

Title: Kinetic theory of collisionless self-gravitating systems

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Series: Cosmology & Gravitation

Date: July 10, 2023 - 12:00 PM

URL: <https://pirsa.org/23070046>

Abstract: Abstract and Zoom Link: TBD

Kinetic Theory of Self-gravitating collisionless systems

Collisionless Boltzmann Eqⁿ: $\frac{\partial f}{\partial t} + [f, H] = 0$

Poisson Eqⁿ: $\nabla^2 \Phi = 4\pi G \int f d^3v$

Poisson bracket
 $[f, H] = \frac{\partial f}{\partial v} \frac{\partial H}{\partial \mathbf{r}} - \frac{\partial f}{\partial \mathbf{r}} \frac{\partial H}{\partial v}$

$H = H_0 + \Phi_p + \Phi$

↑ Est. perturbation ↓ Due to galaxy response



Perimeter-A



Kinetic Theory of Self-gravitating collisionless systems

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Poisson Eqⁿ: $\nabla^2 \Phi = 4\pi G \int f d^3v$

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 $[f, H] = \frac{\partial f}{\partial \vec{r}} \frac{\partial H}{\partial \vec{I}} - \frac{\partial f}{\partial \vec{I}} \frac{\partial H}{\partial \vec{r}}$

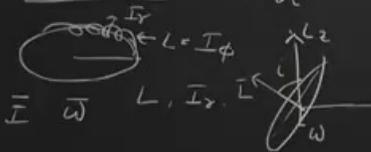
$H = H_0 + \Phi_p + \Phi$
↑
E_{int} perturbation
↓
due to galaxy response

Galaxies
CDM halos

$dN = f d\vec{r} d^3p$

Action-angle

$\frac{\partial f}{\partial t} = 0 \rightarrow [f, H] = 0 \rightarrow f = f[H, \vec{I}]$



Kinetic Theory of Self-gravitating collisionless systems

Collisionless Boltzmann Eqⁿ: $\frac{\partial f}{\partial t} + [F, H] = 0$ Poisson Eqⁿ: $\nabla^2 \Phi = 4\pi G \int f d^3v$

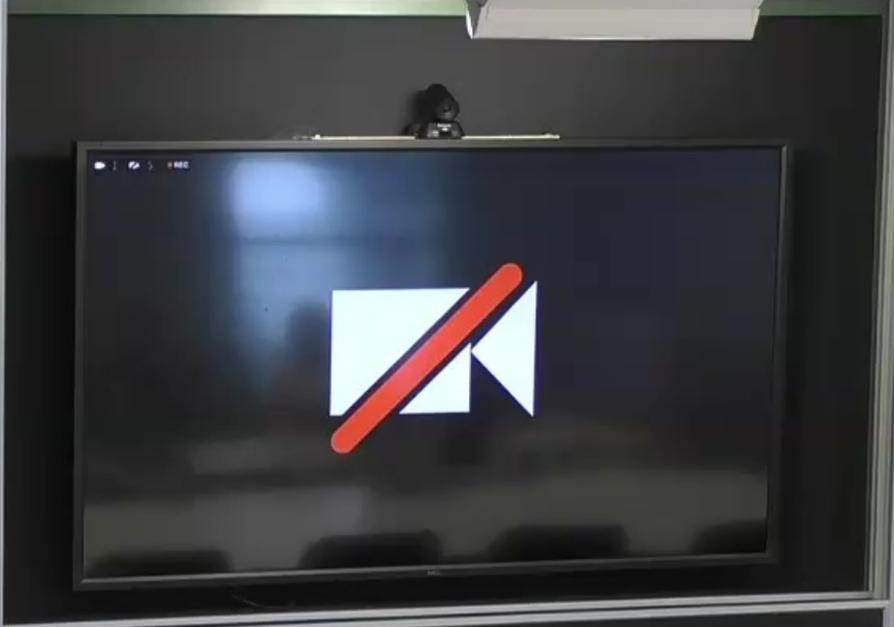
Poisson bracket: $[f, H] = \frac{\partial f}{\partial v} \frac{\partial H}{\partial \mathbf{r}} - \frac{\partial f}{\partial \mathbf{r}} \frac{\partial H}{\partial v}$

