Title: Talk 110 - NoRA: A Tensor Network Ansatz for Volume-Law Entangled Equilibrium States of Highly Connected Hamiltonians

Speakers: ValÃ©rie Bettaque

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Abstract: Motivated by the ground state structure of quantum models with all-to-all interactions such as mean-field quantum spin glass models and the Sachdev-Ye-Kitaev (SYK) model, we propose a tensor network architecture which can accomodate volume law entanglement and a large ground state degeneracy. We call this architecture the non-local renormalization ansatz (NoRA) because it can be viewed as a generalization of MERA, DMERA, and branching MERA networks with the constraints of spatial locality removed. We argue that the architecture is potentially expressive enough to capture the entanglement and complexity of the ground space of the SYK model, thus making it a suitable variational ansatz, but we leave a detailed study of SYK to future work. We further explore the architecture in the special case in which the tensors are random Clifford gates. Here the architecture can be viewed as the encoding map of a random stabilizer code. We introduce a family of codes inspired by the SYK model which can be chosen to have constant rate and linear distance at the cost of some high weight stabilizers. We also comment on potential similarities between this code family and the approximate code formed from the SYK ground space.

Brandeis
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NoRA

A Renormalization Ansatz for Non-Local Many-Body Quantum Systems

Valérie Bettaque

[arXiv:2303.16946 \w Brian Swingle]

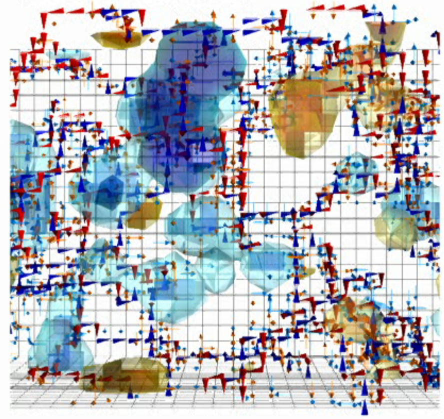
It from Qubit, August 1 2023

What is a renormalization ansatz?

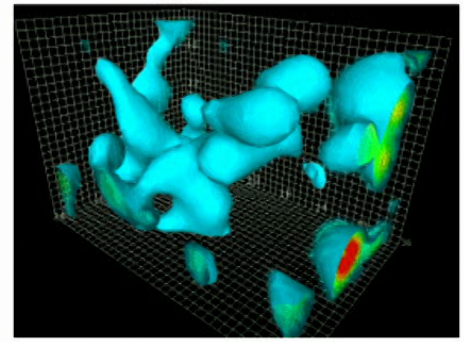
And why should we care?

- Perturbative methods **fail** to describe strongly coupled many-body systems and quantum field theories accurately.
- Approximating such systems as **discrete lattices** introduces a "natural" UV cutoff $1/a$ (inverse lattice spacing).
- States on any such lattice (and their entanglement structure) can then be approximated using **tensor networks.**
  - $D = 1$: **Matrix product states (MPS)**
  - $D > 1$: **Projected entangled pair states (PEPS)**
- Local observables are efficiently computable, but only exactly for MPS. PEPS can only be approximated.
- Hilbert space still grows exponentially with lattice size.
- **Ansatz**: Use **density matrix renormalization group (DMRG)** to construct a **variational** tensor network:
  - Parametrize different (discrete) energy scales of the system individually, connected by **fine/coarse-graining** isometries.

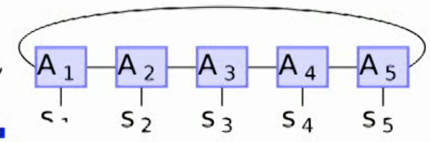
[Biddle et al; 2019]

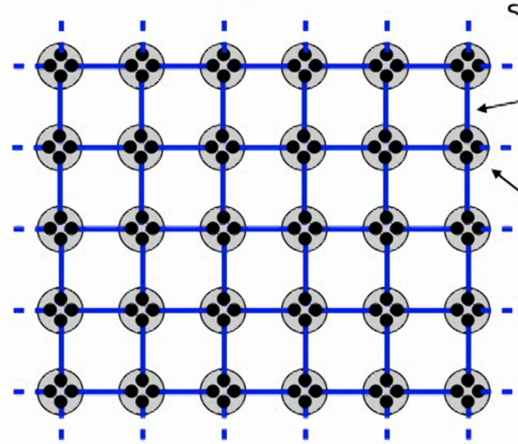


[CSSM, U. Adelhaide: 2003]



$$|\Psi\rangle = \sum_{\{s\}} \mathrm{Tr}\left[A_1^{(s_1)} A_2^{(s_2)} \cdots A_N^{(s_N)}\right] |s_1 s_2 \ldots s_N\rangle \sim$$





maximally entangled state

$$|I\rangle = \sum_{i=1}^{D} |ii\rangle$$

projection onto lattice site

$$\hat{T} = \sum_{l,r,u,d=1}^{D} \sum_{p}^{d} T^{p}_{lrud} |p\rangle\langle l\, r\, u\, d|$$

[Verstraete, Cirac; 2019]

