Title: Linear waves in spacetimes - Sharmila Gunasekara, The Fields Institute for Research in Mathematical Sciences

Speakers:

Series: Cosmology & Gravitation

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Abstract: I will introduce the stability problem for spacetimes from the initial value formulation perspective in general relativity. After introducing some notions on how to quantitatively characterize (in)stability, I will present a result for a class of spacetimes called gravitational solitons which exhibit slower decay compared to black holes. This is joint work with Hari Kunduri (McMaster University).

Zoom Link: TBD

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Linear waves in spacetimes

Sharmila Gunasekaran

- joint work with Hari Kunduri (McMaster University)

[Annales Henri Poincare 22 (2021) 3, 821-872]

Cosmology group meeting
Perimeter Insitute

July 17, 2023

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Overview - stability

- Framework : General Relativity

$$\mathrm{Ric}_{\mu\nu}[g] - \frac{1}{2} g_{\mu\nu} \mathrm{R}[g] = \mathrm{T}_{\mu\nu}$$

+ equations for matter

In particular, the focus is on asymptotically flat, stationary solutions to Einstein Field Equations (EFE)

- Goal : understand stability of spacetimes as solutions to EFE - what happens to small perurbations ?

- What does stability mean here ? EFE as a second order PDE system - initial/boundary conditions depending on the PDE type (elliptic/hyperbolic/parabolic)

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Existence result

- For EFEs, initial data : spacelike hypersurface and it's derivative (extrinsic curvature) - (Σ, h, K)
- Initial data is constrained : Gauss and Codazzi equations
  If (M, g) is a spacetime satisfying the Einstein field equations and $\Sigma \hookrightarrow M$ is a spacelike hypersurface with induced Riemannian metric h and second fundamental form K then

$$R(h) - |K|_h^2 - (\mathrm{tr}_h K)^2 = 0$$
$$(\mathrm{div} K)_i - \nabla_i{}^{\Sigma}(\mathrm{tr}_h K) = 0$$

Existence and uniqueness result :

Given Σ, h, K satisfying constraint equations, $\exists$ a unique, globally hyperbolic, maximal spacetime $(\mathcal{M}, g)$ where $\Sigma \hookrightarrow \mathcal{M}$ with induced metric h_{ij} and extrinsic curvature K_{ij}.

Summary : we have initial value formulation for EFEs - initial data (w/ constraints) + evolution equations + existence and uniqueness

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Stability

- Stability - how much does the geometry change if the initial data is perturbed ?

- EFE - nonlinear, coupled, so a simple model to study global behaviour of perturbations : $\Box_g \Phi = 0$

$$\Box_g \Phi = 0 \text{ with}$$
$$\Phi|_\Sigma = \Phi_0 \, , \Phi_t|_\Sigma = \Phi_1$$

- Φ is a perturbation (ignoring backreaction) and we ask :
  - Does Φ remain bounded (boundedness) ?
  - Does $\Phi \to 0$ eventually (decay) ?

Review of stability problem (non-exhaustive!!)

- Mode stability [Regge 1957, Whiting 1958]
- Minkowski [Christodoulou–Klainerman]
- Waves - Schwarzschild [Kay–Wald, Dafermos–Rodnianski, Blue-Soffer..] ; (sub-extremal) Kerr [Finster–Kamran–Smoller–Yau], [Dafermos–Rodnianski–Shlapentokh-Rothman], extremal [Angelopoulos–Aretakis–Gajic]
- Schwarzschild - polarized perturbations [Klainerman–Szeftel]; Teukolsky equation on Kerr [Shlapentokh-Rothman–Teixeira da Costa], [Dafermos–Holzegel–Rodnianski]; slowly rotating Kerr [Hafner–Hintz–Vasy]
- Recent progress : nonlinear stability of Schwarzschild [Dafermos–Holzegel–Rodnianski–Taylor]; nonlinear stability of Kerr [Giorgi–Klainerman–Szeftel]

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Background - gravitational solitons

- Solitons - globally regular, stationary, asymptotically flat spacetimes with positive energy (horizonless) - 5d and higher

Theorem (Lichnerowiscz theorem)
$d = 4$ Einstein-Maxwell doesn't admit solitons

(M, g, F) - 4d AF spacetime with timelike KVF K.

$$\mathrm{Ric}(g)_{\mu\nu} = 2\left(F_{\mu}{}^{\rho}F_{\nu\rho} - \frac{1}{2(n-2)}g_{\mu\nu}F^2\right)$$

$$\mathrm{d}\star F = 0, \quad \mathrm{d}F = 0$$

- $\mathrm{d}F = 0 \implies$ globally defined electric potential ψ with $\mathrm{d}\psi = -i_K F$
- $\mathrm{d}\star F = 0 \implies$ a closed one form $\Theta = i_K \star F = \mathrm{d}\mu$

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Lichnerowiscz theorem (contd.)

