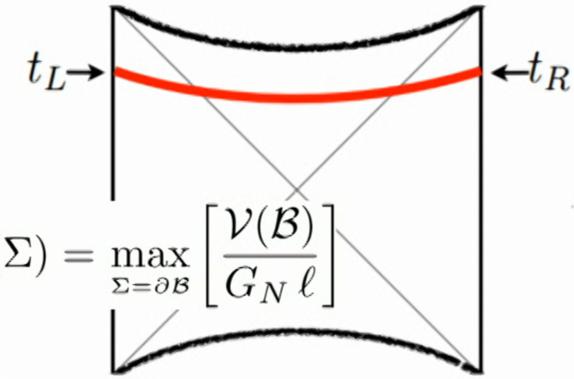

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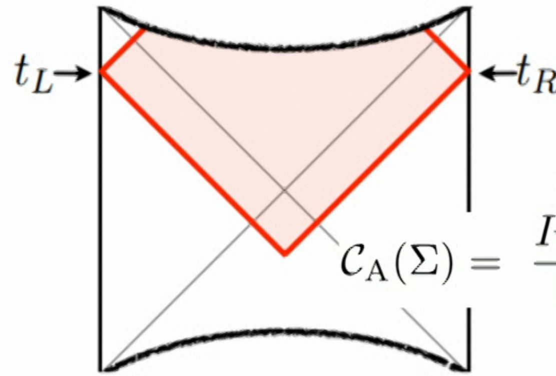


$$\mathcal{C}_{\mathrm{A}}(\Sigma) = \frac{I_{\mathrm{WDW}}}{\pi\,\hbar}$$

WHY COMPLEXITY??

- linear growth (at late times)

(d = boundary dimension)

$$\left.\frac{d\mathcal{C}_{\mathrm{V}}}{dt}\right|_{t\to\infty} = \frac{8\pi}{d-1}\,M \quad \text{(planar)} \qquad\qquad \left.\frac{d\mathcal{C}_{\mathrm{A}}}{dt}\right|_{t\to\infty} = \frac{2M}{\pi}$$

Susskind, Brown, . . .

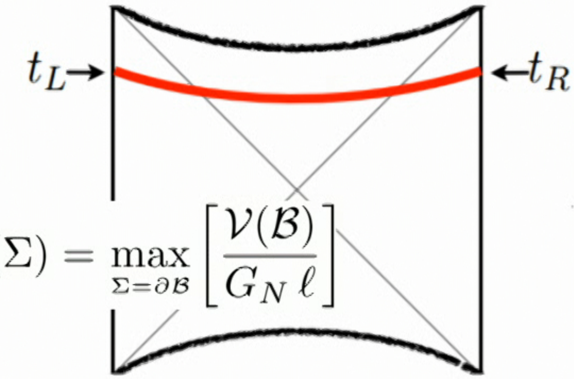
- “switchback” effect: complexity $\propto \Sigma\,|t_i - t_{i+1}| - \underline{2\,n\,t_*}$

$\longrightarrow$ probe black holes with shock waves

Stanford, Susskind, . . .

Holographic Complexity:

Complexity = Volume



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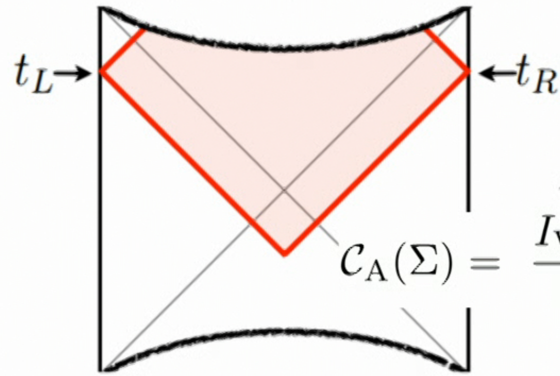


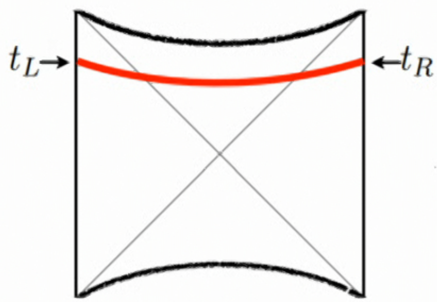
$$\mathcal{C}_{\mathrm{A}}(\Sigma) = \frac{I_{\mathrm{WDW}}}{\pi\,\hbar}$$

But why does Complexity = Volume? or Action?
(or Spacetime Volume?)

Complexity=Volume Revisited:

- complexity=volume: evaluate proper volume of extremal codim-one surface connecting Cauchy surfaces in boundary theory (cf holo EE)

(Stanford & Susskind)



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Two steps: 1) find a special surface

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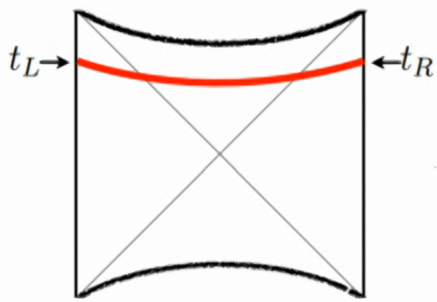

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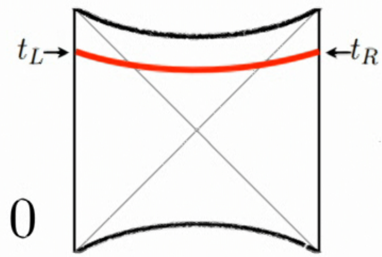
Two steps: 1) find a special surface

2) Evaluate geometric feature of surface

Generalize two step procedure:



1) find a special surface Σ :

$$\delta_X \left(\int_\Sigma d^d\sigma \sqrt{h}\ F_2(g_{\mu\nu}; X^\mu) \right) = 0$$

- F_2 is *scalar* function of bkgd metric $g_{\mu\nu}$ and embedding $X^\mu(\sigma)$

2) evaluate geometric feature of surface:

$$O_{F_1,\Sigma_{F_2}}(\Sigma_{CFT}) = \frac{1}{G_N\, L} \int_{\Sigma_{F_2}} d^d\sigma \sqrt{h}\ F_1(g_{\mu\nu}; X^\mu)$$

- F_1 is *scalar* function of bkgd metric $g_{\mu\nu}$ and embedding $X^\mu(\sigma)$

- yields “nice” diffeomorphism invariant observables

So what?

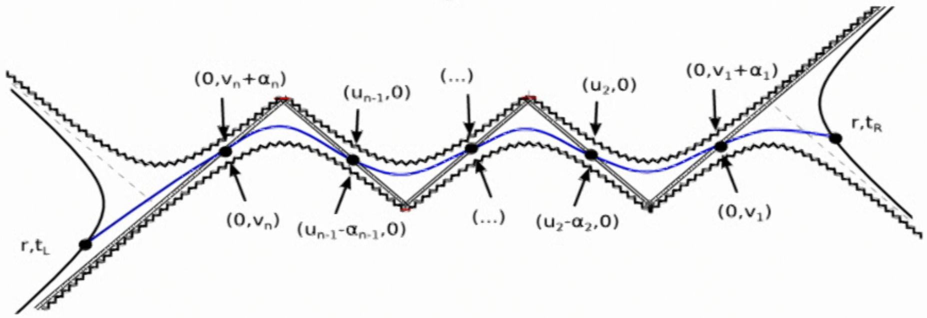
So what?