$H = H_0 + \Phi_p + \Phi$
↑
E.g. perturbation ↑
due to galaxy response

$d^6N = f d^3q d^3p$

Action-angle $\frac{\partial f_0}{\partial t} = 0 \rightarrow [f_0, H_0] = 0 \rightarrow f_0 = f_0[H_0, \mathbf{I}] \leftarrow \text{Jeans' theorem}$

(Galaxies
CDM halos)



Perimeter-A

Kinetic Theory of Self-gravitating collisionless systems

Collisionless Boltzmann Eqⁿ: $\frac{df}{dt} = 0$ Poisson Eqⁿ: $\nabla^2 \Phi = 4\pi G \int f d^3v$

Poisson bracket: $\{f, H\} = \frac{\partial f}{\partial \mathbf{r}} \cdot \frac{\partial H}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial H}{\partial \mathbf{r}}$

$\frac{dN}{dt} = \int \frac{\partial f}{\partial t} d^3v$

Action-angle: $\{f_0, H_0\} = 0 \rightarrow f_0 = f_0[H_0, \mathbf{I}]$ ← Jeans theorem

$H_0 = \frac{p^2}{2m} + \Phi_0$ Sph
 $\nabla^2 \Phi_0 = \int f_0 d^3v$ Isotropy
 $\nabla^2 \Phi_0 = \int \sqrt{2(E - \Phi_0)} f_0 dE$ $f = f(E)$

*Galaxies
CDM halos*

Due to galaxy response

Perimeter-A



Kinetic Theory of Self-gravitating collisionless systems

Collisionless Boltzmann Eqⁿ: $\frac{\partial f}{\partial t} + [f, H] = 0$ Poisson Eqⁿ: $\nabla^2 \Phi = 4\pi G \int f d^3v$

Poisson bracket: $[f, H] = \frac{\partial f}{\partial \mathbf{w}} \frac{\partial H}{\partial \mathbf{I}} - \frac{\partial f}{\partial \mathbf{I}} \frac{\partial H}{\partial \mathbf{w}}$

$H = H_0 + \Phi_p + \Phi$
 ↑
 Ext. perturbation due to galaxy response

$d^6N = f d^3q d^3p$ Galaxies CDM halos

Action-angle $\frac{\partial f_0}{\partial \mathbf{I}} = 0 \rightarrow [f_0, H_0] = 0 \rightarrow f_0 = f_0[H_0, \mathbf{I}] \leftarrow$ Jeans Theorem

Perturbation theory \rightarrow

$$\begin{cases} f = f_0 + \delta f_1 + \delta f_2 + \dots \\ \Phi = \Phi_0 + \delta \Phi_1 + \delta \Phi_2 + \dots \\ \Phi_p = \Phi_p + \delta \Phi_p \end{cases}$$


Perimeter-A

Kinetic Theory of Self-gravitating collisionless systems

Collisionless Boltzmann Eq: $\frac{\partial f}{\partial t} + [f, H] = 0$ Poisson Eq: $\nabla^2 \Phi = 4\pi G \int f d^3v$

Poisson bracket: $[f, H] = \frac{\partial f}{\partial \mathbf{v}} \frac{\partial H}{\partial \mathbf{r}} - \frac{\partial f}{\partial \mathbf{r}} \frac{\partial H}{\partial \mathbf{v}}$

$H = H_0 + \Phi_p + \Phi$
 ↑ due to galaxy response
 ↑ external perturbation

$d^6N = f d^3q d^3p$ Galaxies CDM halos

Action-angle: $\frac{\partial f}{\partial t} = 0 \rightarrow [f_0, H_0] = 0 \rightarrow f_0 = f_0[H_0, \mathbf{I}]$ ← Jeans theorem

$\frac{\partial f_1}{\partial t} + [f_1, H_0] + [f_0, \Phi_p] + \sum_j [f_{1-j}, \Phi_j] = 0$ Perturbation theory

$\nabla^2 \Phi_c = 4\pi G \int f_1 d^3v$ f_0 known

$f = f_0 + \delta f_1 + \delta f_2 + \dots$
 $\Phi = \Phi_0 + \delta \Phi_1 + \delta \Phi_2 + \dots$
 $\Phi_p \sim \mathcal{E} \Phi_p'$