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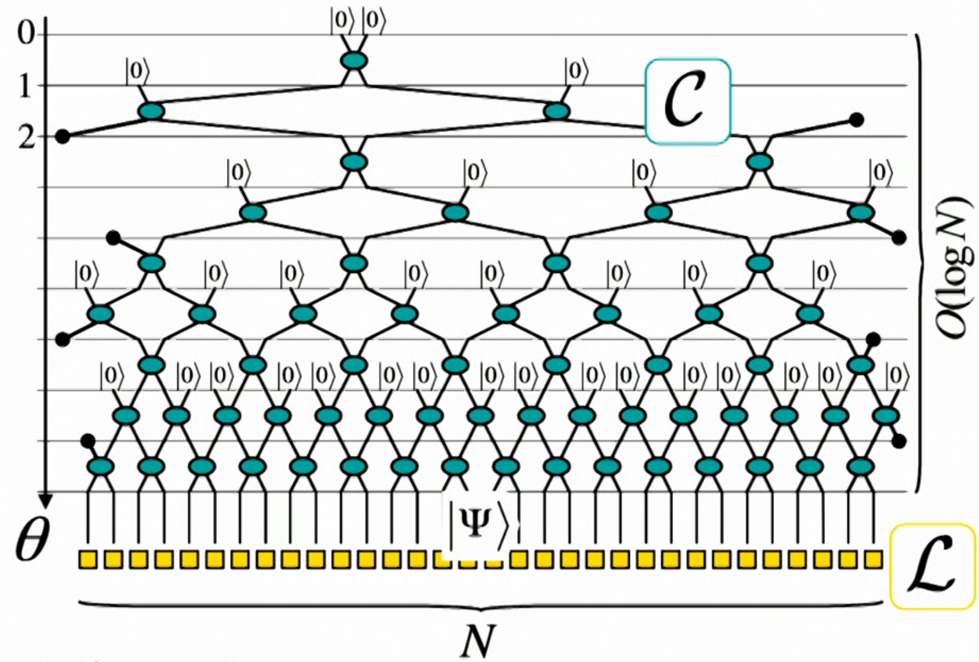
What is a renormalization ansatz?

Example: MERA [Vidal, 2006]

- **Multi-Scale Entanglement Renormalization Ansatz**
- "Time" variable θ goes from top of the network (IR) to the bottom (UV) in $\mathcal{O}(\log N)$ steps.

IR (ground state)

$\mathcal{O}(k)$ DOFs, short-range entanglement, information localized on the "lattice".



UV (lattice state)

$\mathcal{O}(N \gg k)$ DOFs, long range entanglement, information spread out over lattice.

3

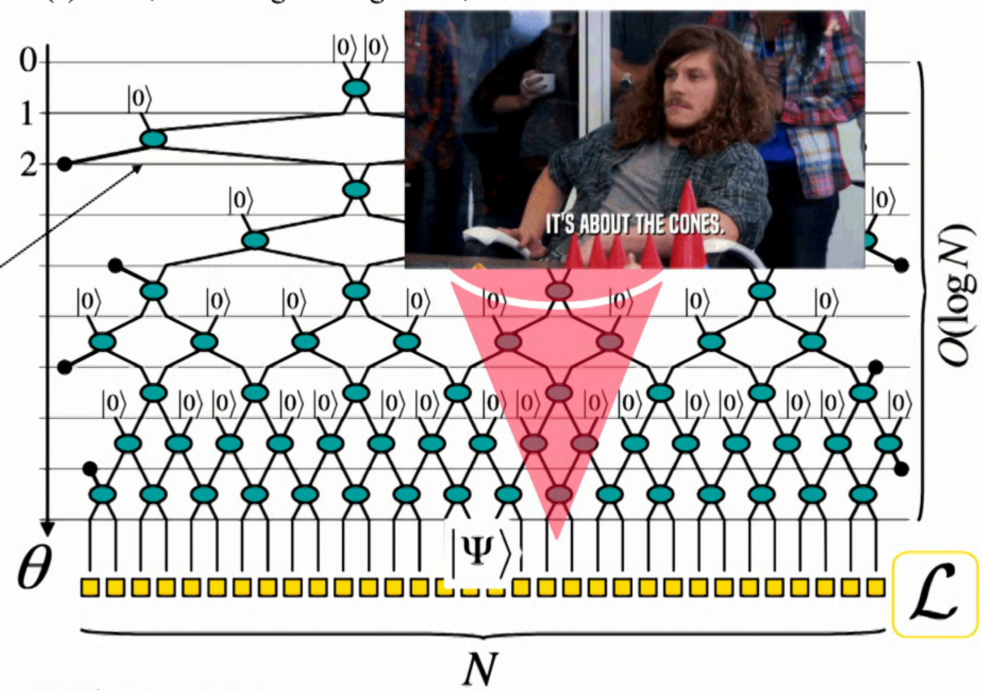
What is a renormalization ansatz?

Example: MERA [Vidal, 2006]

- **Multi-Scale Entanglement Renormalization Ansatz**
- "Time" variable θ goes from top of the network (IR) to the bottom (UV) in $\mathcal{O}(\log N)$ steps.
- Local fine-graining isometries (green) "create" entanglement and encode theory.
- Hence MERAs exhibit locality and causal structure (red cone).
    - Network can be efficiently simulated (**"contracted"**).
    - Local observables can be exactly computed.
- Self-similarity of circuit reflects renormalization invariance (demanded by DMRG).

IR (ground state)

$\mathcal{O}(k)$ DOFs, short-range entanglement, information localized on the "lattice".