1) Observables grow linearly with time at late times:

$$\lim_{\tau\to\infty} O_{F_1,\Sigma_{F_2}}(\tau) \sim P_\infty\, \tau$$

where in large *T* limit, the constant $P_\infty \propto$ mass

2) Observables exhibit "switchback effect", ie, universal time delay in response to shock waves falling into the dual black hole



- **universality displayed by observables suggests that all of them are equally viable candidates for holographic complexity!!**

Simple Example: $F_1 = F_2 = 1 + \lambda L^4 C_{abcd}C^{abcd}$

- generalized "volume": $\mathcal{C}_{\rm gen} = \frac{V_x}{G_N L} \int_\Sigma d\sigma \left(\frac{r}{L}\right)^{d-1} \sqrt{-f(r)\dot{v}^2 + 2\dot{v}\,\dot{r}}\; a(r)$

where $f(r) = \frac{r^2}{L^2}\left(1 - \frac{r_h^d}{r^d}\right)$ and $a(r) = 1 + \tilde{\lambda}\left(\frac{r_h}{r}\right)^{2d}$

$\tilde{\lambda} = d(d-1)^2(d-2)\,\lambda$

- "gauge fix" worldvolume coordinate: $\sqrt{-f(r)\dot{v}^2 + 2\dot{v}\,\dot{r}} = a(r)\left(\frac{r}{L}\right)^{d-1}$

- conserved "momentum": $P_v = \dot{r} - f(r)\,\dot{v}$ ➡ $\frac{d\mathcal{C}_{\rm gen}}{d\tau} = \frac{1}{2} P_v$

- profile determined by classical mechanics problem

$$\dot{r}^2 + \widetilde{U}(r) = P_v^2 \quad \text{with} \quad \widetilde{U}(r) = -f(r)a^2(r)\left(\frac{r}{L}\right)^{2(d-1)},$$

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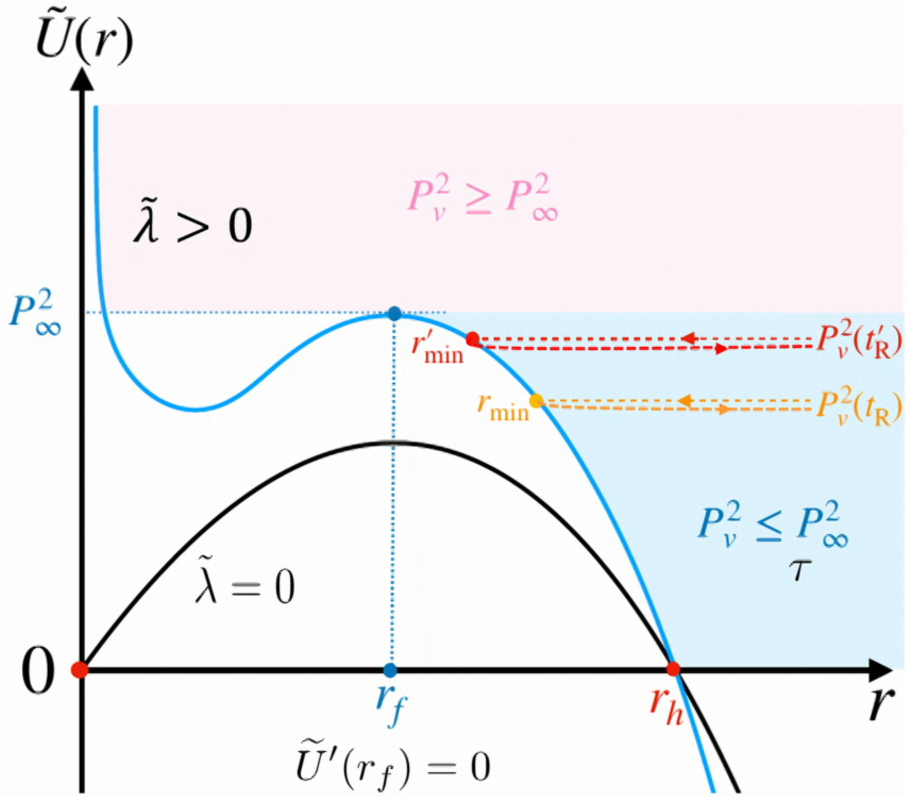
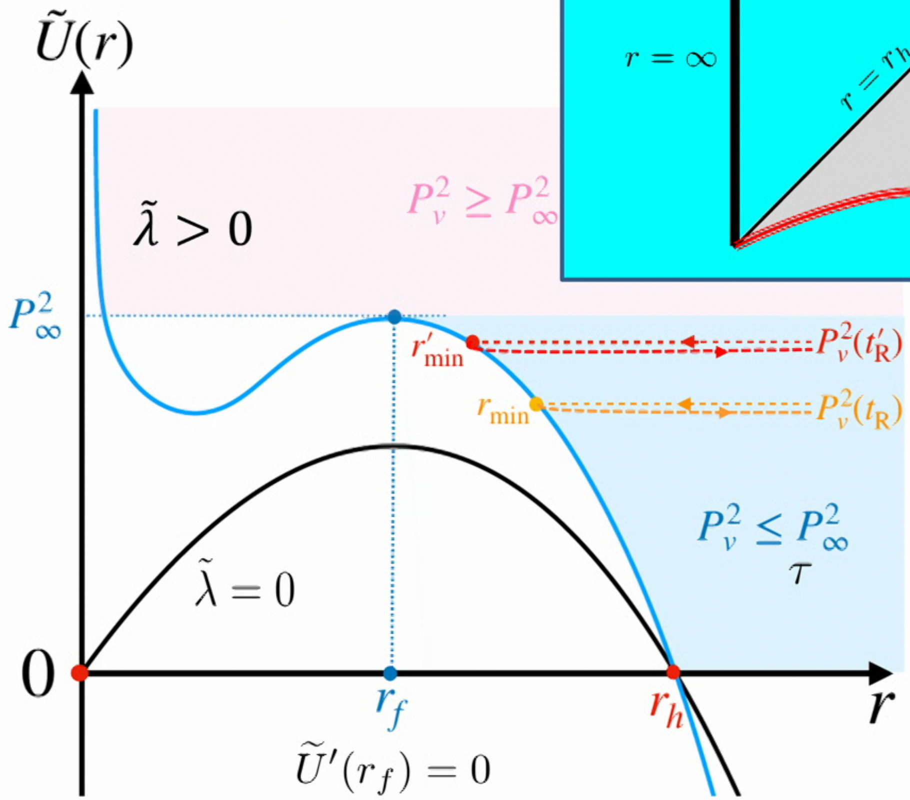


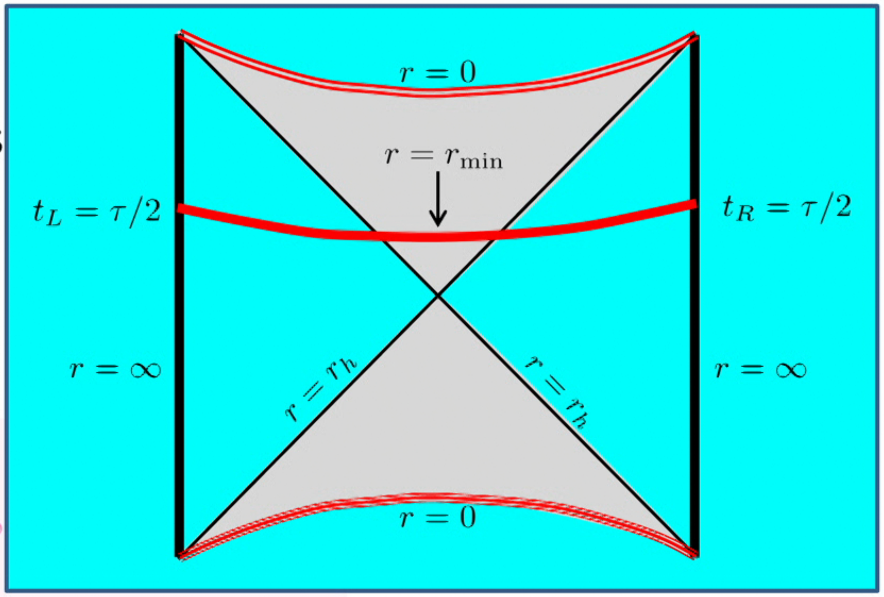
- turning point: $P_v^2 = \widetilde{U}(r_{min})$