Perimeter-A

Action-angle $\frac{\partial f}{\partial t} = 0$

$$\frac{\partial f_i}{\partial t} + [f_i, H_0] + [f_i, \Phi_P] + \sum [f_{i-1}, \Phi_i] = 0$$

f_0 known

Perturbation theory \rightarrow

$$\begin{cases} f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots \\ \Phi = \Phi_0 + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \dots \\ \Phi_P \sim \epsilon \Phi_P' \end{cases}$$

$[\dots, \dots] \leftarrow$ Jeans' theorem

1. $\frac{\partial f_1}{\partial t} + [f_1, H_0] + [f_0, \Phi_P] + [f_0, \Phi_1] = 0$ $\nabla^2 \Phi_1 = 4\pi G \int f_1 d^3v$

2. $\frac{\partial f_2}{\partial t} + [f_2, H_0] + [f_1, \Phi_P] + [f_1, \Phi_1] + [f_0, \Phi_2] = 0$ $\nabla^2 \Phi_2 = 4\pi G \int f_2 d^3v$

\rightarrow Easy \rightarrow PhD

Linear

- ① Phase-mixing
- ② Jeans instability
- ③ Landau damping

Hard \rightarrow Now



Action-angle $\frac{\partial f_0}{\partial t} = 0$

$$\frac{\partial f_1}{\partial t} + [f_1, H_0] + [f_1, \Phi_P] + \sum [f_{1-j}, \Phi_j] = 0$$

Perturbation theory \rightarrow

$$\begin{cases} f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots \\ \Phi = \Phi_0 + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \dots \\ \Phi_P \sim \epsilon \Phi'_P \end{cases}$$

f_0 known

$$\nabla^2 \Phi_i = 4\pi G \int f_i d^3v$$

$[\bar{I}] \leftarrow$ Jeans' theorem

$$1. \frac{\partial f_1}{\partial t} + [f_1, H_0] + [f_0, \Phi_P] + [f_0, \Phi_1] = 0 \quad \nabla^2 \Phi_1 = 4\pi G \int f_1 d^3v$$

$$\frac{\partial f_2}{\partial t} + [f_2, H_0] + [f_1, \Phi_P] + [f_1, \Phi_1] + [f_0, \Phi_2] = 0 \quad \nabla^2 \Phi_2 = 4\pi G \int f_2 d^3v$$

Action-angle space

$$1. \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial \bar{\omega}} \frac{\partial H_0}{\partial \bar{I}} - \frac{\partial f_0}{\partial \bar{I}} \frac{\partial \Phi_P}{\partial \bar{\omega}} - \frac{\partial f_0}{\partial \bar{I}} \frac{\partial \Phi_1}{\partial \bar{\omega}} = 0$$

$$H_0 = H_0(\bar{I})$$

$$f_0 = f_0(\bar{I})$$



Perimeter-A

Action-angle $\frac{\partial f_i}{\partial t} = 0$ $[\bar{I}] \leftarrow$ Jeans' theorem

$$\frac{\partial f_i}{\partial t} + [f_i, H_0] + [f_{i+1}, \Phi_P] + \sum [f_{i-j}, \Phi_j] = 0$$

Perturbation theory \rightarrow

$$\begin{cases} f = f_0 + \delta f_1 + \delta f_2 + \dots \\ \Phi = \Phi_0 + \delta \Phi_1 + \delta \Phi_2 + \dots \\ \Phi_P \sim \mathcal{E} \Phi_P' \end{cases}$$

f_0 known

$$1. \frac{\partial f_1}{\partial t} + [f_1, H_0] + [f_0, \Phi_P] + [f_0, \Phi_1] = 0 \quad \nabla^2 \Phi_1 = 4\pi G \int f_1 d^3v$$

$$2. \frac{\partial f_2}{\partial t} + [f_2, H_0] + [f_1, \Phi_P] + [f_1, \Phi_1] + [f_0, \Phi_2] = 0 \quad \nabla^2 \Phi_2 = 4\pi G \int f_2 d^3v$$

Action-angle space

$$1. \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial \bar{w}} \left(\frac{\partial H_0}{\partial \bar{I}} \right) - \frac{\partial f_0}{\partial \bar{I}} \frac{\partial \Phi_P}{\partial \bar{w}} - \frac{\partial f_0}{\partial \bar{I}} \frac{\partial \Phi_1}{\partial \bar{w}} = 0$$

$$\Rightarrow \left[\frac{\partial f_{1i\ell}}{\partial t} + i\bar{\omega}_{i\ell} f_{1i\ell} = i\bar{\omega}_{i\ell} \frac{\partial f_0}{\partial \bar{I}} (\Phi_{P\ell}^{(i)} + \Phi_{1\ell}^{(i)}) \right]$$