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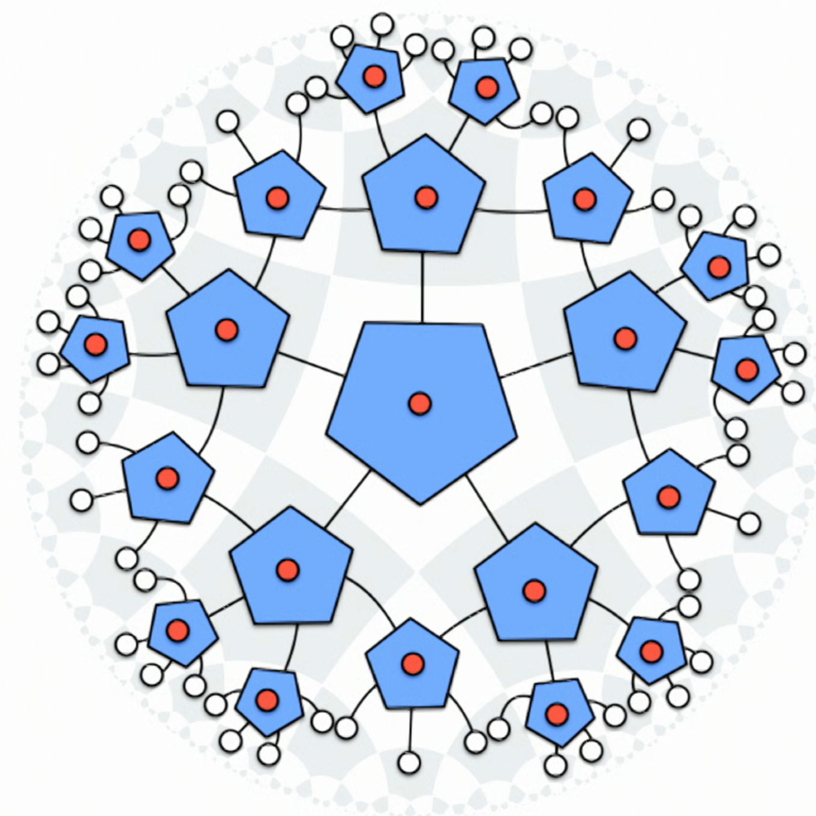
What is a renormalization ansatz?

Example: HaPPY [Harlow et al. 2015]

- MERA-like toy model for AdS/CFT.
  - IR bulk (D-dim AdS) is dual to UV boundary (($D-1$)-dim CFT).
  - Can also be interpreted as a PEPS.
- Hyperbolic tessellation of AdS Cauchy slice produces a RG-conform tensor network.



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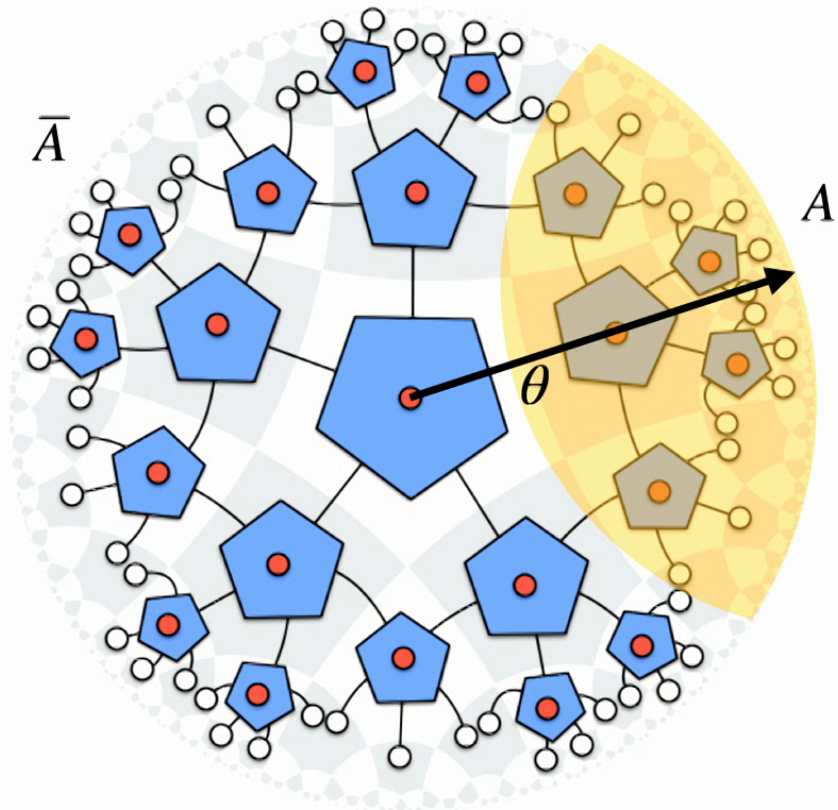
    - Radial direction θ corresponds to renormalization scale.
- Resulting “causal structure” of HaPPY allows for exact entanglement wedge reconstruction.