Simple Example: F_1

- profile determined by class

$$\dot{r}^2 + \widetilde{U}(r) = P_v^2$$





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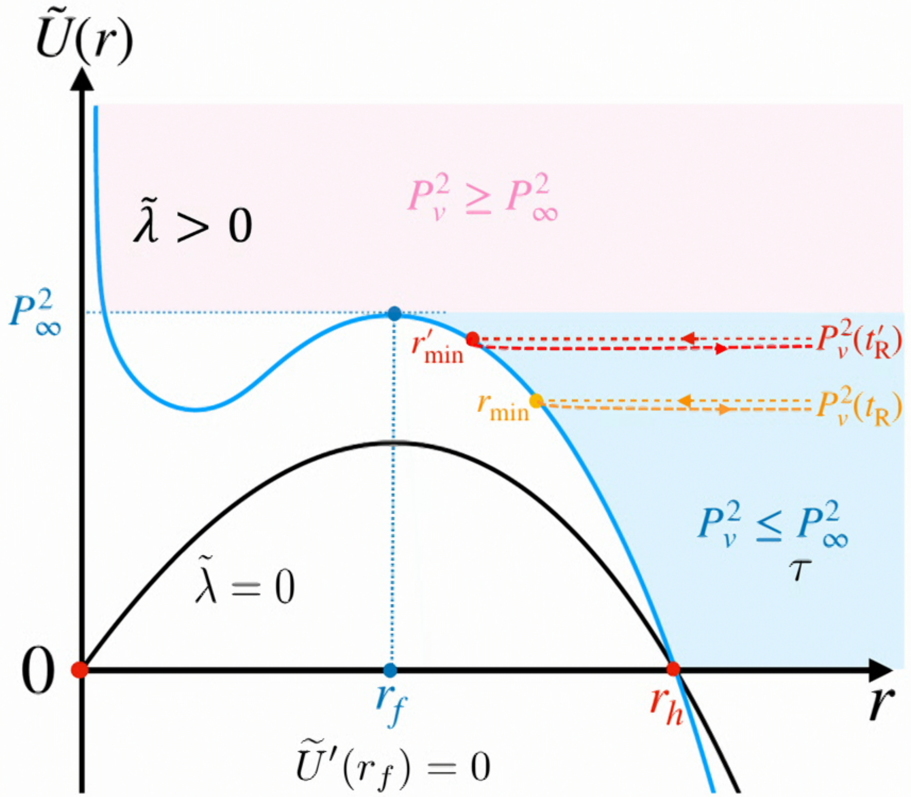

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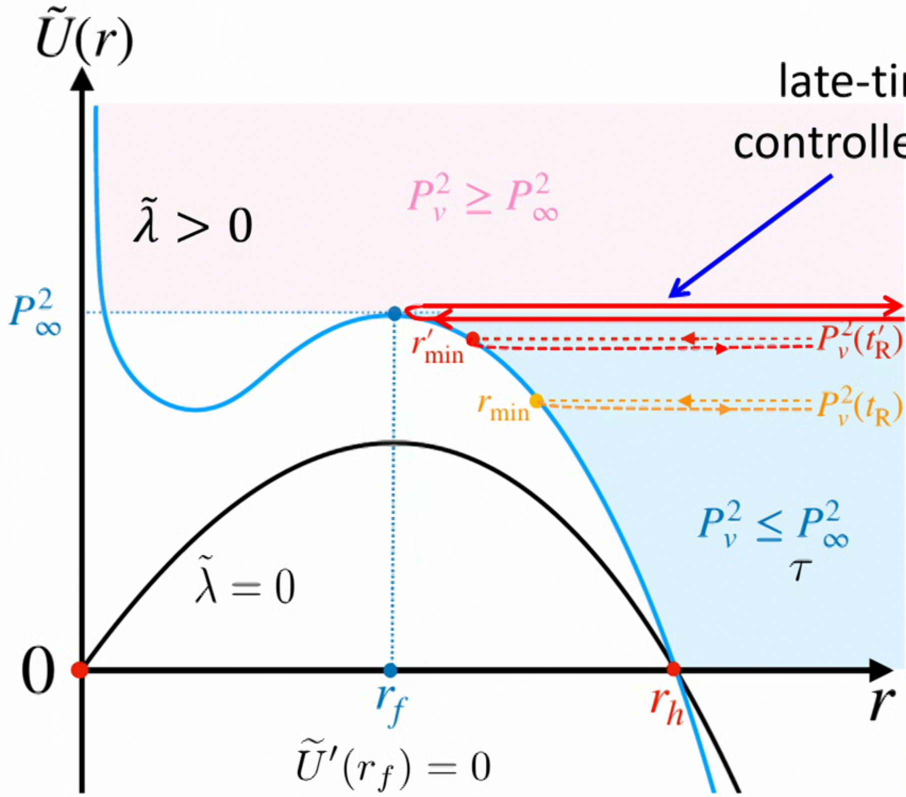