$f_{1i\ell} \sim e^{-i\bar{\omega}_{i\ell} t}$

$$H_0 = H_0(\bar{I})$$

$$f_0 = f_0(\bar{I})$$

$$f_1(\bar{I}, \bar{w}) = \sum f_{1i\ell}(\bar{I}) e^{i\bar{I}\bar{w}}$$

$$\Phi_1 = \sum \Phi_{1i\ell}(\bar{I}) e^{i\bar{I}\bar{w}}$$

$$\Phi_P = \dots$$



Action-angle $\frac{\partial f_0}{\partial t} = 0$

$$\frac{\partial f_i}{\partial t} + [f_i, H_0] + [f_{i-1}, \Phi_P] + \sum [f_{i-1}, \Phi_i] = 0$$

Perturbation theory \rightarrow

$$\begin{cases} f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots \\ \Phi = \Phi_0 + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \dots \\ \Phi_P \sim \epsilon \Phi_P' \end{cases}$$

$\nabla^2 \Phi_i = 4\pi G \int f_i d^3v$ f_0 known

$[\bar{I}] \leftarrow$ Jeans' theorem

- $\frac{\partial f_1}{\partial t} + [f_1, H_0] + [f_0, \Phi_P] + [f_0, \Phi_1] = 0$ $\nabla^2 \Phi_1 = 4\pi G \int f_1 d^3v$
- $\frac{\partial f_2}{\partial t} + [f_2, H_0] + [f_1, \Phi_P] + [f_1, \Phi_1] + [f_0, \Phi_2] = 0$ $\nabla^2 \Phi_2 = 4\pi G \int f_2 d^3v$

Action-angle space

$$1. \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial \bar{w}} \left(\frac{\partial H_0}{\partial \bar{I}} \right) - \frac{\partial f_0}{\partial \bar{I}} \frac{\partial \Phi_P}{\partial \bar{w}} - \frac{\partial f_0}{\partial \bar{I}} \frac{\partial \Phi_1}{\partial \bar{w}} = 0$$

$$\Rightarrow \left[\frac{\partial f_{1i\bar{c}}}{\partial t} + i\bar{c} \bar{\omega} f_{1i\bar{c}} = i\bar{c} \frac{\partial f_0}{\partial \bar{I}} (\Phi_{P\bar{c}}^{(i)} + \Phi_{i\bar{c}}^{(i)}) \right]$$

$\rightarrow f_{1i\bar{c}} \sim e^{-i\bar{c}\bar{\omega}t}$ Phase-mixing

$$H_0 = H_0(\bar{I})$$

$$f_0 = f_0(\bar{I})$$

$$f_1(\bar{I}, \bar{w}) = \sum f_{1i\bar{c}}(\bar{I}) e^{i\bar{c}\bar{w}}$$

$$\Phi_1 = \sum \Phi_{1i\bar{c}}(\bar{I}) e^{i\bar{c}\bar{w}}$$

$$\Phi_P = \dots$$



Action-angle $\frac{\partial f_i}{\partial t} = 0$

$\frac{df_i}{dt} + [f_i, H_0] + [f_i, \Phi_P] + \sum [f_i, \Phi_j] = 0$ Perturbation theory \rightarrow

$\nabla^2 \Phi_i = 4\pi G \int f_i d^3v$ f_0 known

$[I] \leftarrow$ Jeans theorem

$$\begin{cases} f = f_0 + \delta f_1 + \delta f_2 + \dots \\ \Phi = \Phi_0 + \delta \Phi_1 + \delta \Phi_2 + \dots \\ \Phi_P \sim \epsilon \Phi_P' \end{cases}$$

No $\frac{df_i}{dt} \ll \frac{df_i}{dt}$

$$f_{iz}(t) = f_{iz}(0) e^{-i\bar{\omega}t} + i \frac{df_0}{dI} \int_0^t d\tau e^{-i(\bar{\omega}-m\Omega)\tau} \Phi_{PL}(t-\tau)$$