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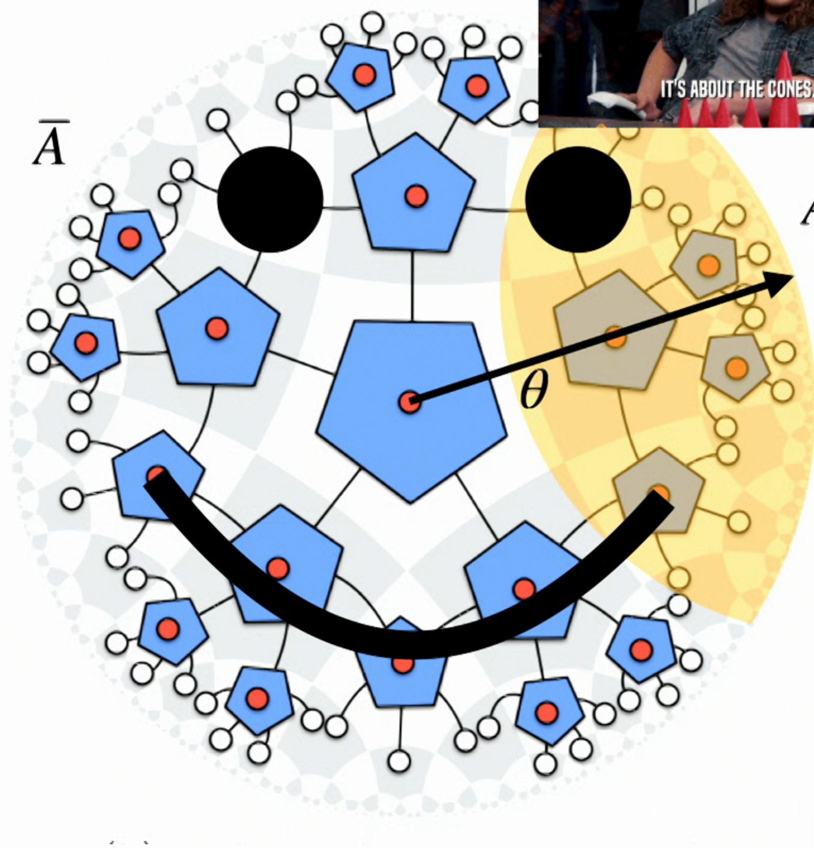
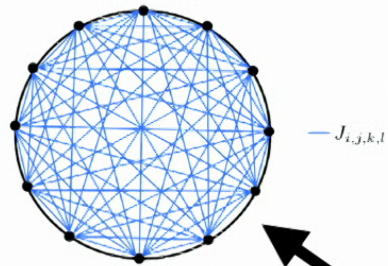
  - Radial direction θ corresponds to renormalization scale.
- Resulting "causal structure" of HaPPY allows for exact entanglement wedge reconstruction.
  - Concrete bulk-boundary dictionary.
  - Reproduces Ryu-Takayanaki (RT) etc.



Our goal: Get rid of locality!

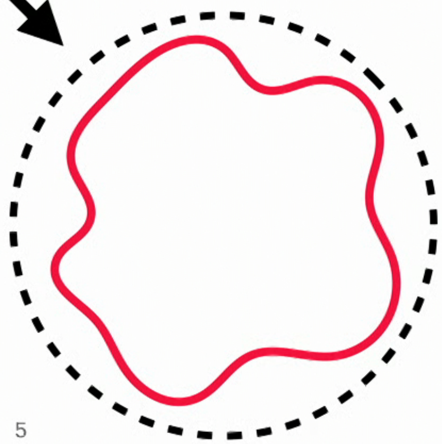
(also world domination...)

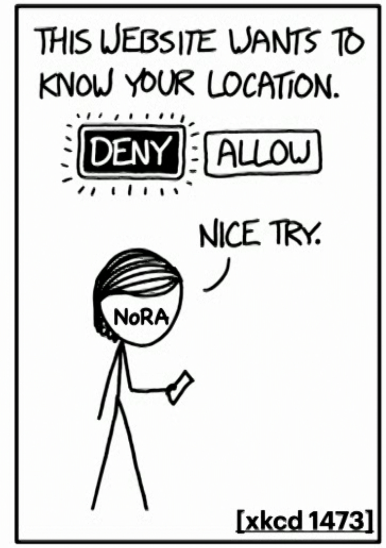
- Generalizing the renormalization ansatz like this opens up applications for a larger variety of models.
- Main interest: **SYK (Sachdev-Ye-Kitaev) Model**
  - $(0+1)$-dim. CFT sector in the large N limit.
  - CFT-breaking sector dual to deformations of $(1+1)$-dim. **JT (Jackiw-Teichelboim) Gravity** to leading order (**Schwarzian**).
- Many other systems in physics can be approximated as having no spatial DOFs:
  - **Spin glasses, nucleons, thermalizing systems**.
- However: Lack of causal structure means the circuit **can not be efficiently contracted** (classically).



$$H = (i)^{q/2} \sum_{1 \le i_1 < i_2 < \ldots < i_q \le N} J_{i_1 i_2 \ldots i_q} \chi_{i_1} \chi_{i_2} \cdots \chi_{i_q}$$