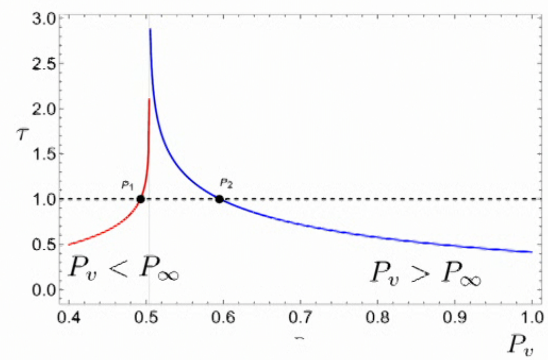

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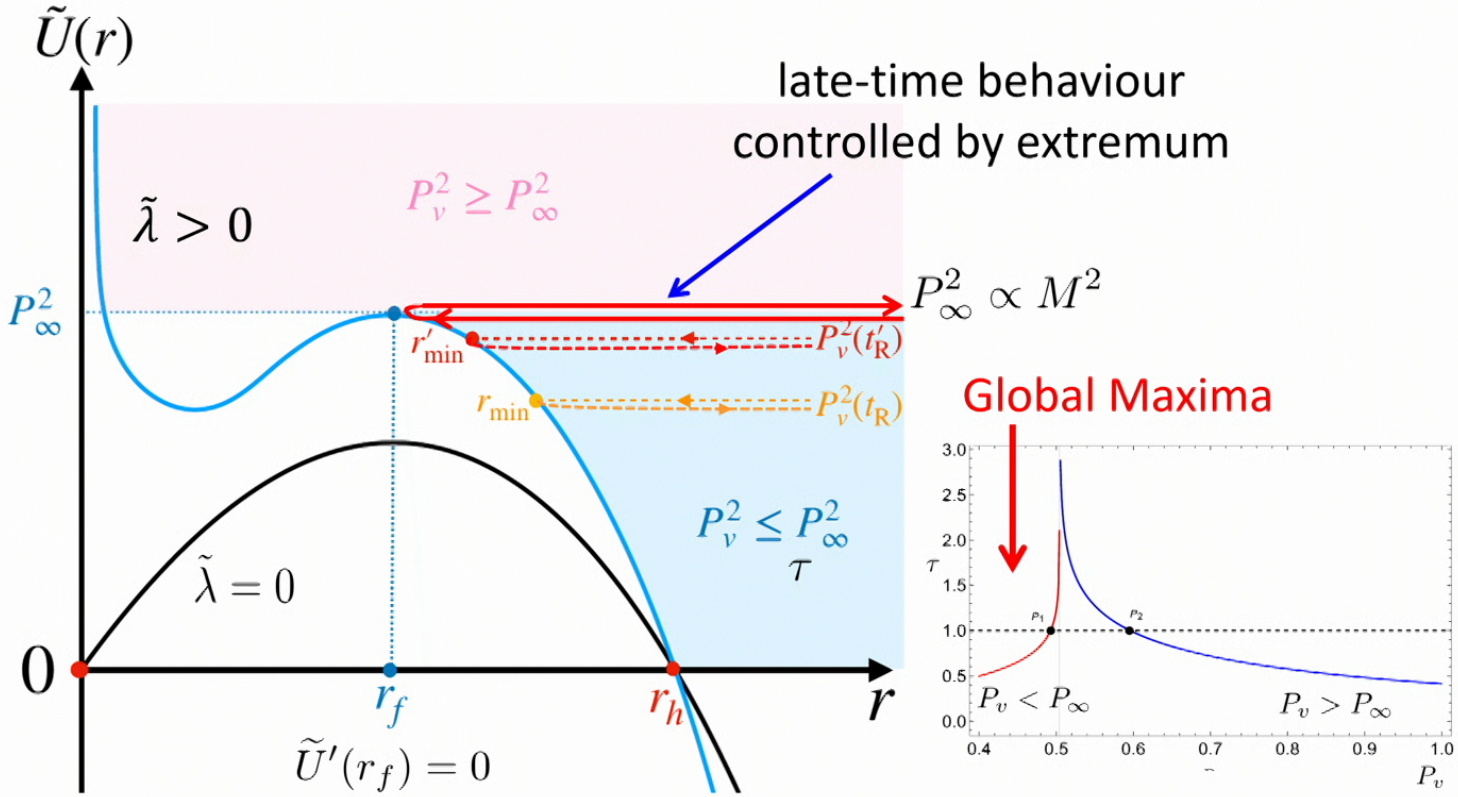
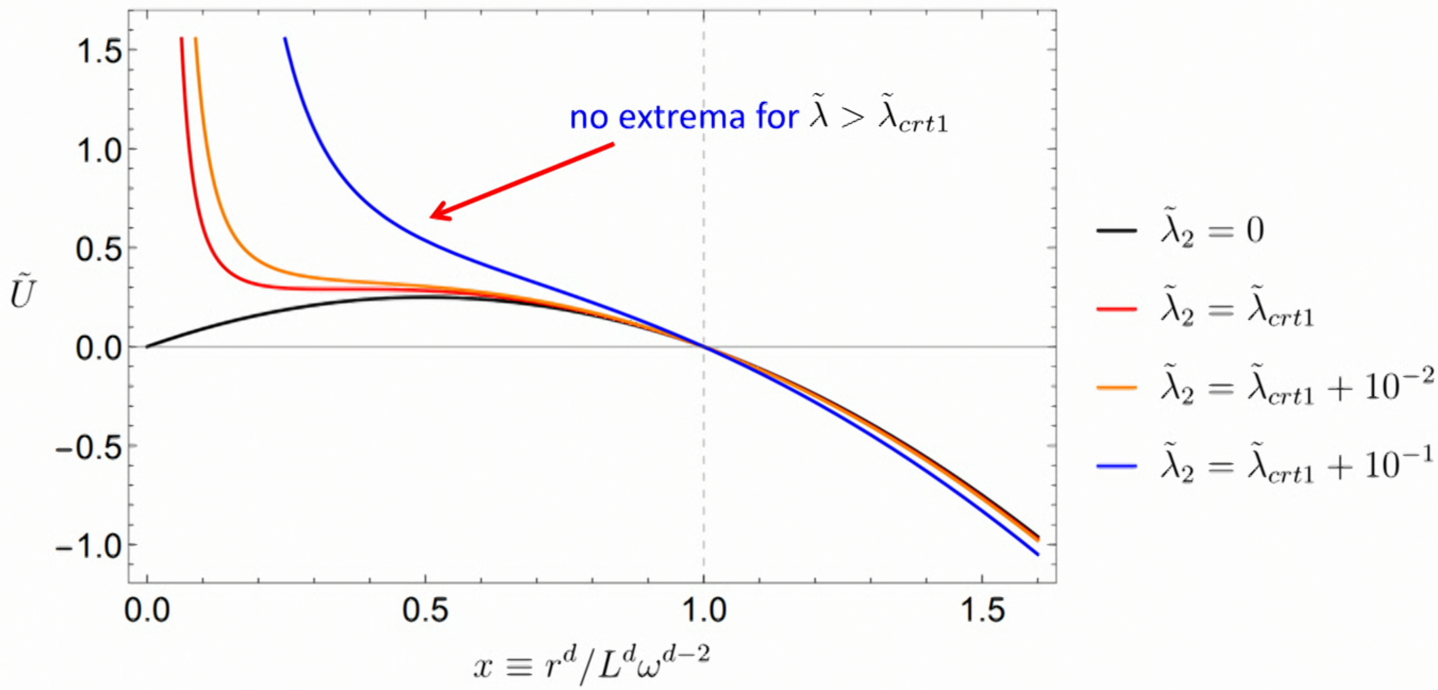


Complexity = Almost Anything:

- recall simple example: $F_1 = F_2 = 1 + \lambda\, L^4\, C_{abcd} C^{abcd}$
- coupling cannot be "too large", ie,

$\tilde{\lambda} = d(d-1)^2(d-2)\,\lambda$

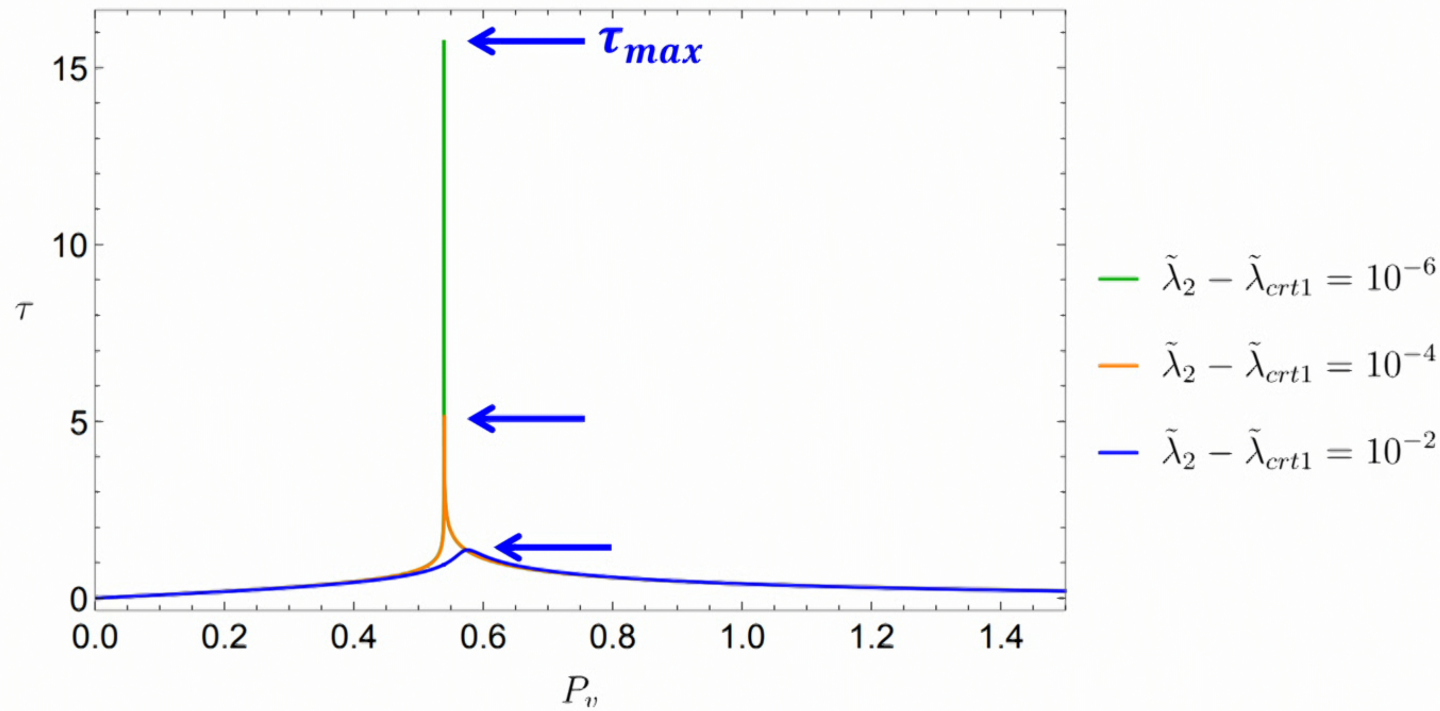
$$-1 < \tilde{\lambda} < \tilde{\lambda}_{crit} = \frac{47 - 13\sqrt{13}}{8} \simeq .016$$