- Komar formula, field equations + Stokes' theorem gives :

$$
\begin{aligned}
M &= -\frac{1}{8\pi}\int_{S^2_\infty} \star \mathrm{d}K = \frac{1}{4\pi}\int_\Sigma \mathrm{d}\star \mathrm{d}K \\
&= \frac{1}{4\pi}\int_\Sigma \star \mathrm{Ric}(K) = \frac{1}{4\pi}\int_\Sigma \Theta \wedge F \\
&= \frac{1}{4\pi}\int_\Sigma \mathrm{d}(\mu F)
\end{aligned}
$$

- In $d = 4$, $\Theta = \mathrm{d}\mu$ is exact $\implies M = 0$ when there is no inner boundary (horizon)
- Rigidity of positive mass theorem leaves only Minkowski [Shoen–Yau 1981]

Summary : there are no asymptotically flat gravitational solitons in $4d$ Einstein Maxwell theory

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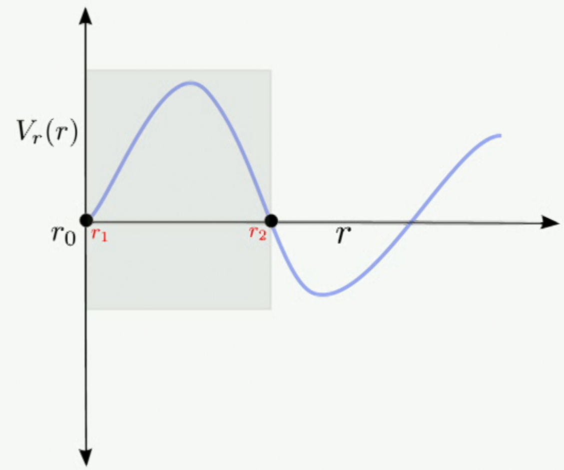
Behavior of null geodesics

- Understand waves in high frequency regime.
- Gravitational solitons - isometries + Killing tensor $\rightarrow$ four constants of motion :

$$\dot{r}^2 = V_r(r)$$

Trapping - (not MOTS!)



Compare with Schwarzschild $\rightarrow$ photon sphere is $r = 3M$

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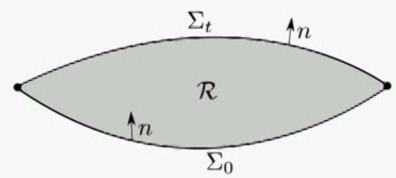
Back to waves

Evolution - start with a foliation - Σ_t and prescribe data on Σ_0

- Stress energy tensor for massless scalar field :

$$Q_{\alpha\beta} = \nabla_\alpha \Phi \nabla_\beta \Phi - \frac{1}{2} g_{\alpha\beta} \nabla^\gamma \Phi \nabla_\gamma \Phi.$$

- If T and n are future directed and timelike, $Q(T,n) \sim \sum_\alpha (\partial_\alpha \Phi)^2$
- divergence free Q + timelike KVF $T \implies \nabla^\mu Q_{\mu\nu} T^\nu = 0$
- Integrate this identity on $\mathcal{R}$



Stokes theorem : $\int_{\Sigma_t} Q_{\mu\nu} T^\nu n^\mu_{\Sigma_t} = \int_{\Sigma_0} Q_{\mu\nu} T^\nu n^\mu_{\Sigma_0} \left(\sim \int_\Sigma |D\Phi|^2 \right)$

We have some t dependent quantity that relates to initial data.

Measures of stability - energy

Can construct (conserved)quantities - energies on Σ_t/Σ_0 :

1. Energy

$$E[\Phi](t) \sim \sum_{|\alpha|=1} \int_{\Sigma_t} \left|D^\alpha \Phi\right|^2$$

2. Higher order energy

$$E_k[\Phi](t) \sim \sum_{|\alpha|\leq k} \int_{\Sigma_t} \left|D^\alpha \Phi\right|^2$$

3. $E_\Omega[\Phi](t)$ (or $E_{loc}[\Phi](t)$):

$$E_\Omega[\Phi](t) \sim \sum_{|\alpha|=1} \int_{\Sigma_t \cap \Omega} \left|D^\alpha \Phi\right|^2$$