$$S = \frac{1}{\kappa} \int d^2x \sqrt{-g}\, \Phi\, (R - \Lambda)$$





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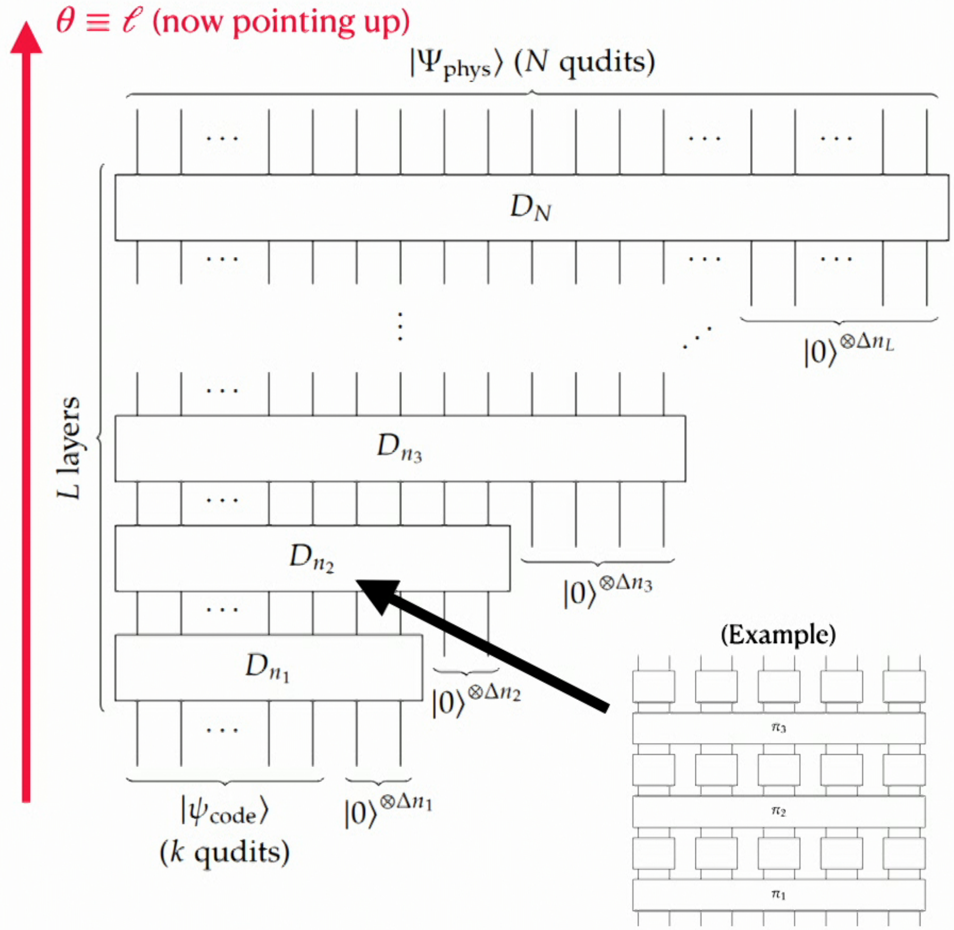
Introducing: NoRA

Also known as RAMEN (sorry Brian :P)

- Each layer $1 \leq \ell \leq L$ has a depth D (**const.**) q-local isometry D_{n_ℓ} acting on the previous $n_{\ell-1}$ qudits and Δn_ℓ new "thermal" ancillary qudits $|0\rangle^{\otimes \Delta n_\ell}$
  - Like in MERA, D_{n_ℓ} can encode scale-dependent details of the theory i.e. its Hamiltonian description and entanglement structure.
- Renormalization invariance requires $\Delta n_\ell \propto e^\ell$
  - $n_\ell \equiv k + r^\ell$ and hence $\Delta n_\ell = r^\ell - r^{\ell-1} \approx r^{\ell-1}$.
  - $N \equiv n_L = k + r^L \quad \Longrightarrow \quad L \propto \log N$ (same as MERA)
- Highly degenerate ground space: $\dim \mathscr{H} = d^k$.
  - Scaling $k \propto N$ required to align ansatz with (super-symmetric) SYK properties.
  - For fixed k, NoRA can be interpreted as a (stabilizer) **quantum error-correcting code (QECC)**.



NoRA as a QECC

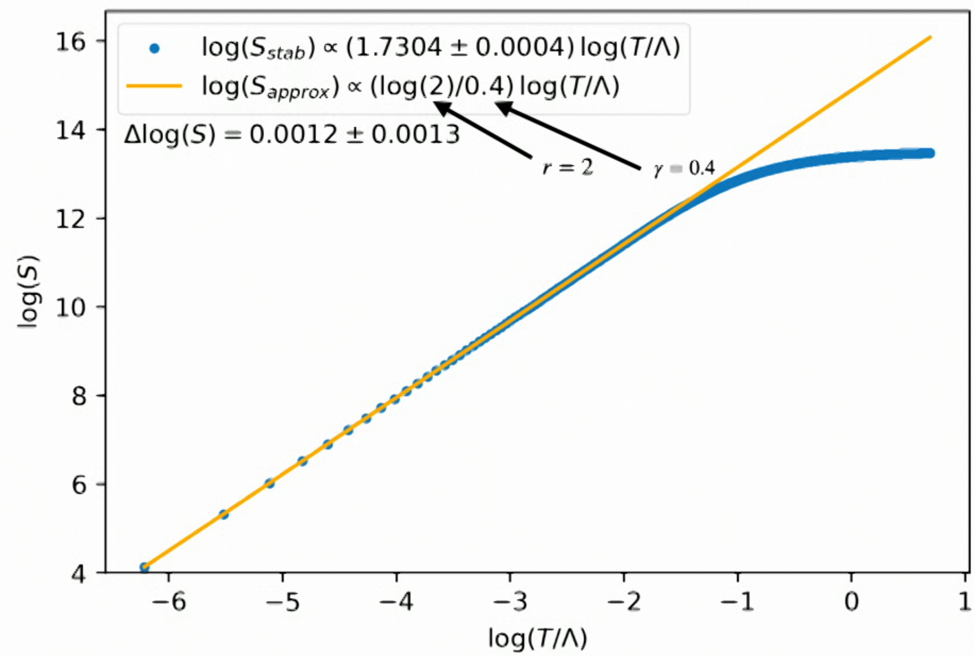
A stable basis (get it?) for the future of quantum computing.