Complexity = Almost Anything:

- coupling for curvature invariants should not be too large

recall $-1 < \tilde{\lambda} < \tilde{\lambda}_{crit} = \dfrac{47 - 13\sqrt{13}}{8} \simeq .016$

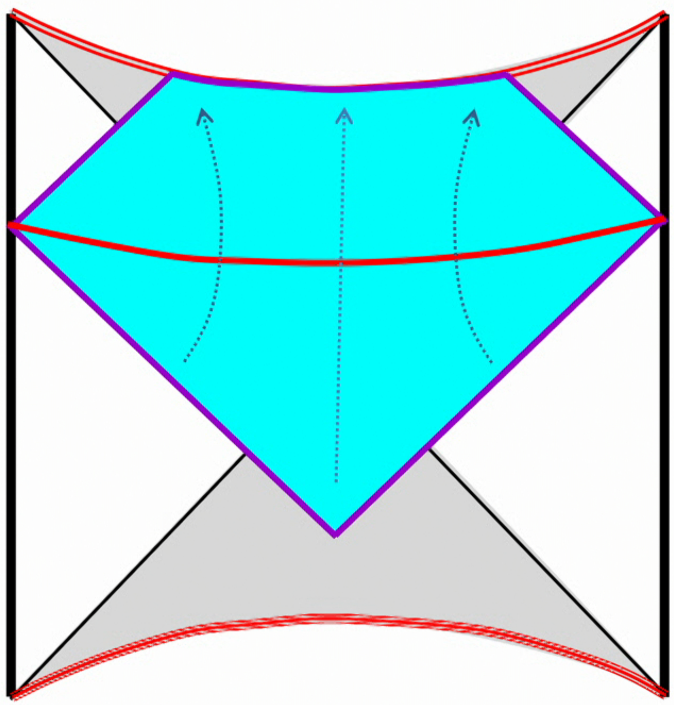


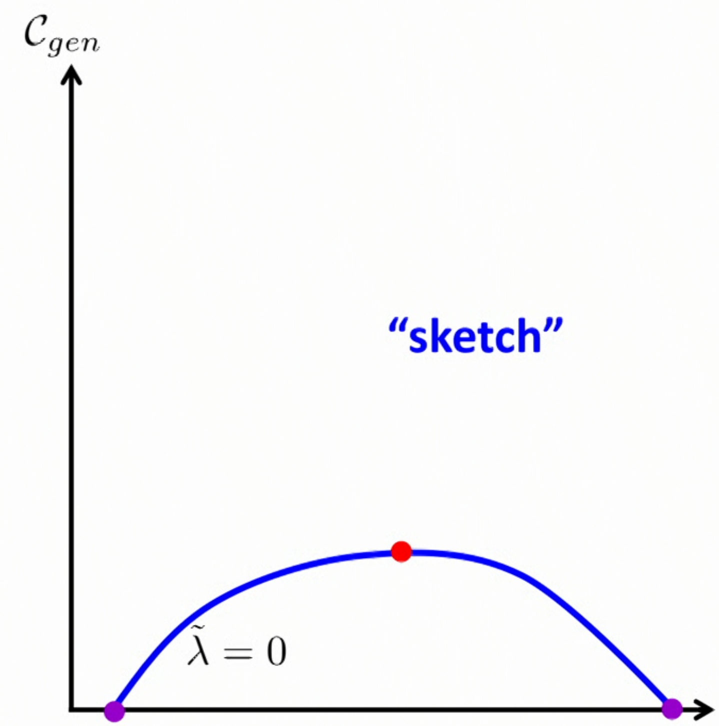
only reach some maximum τ_{max} as P_v is scanned

but what happens for $\tau > \tau_{max}$???

Complexity = Almost Anything:

- where is surface yielding maximal value of $\mathcal{C}_{gen}$ beyond $\boldsymbol{\tau_{max}}$??





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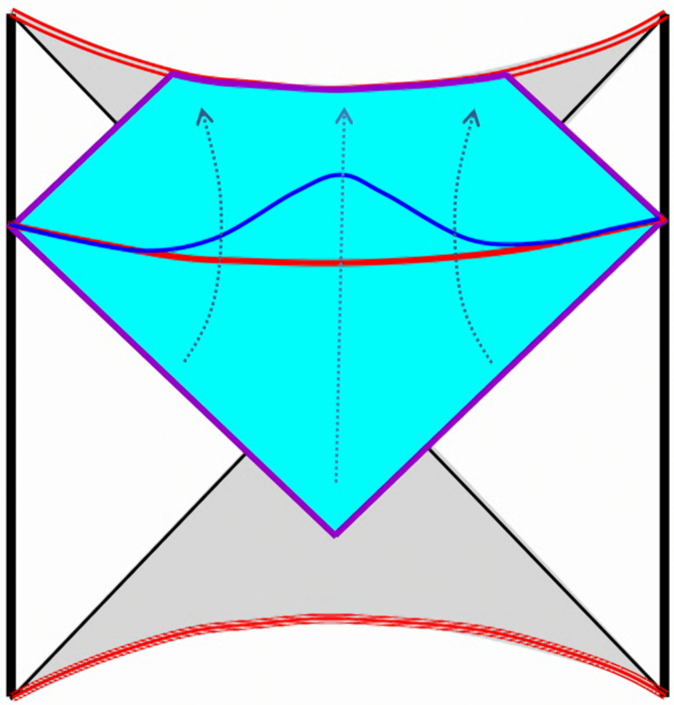