Aim : optimally relate $\boldsymbol{E[\Phi](0)}$ and $\boldsymbol{E[\Phi](t)}$

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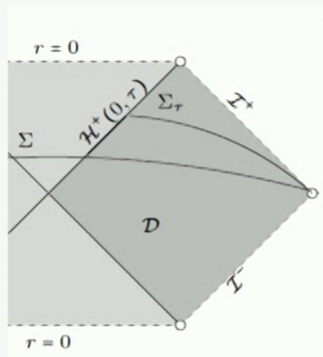
Stability questions

- Boundedness : $E[\Phi](t) \leq_? E[\Phi](0)$
- Decay : $E_\Omega[\Phi](t) \leq_? \delta(t) E_k[\Phi](0)$,
  where $\Omega \subset \Sigma_t, \delta(t) \to 0$ as $t \to \infty$ and $k \in \mathbb{N}$

Solitons : ∂_t is globally stationary $\implies E[\Phi](t) = E[\Phi](0)$

Compare with the case for black holes below : part of Σ_t or Σ_0 is null



Boundedness ✓

What about decay ?

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General decay result for waves

Theorem (Moschidis, 2015)

Let $(\mathcal{M}^{d+1}, g)$, $d \geq 3$, be a globally hyperbolic spacetime, which is stationary and asymptotically flat and which can possibly contain black holes with a non-degenerate horizon and a small ergoregion. Moreover, suppose that an energy boundedness statement is true for solutions Φ of the linear wave equation on the domain of outer communications $\mathcal{D}$ of the spacetime. Then the local energy of Φ on $\mathcal{D}$, $E_\Omega[\Phi](t)$ decays at least with a logarithmic rate :

$$E_\Omega[\Phi](t) \leq C \frac{1}{\{\log(2+t)\}^2} E_2[\Phi](0)$$

where t is a suitable time function on $\mathcal{D}$ and C are constants.

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Decay - uniform lower bound

Theorem (S.G.-Kunduri 2021)
Let Φ be a solution to the wave equation in a gravitational soliton. There exists an open set Ω and positive constant C such that,

$$\limsup_{t\to\infty} \sup_{\Phi\neq 0} (\log(2+t))^2 \frac{E_\Omega[\Phi](t)}{E_2[\Phi](0)} \geq C$$

Consequence : $E_\Omega[\Phi](t) \leq \frac{C}{\log(2+t)^2} E_2[\Phi](0)$ is sharp for the special case of gravitational solitons i.e., $E_\Omega[\Phi](t) \leq \frac{C\delta(t)}{\log(2+t)^2} E_2[\Phi](0)$ where $\delta(t) \to 0$ is false.

- Good decay : $\delta(t)$ is (inverse) polynomial (black holes)
- Slow decay : $\delta(t)$ is (inverse) logarithmic (Kerr-AdS_4, black rings)

Slow decay is due to stable trapping of null geodesics

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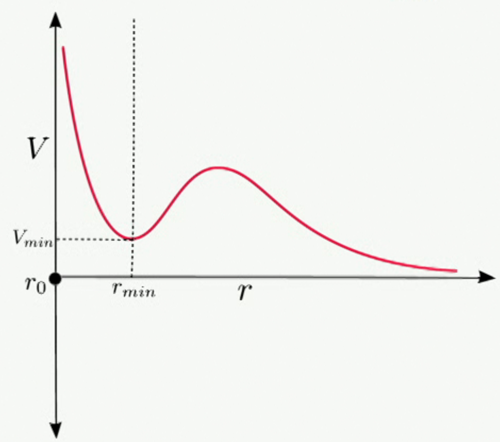
Stable trapping and decay of waves (rough idea)

Model problem : Consider solutions of the form $\Psi_n(x,t) = e^{-i\omega_n t}u(x)$ with $u(x)$ solving

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + (\mathbf{V} - \omega_n^2)u = 0$$



- Stable trapping $\implies V_{min}$
- Slow decay : one can find bound state solutions Ψ_n in a region Ω about V_{min}.
- Idea : Use the bound state solutions to find a lower bound for $E_\Omega[\Phi](t)$

Related work

- Kerr-AdS$_4$ [Holzegel-Smulevici 2013], microstate geometries, ultracompact neutron stars [Keir 2014], black strings and rings [Benomio 2018]

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Quasimodes construction 1

Separation of variables and isometries leave us with a 1-d Schrödinger type equation for radial function $u(r)$:

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \left(n^2 V - nf\omega - \omega^2\right) u = 0 \tag{1}$$