- A $[[N, k, d]]$ quantum error-correcting code ...
  - ... consists of N **physical** qudits ...
  - ... which encode k **logical** qudits ...
  - ... such that the logical state (**code word**) is protected from up to $d-1$ **localized** physical qudit flip errors (**Hamming code distance**).
- Classical ECCs are used in all modern electronic devices.
  - Those codes are not easy to translate to quantum due to their dependence on cloning and measurements.
- **Stabilizer codes** [Gottesmann; 1997] are the current state of the art for QECCs.
  - $N-k$ mutually commuting independent Pauli strings S_i satisfying $S_i|\psi\rangle = |\psi\rangle$ for any codeword $|\psi\rangle$. Those strings form the **stabilizer basis** i.e. any linear combination has the same properties.
  - Should an error occur such that $S_i|\psi\rangle \neq |\psi\rangle$ for some i, then it can be corrected using the associated (well-defined) error operators E_i that satisfy $[E_i, S_j] = 0$ when $i \neq j$, and $\{E_i, S_i\} = 0$ otherwise.
  - The space of **logical operators** that commute with all stabilizer basis elements S_i hence must map between code words.
  - The **weight** of a stabilizer code is the largest number of non-trivial operators (Pauli X, Y, Z) that occur in a single basis element.
- Any stabilizer basis of NoRA can be written as $S_i = U^\dagger Z_{k+i} U$ where Z_i acts on the ith ancillary qubit $|0\rangle$, and U encodes the circuit.

NoRA as a QECC

Don't get too excited!

- Any stabilizer QECC can be written as a projective Hamiltonian with the code space as its ground space:
  - $H = -\sum_i J_i \cdot \Pi_i \quad J_i > 0, \quad \Pi_i = (S_i + I)/2$
  - J_i are free parameters determining the energy scales associated to a given stabilizer basis S_i (or equivalent projector basis Π_i).
  - **Correctable** errors are interpreted as excitations on top of the ground state.
- For NoRA: $J_i \to J_\ell = \Lambda \cdot e^{-\gamma(L-\ell)}$ with $\Lambda, \gamma > 0$ due to renormalization invariance.
- Entropy of associated Gibbs state $\rho = e^{-\beta H}/Z$ obeys power law at low relative temperatures T/Λ.
  - Growth determined by α/γ where $\alpha = \log r$



$$S \lessapprox k\log(d) + (d-1)(N-k)\cdot\frac{\alpha}{\gamma}\cdot\Gamma\left(\frac{\alpha}{\gamma}+1\right)\cdot(\beta\Lambda)^{-\alpha/\gamma} \propto (T/\Lambda)^{\alpha/\gamma}$$

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NoRA as a QECC

Some heuristics

- Case of scaling k can be expressed as $k = r^a$ and $L = a + b$, where a, b integers and b fixed
  - Hence $N = r^a + r^{a+b}$ and $\dfrac{k}{N} = \dfrac{1}{1 + r^b}$ constant, implying $k \propto N$
- Circuit complexity estimated by $$\text{total gates} = \frac{D}{q}\sum_{\ell=1}^{L} n_\ell = \frac{D}{q}\left(L \cdot k + \frac{r^{L+1} - r}{r - 1}\right) = \begin{cases} \dfrac{D}{q}\dfrac{r}{r-1} \cdot N + O(\log N) & k \text{ fixed} \\ \dfrac{D}{q \cdot (1 + r^b)} \cdot N \log_r N + O(N) & k \propto N \end{cases}$$
  - Scaling k case reproduces holographic complexity of JT gravity.
- Qudit ratio at subsequent layers: $R_\ell = \dfrac{k + r^\ell}{k + r^{\ell-1}} \lessapprox r$
- Weight ratio at subsequent approximately g^D, where $g \approx \dfrac{d^2 - 1}{d^2} \cdot q > 1$
- Hence, in the limit $a \to \infty$ we expect a phase transition around $g^D \approx r$.
  - $g^D > r$: Weight growth dominates over qudit growth => stabilizer weights will be approach $\langle w \rangle = \dfrac{q^2 - 1}{q^2} \cdot N$.