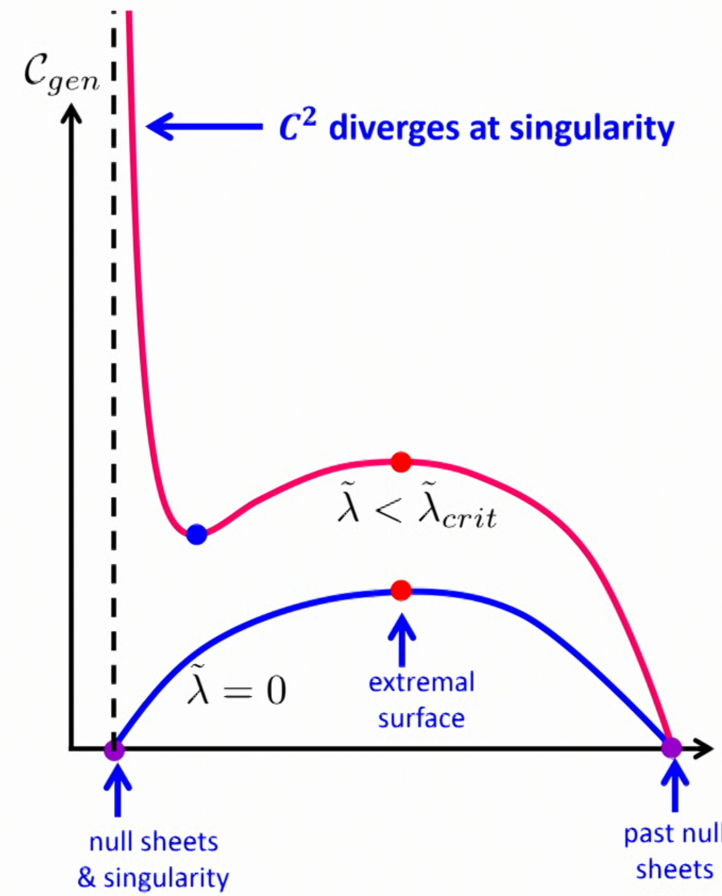

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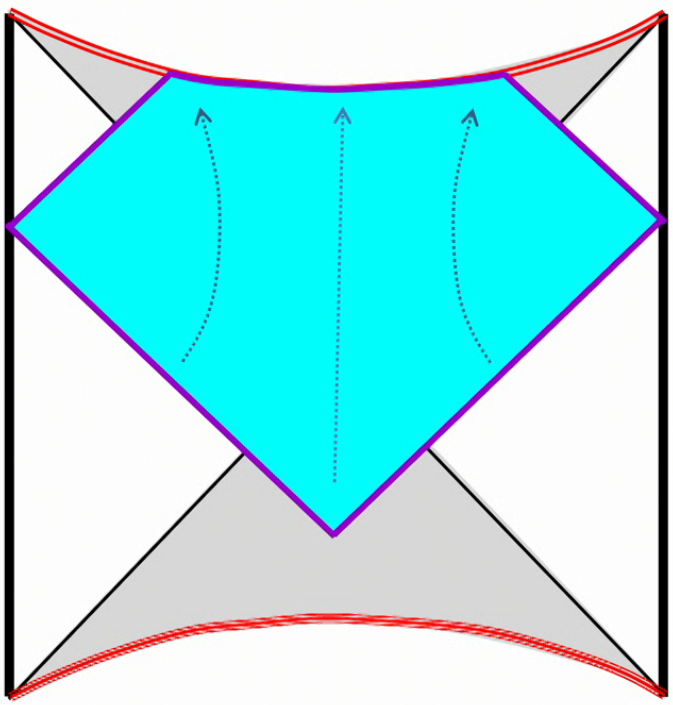
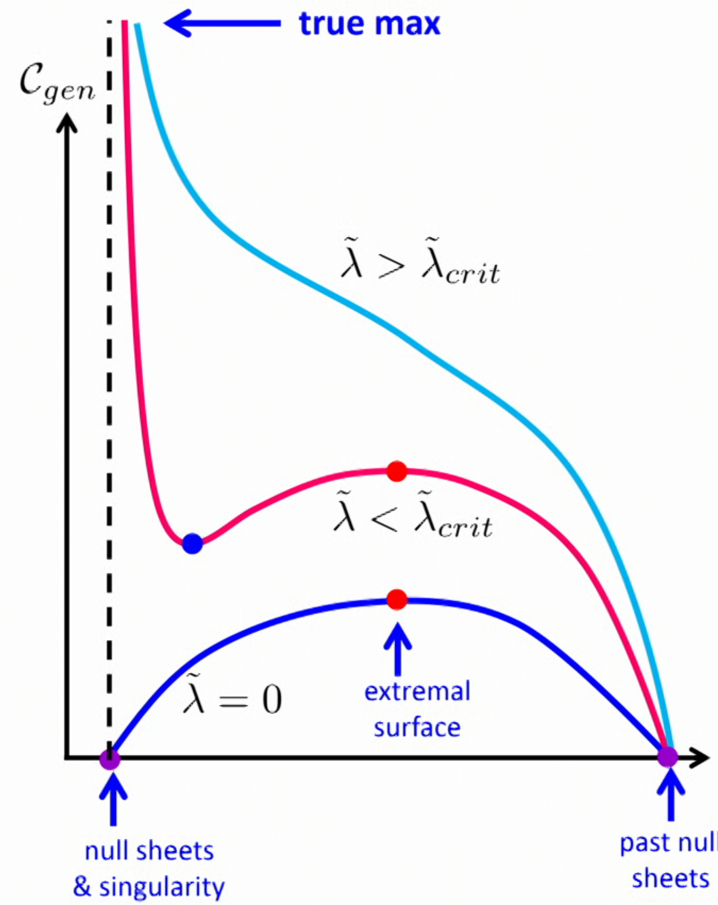
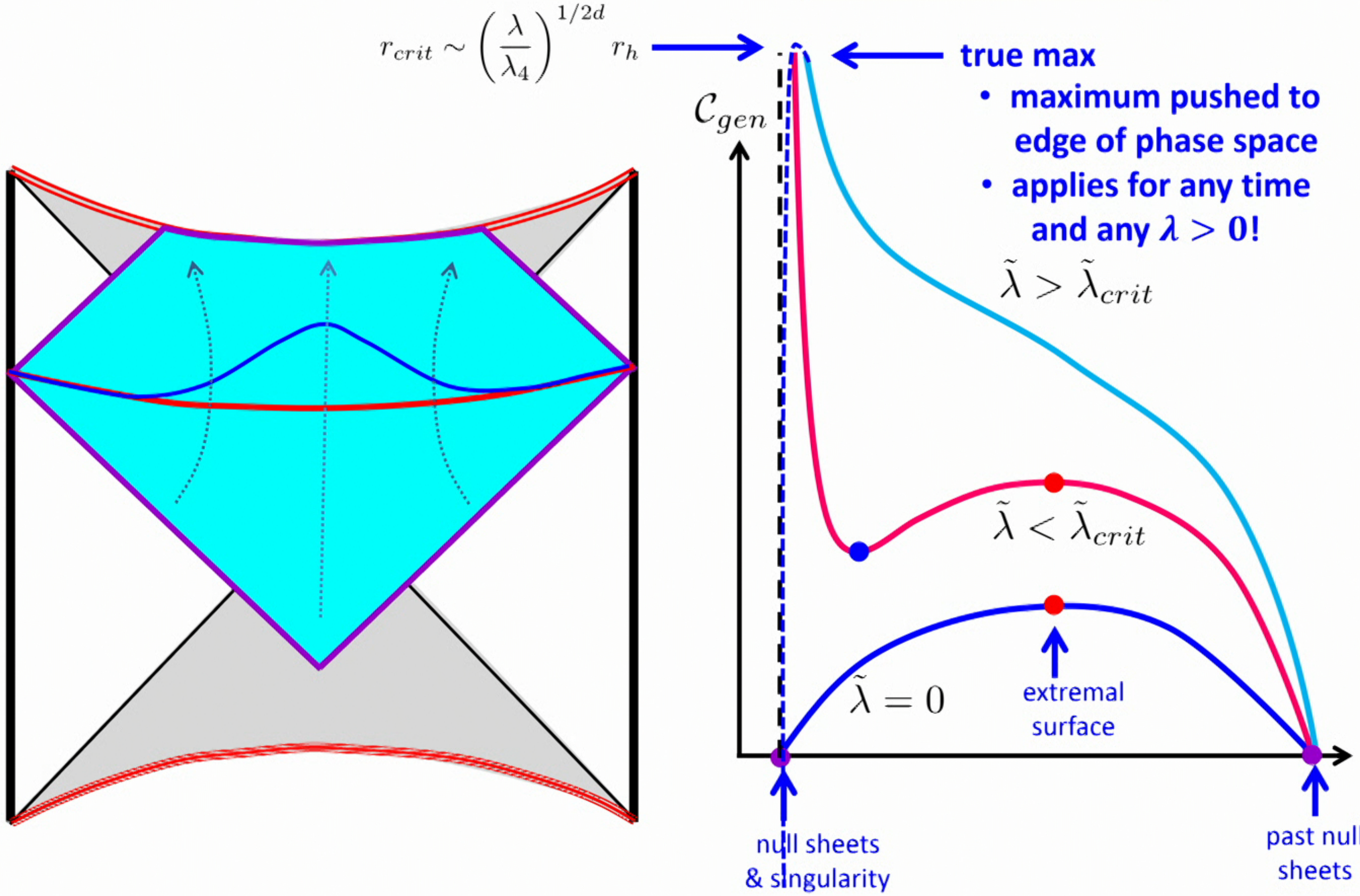