$(\Phi(t,r,\theta,\psi,\phi) = e^{-i\omega_n t} e^{in\psi} u(r) Y(\theta,\phi))$

- Trapped region := $[r_-, r_+]$ about $V_{\min}$ which doesn't contain any other minima
- Goal : find eigenfunctions to (1) with $u(r_\pm) = 0$
- Issues : nonlinear eigenvalue problem due to $-nf\omega$

Existence of eigenfunctions ?

$$\mathcal{P}_\beta \quad : -\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \left(n^2 V - \beta n\omega - \omega^2\right) u = 0 + \text{b.c.}$$

$$\text{Linear } (\mathcal{P}_0) : -\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \left(n^2 V - \omega^2\right) u = 0 + \text{b.c.}$$

$$\text{Nonlinear } (\mathcal{P}_1) : -\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \left(n^2 V - nf\omega - \omega^2\right) u = 0 + \text{b.c.}$$

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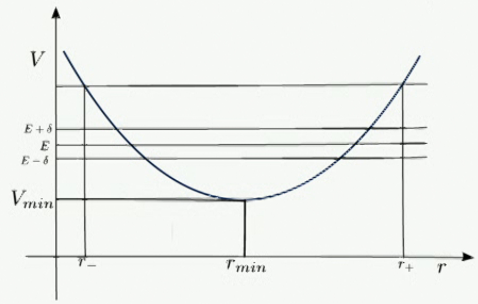
Quasimodes construction 2 - results

- Linear problem : (Define $h := 1/n$) : $-h^2 \frac{d^2 u_n}{dr^2} + V u_n = \omega_n^2 h^2 u_n$
  There are eigenvalues κ_n converging to any fixed $E > V_{min}$ as $n \to \infty$ (scale as h^{-1} - : Weyl asymptotics)



- Nonlinear problem : Given eigenvalues $\omega_{\text{lin},n}^2$ for the linear eigenvalue problem where $\omega_{\text{lin},n}^2 > 0$, there exists an eigenvalue ω_n^2 to the nonlinear eigenvalue problem for large enough n (Implicit function theorem)

Summary : We have bound state solutions localized in the trapped region (i.e., existence of solutions to the eigenvalue problem)

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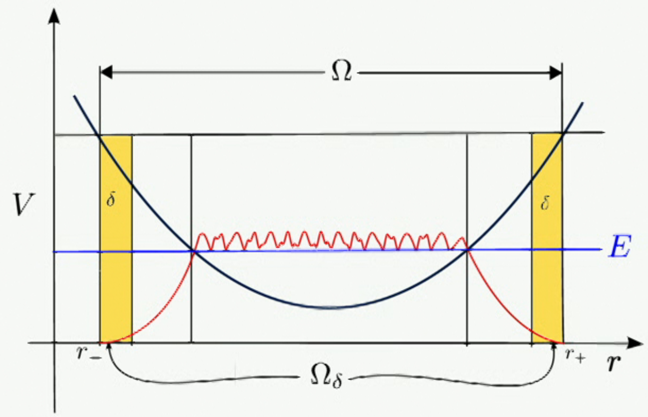
Quasimodes - error

The solution need to be extended to the whole spacetime. These " nicely extended" solutions are quasimodes.



Quasimodes - solve the wave equation everywhere except in the cut-off region.

$$\Box_g \Psi_n = \mathrm{err}(\Psi_n)$$

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Slow decay

1. Extend the solution in the trapped region to the entire spacetime.
2. Duhamel's formula to solve $\Box_g \Psi = \mathrm{err}_n$ through homogenous solution, $\Box_g \Phi = 0$

$$\Psi(t) = \Phi(t) + \int_0^t \Phi(t,s)\, \mathrm{err}_n(s)\, \mathrm{d}s$$

   Φ is solution to the homogenous equation with initial data at s.

3. Energy of Ψ and Φ are close if err_n is very small.

$$E_\Omega[\Phi](t) \geq \frac{C}{n^2} E_2[\Phi](0) \text{ for } t \leq \frac{e^{Cn}}{2C}$$

   The above result prevents the possibility of a local uniform decay statement faster than logarithmic.

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Remarks

1. The above results do not prevent smooth solutions from decaying faster than logarithmically. It simply states that there will always exist solutions that decay slower - a faster than logarithmic decay rate wouldn't be an uniform bound.
2. The quasimode construction is applicable to any spacetime which exhibits stable trapping. [Benomio, 2018]. Hence, stable trapping $\Longrightarrow$ slow decay (universal).
3. There are bound states of solitons and black holes - structure of trapping and nature of decay ? (open)
4. Instability caused by stable trapping ? (open)

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Thank you!