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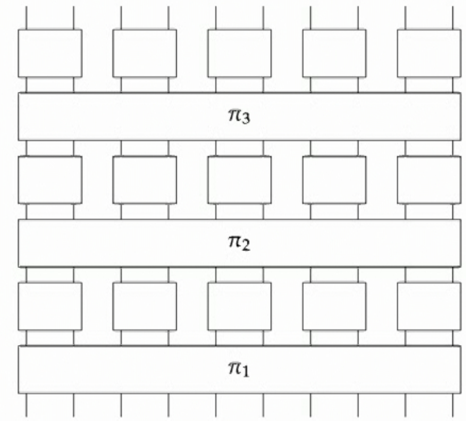
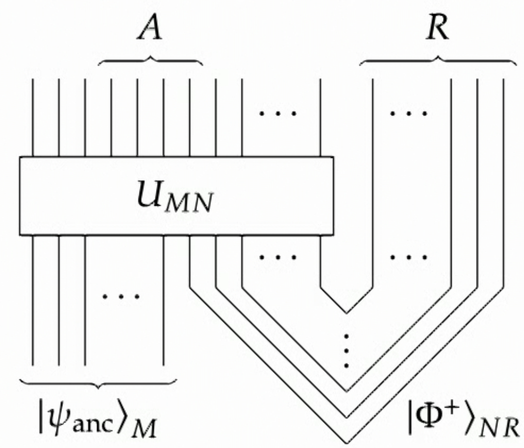
  - $g^D > r$: Weight growth dominates over qudit growth => stabilizer weights will be approach $\langle w \rangle = \dfrac{q^2 - 1}{q^2} \cdot N$.
  - $g^D < r$: More qudits are being added than can be scrambled => relative stabilizer weights become vanishingly small.

Numerical Analysis

Our Setup

- Our first goal was to analyze the **stabilizer weights and the code distance** for a simple implementation of NoRA.
  - We used **qutrits** (bond dimension $d = 3$), a growth rate of $r = 2$, and 2-local random Clifford brick wall circuits at each layer.
- The stabilizer weights were simply determined by counting.
- The code distance was approximated using the **adversarial approach**.
  - Assume the code word $|\Phi^+\rangle_{NR}$ is maximally entangled with an additional system R.
  - The distance of the code defined by U_{MN} is then the **smallest number of additional qudits** A **required to get non-zero mutual information** $I(A, R) = S(A) + S(R) - S(AR) > 0$.
  - Finding A is **hard**, but can be approximated efficiently.
- We considered both **fixed** k, and k **scaling with** N.