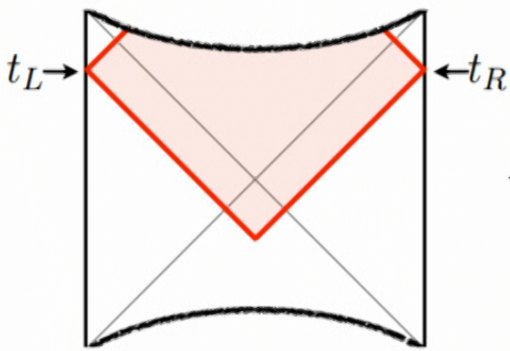
Complexity = Almost Anything:

"regulate": $F_1 = F_2 = 1 + \lambda L^4 C_{abcd}C^{abcd} - \lambda_4 L^8 (C_{abcd}C^{abcd})^2$



Generalizing CA and CV2.0:

- complexity=action: evaluate gravitational action for Wheeler-DeWitt patch = domain of dependence of bulk time slice connecting boundary Cauchy slices in CFT (Brown, Roberts, Swingle, Susskind & Zhao)
- complexity=volume**2.0**: evaluate spacetime volume of WDW patch (Couch, Fischler & Nguyen)



$$\mathcal{C}_A(\Sigma) = \frac{I_{WDW}}{\pi\,\hbar}$$

$$\mathcal{C}'_V(\Sigma) = \frac{V_{WDW}}{G_N\,\ell^2}$$

Two steps: 1) find a special surfaces bounding codim.-0 region

2) evaluate geometric feature of codim.-0 region (& boundary surfaces)

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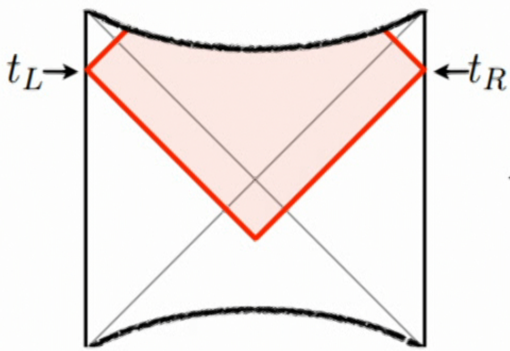

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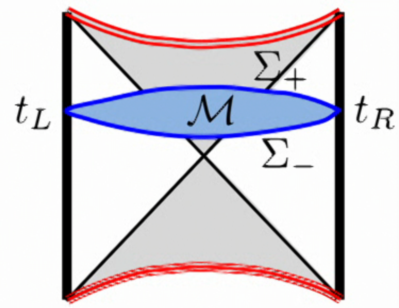


1) find a bounding surfaces $\Sigma_\pm$:

$$\delta_{\{X_-,X_+\}}\Big(\int_{\mathcal{M}} d^{d+1}\sigma\,\sqrt{-g}\;F_6(g_{\mu\nu})\;+$$

$$\int_{\Sigma_+} d^d\sigma\,\sqrt{h}\;F_4(g_{\mu\nu};X^\mu_+)+\int_{\Sigma_-} d^d\sigma\,\sqrt{h}\;F_5(g_{\mu\nu};X^\mu_-)\Big)=0$$

- F_4 and F_5 are scalar functions of bkgd metric $g_{\mu\nu}$ and embeddings $X^\mu_\pm\,(\sigma)$ respectively, while F_6 is scalar function of bkgd metric $g_{\mu\nu}$

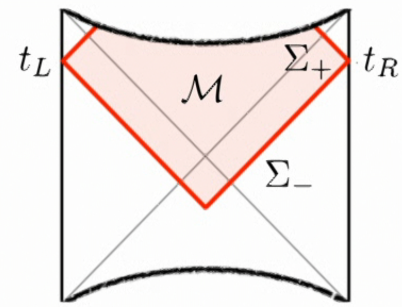
2) Evaluate geometric feature of corresponding region:

$$O(\Sigma_{CFT})=\frac{1}{G_N\,L^2}\int_{\mathcal{M}} d^{d+1}\sigma\,\sqrt{-g}\;F_3(g_{\mu\nu})\;+$$

$$\frac{1}{G_N\,L}\int_{\Sigma_+} d^d\sigma\,\sqrt{h}\;F_1(g_{\mu\nu};X^\mu_+)+\frac{1}{G_N\,L}\int_{\Sigma_-} d^d\sigma\,\sqrt{h}\;F_2(g_{\mu\nu};X^\mu_-)$$

- yields “nice” diffeomorphism invariant observables, which exhibit linear growth at late times as well as the switchback effect!!

Simplest Example:



- extremize the functional

$$O(\Sigma_{CFT}) = \frac{\alpha_+}{G_N\,L}\int_{\Sigma_+} d^d\sigma\,\sqrt{h} + \frac{\alpha_-}{G_N\,L}\int_{\Sigma_-} d^d\sigma\,\sqrt{h} + \frac{1}{G_N\,L^2}\int_{\mathcal{M}} d^{d+1}\sigma\,\sqrt{-g}$$

- evaluating the volumes of the bounding surfaces $\Sigma_\pm$ weighted by coefficients $\alpha_\pm$, as well as of volume of codim.-0 region $\mathcal{M}$
- extremal equations yields CMC surfaces (eg, see Witten & Kuchar)

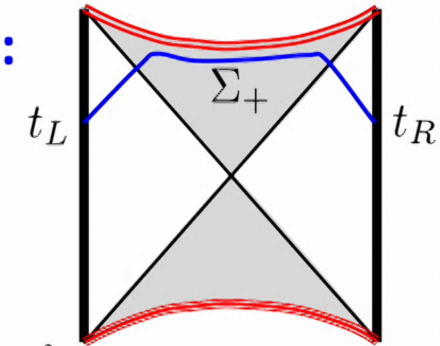
$$K(\Sigma_+) = -\frac{1}{\alpha_+\,L} \qquad K(\Sigma_-) = +\frac{1}{\alpha_-\,L}$$

- in limit $\alpha_\pm \to 0$, these surfaces become the future/past light sheets

  ⟶ $\mathcal{M}$ becomes WDW patch!

- evaluate action (including bdy terms) ⟶ complexity=action
- evaluate volume (same functional) ⟶ complexity = volume2.0

Probe the singularity with simple example:



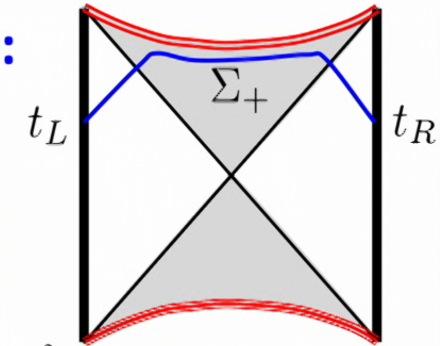
- extremize the functional

$$O(\Sigma_{CFT}) = \frac{\alpha_+}{G_N\, L} \int_{\Sigma_+} d^d\sigma \sqrt{h} + \frac{\alpha_-}{G_N\, L} \int_{\Sigma_-} d^d\sigma \sqrt{h} + \frac{1}{G_N\, L^2} \int_{\mathcal{M}} d^{d+1}\sigma \sqrt{-g}$$

- consider $\alpha_+ \ll 1$ so that future boundary Σ_+ approaches singularity
- evaluate the functional

$$O(\Sigma_{CFT}) = \frac{1}{G_N\, L^2} \int_{\mathcal{M}} d^{d+1}\sigma \sqrt{-g}\; F_3(g_{\mu\nu}) + \frac{1}{G_N\, L} \int_{\Sigma_+} d^d\sigma \sqrt{h}\; F_1(g_{\mu\nu}; X^\mu_+) + \frac{1}{G_N\, L} \int_{\Sigma_-} d^d\sigma \sqrt{h}\; F_2(g_{\mu\nu}; X^\mu_-)$$

Probe the singularity with simple example:



- extremize the functional

$$O(\Sigma_{CFT}) = \frac{\alpha_+}{G_N\, L} \int_{\Sigma_+} d^d\sigma \sqrt{h} + \frac{\alpha_-}{G_N\, L} \int_{\Sigma_-} d^d\sigma \sqrt{h} + \frac{1}{G_N\, L^2} \int_{\mathcal{M}} d^{d+1}\sigma \sqrt{-g}$$

- consider $\alpha_+ \ll 1$ so that future boundary Σ_+ approaches singularity

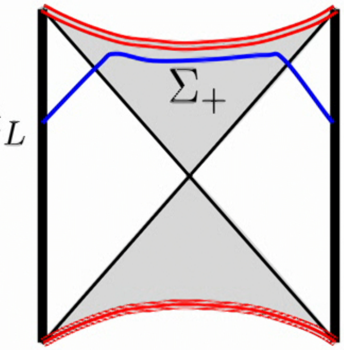
- evaluate the functional

$$O(\Sigma_{CFT}) = \frac{1}{G_N\, L^2} \int_{\mathcal{M}} d^{d+1}\sigma \sqrt{-g}\, F_3(g_{\mu\nu}) + \frac{1}{G_N\, L} \int_{\Sigma_+} d^d\sigma \sqrt{h}\, F_1(g_{\mu\nu}; X^\mu_+) + \frac{1}{G_N\, L} \int_{\Sigma_-} d^d\sigma \sqrt{h}\, F_2(g_{\mu\nu}; X^\mu_-)$$

($F_3(g_{\mu\nu})$ → 0; $F_2(g_{\mu\nu}; X^\mu_-)$ → 0)

$F_1(g_{\mu\nu}; X^\mu_+)$: different choices

Probe the singularity with simple example:



- consider only Σ_+ with $K = -{}^{1}/_{\alpha_+ L}$ and $\alpha_+ \ll 1$

$$O(\Sigma_{CFT}) = \frac{1}{G_N\, L} \int_{\Sigma_+} d^d\sigma \sqrt{h}\; F_1(g_{\mu\nu}; X^\mu_+)$$

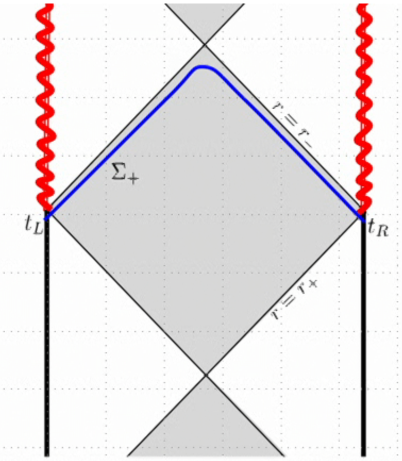
- late time growth dominated by surface $r_f \simeq r_h\, \alpha_+^{2/d}$

$$F_1 = 1 \quad : \quad \frac{d\mathcal{C}_{\rm gen}}{d\tau} \simeq \frac{8\pi\, d}{d-1}\, M\, \alpha_+ \to 0$$

$$F_1 = -L\, K \quad : \quad \frac{d\mathcal{C}_{\rm gen}}{d\tau} \simeq \frac{8\pi\, d}{d-1}\, M$$

$$F_1 = L^4\, C^2 \quad : \quad \frac{d\mathcal{C}_{\rm gen}}{d\tau} \simeq 128\pi\, d^4(d-1)(d-2)\, M\, \frac{1}{\alpha_+^3} \to \infty$$

Probe the singularity with simple example:



- consider only Σ_+ with $K = -{}^1/_{\alpha_+ L}$ and $\alpha_+ \ll 1$

$$O(\Sigma_{CFT}) = \frac{1}{G_N L} \int_{\Sigma_+} d^d\sigma \sqrt{h}\; F_1(g_{\mu\nu}; X^\mu_+)$$

- compare to charged black hole

- late time growth dominated by surface $r_f \simeq r_- + 4\pi L^2 T_- \, \alpha_+^2$

only probe up to Cauchy horizon

$$F_1 = 1 \quad : \quad \frac{d\mathcal{C}_{\rm gen}}{d\tau} \simeq 16\pi\, S_- T_-\, \alpha_+ \to 0$$

$$F_1 = -L\, K \quad : \quad \frac{d\mathcal{C}_{\rm gen}}{d\tau} \simeq 16\pi\, S_- T_-$$

entropy and temperature of inner horizon

$$F_1 = L^4\, C^2 \quad : \quad \frac{d\mathcal{C}_{\rm gen}}{d\tau} \simeq \text{“mess”}\;\, S_- T_-\, \alpha_+ \to 0$$

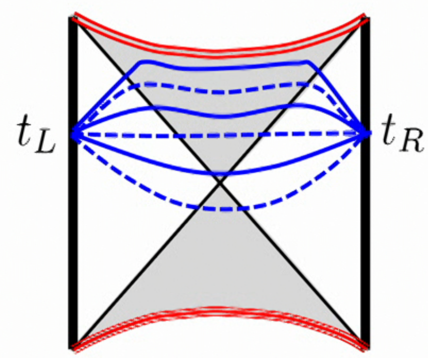
Comments:

- have found an infinite class of gravitational observables which exhibit complexity-like behaviour

→ different observables probe different spacetime regions

→ all information encoded within a given boundary state (encoded within the corresponding WDW patch)

→ in boundary theory, different realizations of complexity make different features of the spacetime geometry manifest



Comments:

- have found an infinite class of gravitational observables which exhibit complexity-like behaviour

⟶ have separated functional which is extremized and functional which is evaluated

⟶ interpretation in boundary theory???

Conclusions/Questions/Outlook:

- simple example but "classical mechanics" analysis readily extends to $F_1(g_{\mu\nu}, \mathcal{R}_{\mu\nu\rho\sigma}, \nabla_\mu)$ and to observables where $F_1 \neq F_2$
- couplings for curvature invariants should not be too large
- similar behaviour appears to hold for functionals including dependence on extrinsic curvature
- **infinite class of holographic observables equally viable candidates for gravitational dual of complexity!!**
- can freedo ... elated to freedom i ... T
- is there so ...
- what is role of extremal solutions which are not global maxima and probe very near to singularity?
- add matter contributions to new observables (eg, CA proposal)
- **precise interpretation of gravitational observables in boundary QFT**

Lots to explore!