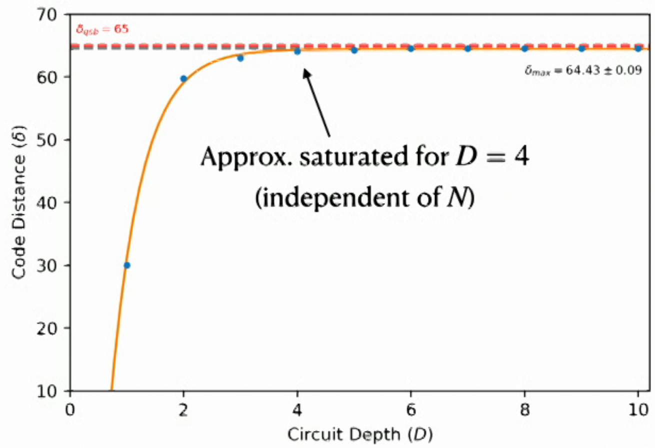


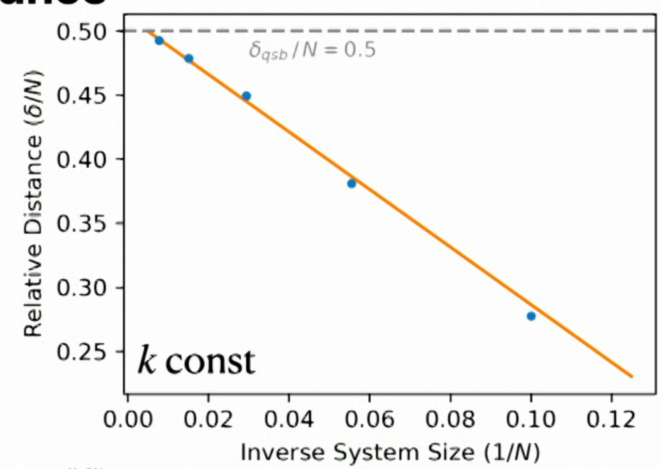


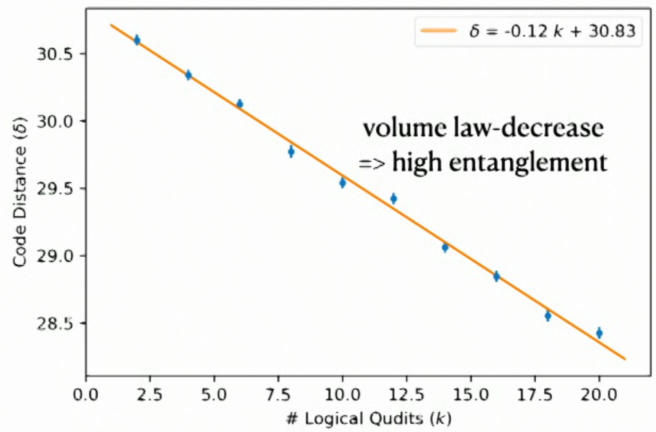
Numerical Analysis
Code Distance

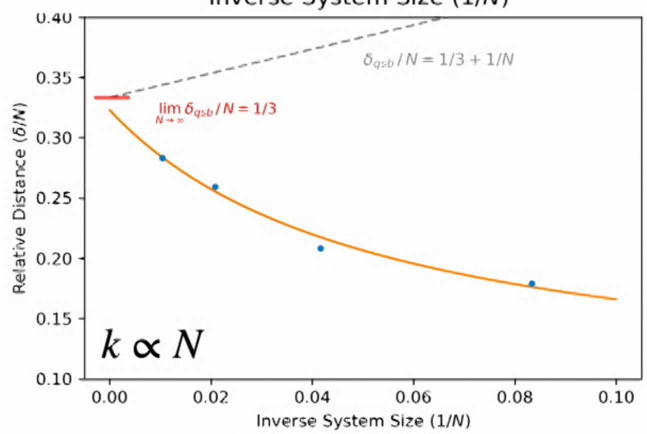
$$\delta_{qsb} = \frac{N-k}{2} + 1$$

Maximum possible distance predicted by **quantum singleton bound.**









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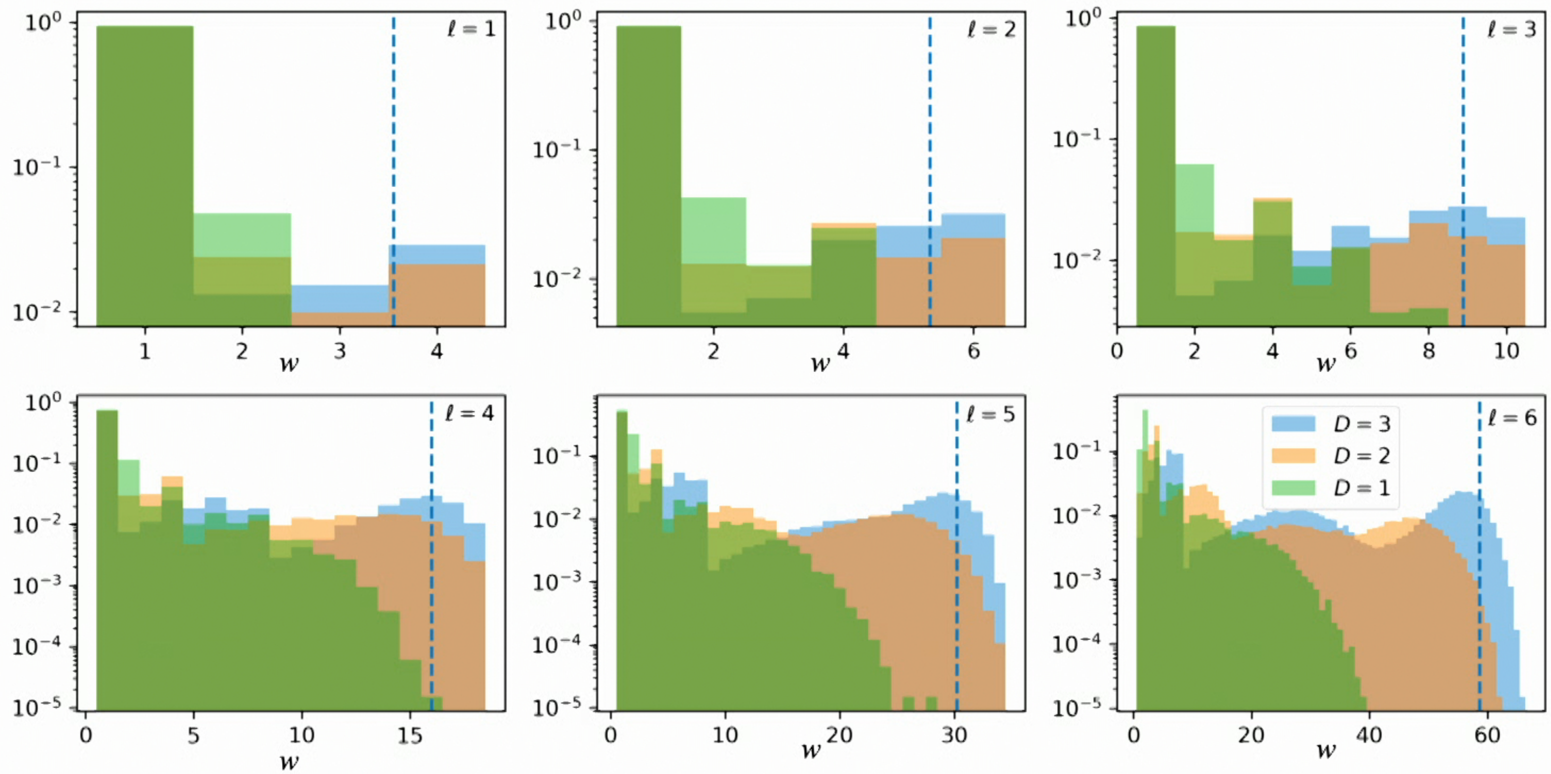
Numerical Analysis

Stabilizer basis weights



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What comes next?

Weight loss and Majorana's.

- Smaller stabilizer weights = better for experiments. But weights of NoRA are high!
  - Explore ways to reduces them (RREF basis, perturbative gadgets, ...)
- All NoRA calculations presented here were made using qudits.
  - Want stabilizer operator strings to be Majoranas like in SYK Hamiltonian.
  - Could be achieved using Jordan-Wigner, but not natural and very "non-local"
- Current focus: Developing a general Clifford stabilizer formalism for Majorana fermions.
  - Naïvely equivalent to Pauli Cliffords i.e. product of Majoranas = tensor product of Paulis.
  - However, Majorana strings have parity that should be preserved by Cliffords. This might lead to interesting properties (e.g. lower weights, different distance scaling).
  - No intuitive interpretation of reduced Majorana stabilizer states, because no tensor product structure.

Thank You!