$\delta f_{iz}(t) \sim e^{-i\bar{\omega}t}$

Dynamical friction $\rightarrow \delta f_{iz}(t) \sim e^{-i\bar{\omega}t}$

$\rightarrow \omega \approx \omega_1 \Omega_1 + \omega_2 \Omega_2 - m\Omega_p < 0$

Resonances

Phase-spiral Graia

Perimeter-A

Action-angle $\frac{\partial f_i}{\partial t} = 0$

$$\frac{\partial f_i}{\partial t} + [f_i, H_0] + [f_i, \Phi_P] + \sum_j [f_i, \Phi_j] = 0$$

Perturbation theory \rightarrow

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots$$

$$\Phi = \Phi_0 + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \dots$$

$$\Phi_P \sim \epsilon \Phi'_P$$

f_0 known

$(\bar{I}) \leftarrow$ Jeans' theorem

$$1. \frac{\partial f_1}{\partial t} + [f_1, H_0] + [f_0, \Phi_P] + [f_0, \Phi] = 0 \quad \nabla^2 \Phi_1 = 4\pi G \int f_1 d^3v$$

$$2. \frac{\partial f_2}{\partial t} + [f_2, H_0] + [f_1, \Phi_P] + [f_1, \Phi] + [f_0, \Phi_2] = 0 \quad \nabla^2 \Phi_2 = 4\pi G \int f_2 d^3v$$

Action-angle

$$1. \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial \bar{w}} \left(\frac{\partial H_0}{\partial \bar{I}} \right) \frac{\partial f_0}{\partial \bar{I}} + \frac{\partial f_0}{\partial \bar{w}} \frac{\partial \Phi_P}{\partial \bar{I}} - \frac{\partial f_0}{\partial \bar{I}} \frac{\partial \Phi_1}{\partial \bar{w}} = 0$$

$$\Rightarrow \frac{\partial f_{1c}}{\partial t} + i\bar{\omega} f_{1c} = i\bar{\omega} \frac{\partial f_0}{\partial \bar{I}} (\Phi_{Pc}^{(1)} + \Phi_{1c}^{(1)})$$

$f_{1c} \sim e^{-i\bar{\omega}t}$ Phase-mixing

No sig $\Phi_{1c} \ll \Phi_{Pc}$

$$f_{1c}(t) = f_{1c}(0) e^{-i\bar{\omega}t} + i\bar{\omega} \frac{\partial f_0}{\partial \bar{I}} \int_0^t dt' e^{-i\bar{\omega}t'}$$

Dynamical friction

Perimeter-A

$$\tilde{f}_{1c}(\gamma) = \int dt e^{-\gamma t} f_{1c}(t)$$

$$\gamma \tilde{f}_{1c} - f_{1c}(0) + i\bar{\omega} \tilde{f}_{1c} = i\bar{\omega} \frac{\partial f_0}{\partial \bar{I}} (\tilde{\Phi}_{Pc} + \tilde{\Phi}_{1c})$$

$$\Rightarrow \tilde{f}_{1c}(\gamma) = i\bar{\omega} \frac{\partial f_0}{\partial \bar{I}} \frac{\tilde{\Phi}_{Pc} + \tilde{\Phi}_{1c}(\gamma)}{\gamma + i\bar{\omega}} + \frac{f_{1c}(0)}{\gamma + i\bar{\omega}}$$

$$\Phi_1(\bar{x}) = \int U(\bar{x}, \bar{x}') \rho(\bar{x}') d^3x'$$

$$\Phi_{1c}(\bar{I}) = (2\pi)^3 \int d\bar{I}' \frac{\Psi_{1c}(\bar{I}, \bar{I}')}{\gamma + i\bar{\omega}} \tilde{f}_{1c}(\bar{I}')$$

$$\tilde{\Phi}_{1c}(\bar{I}) = (2\pi)^3 \int d\bar{I}' \frac{\Psi_{1c}(\bar{I}, \bar{I}')}{\gamma + i\bar{\omega}} f_{1c}(\bar{I}')$$

Action-angle $\frac{\partial f_0}{\partial \bar{I}} = 0$

$$\frac{\partial f_1}{\partial t} + [f_1, H_0] + [f_1, \Phi_P] + \sum [f_{1-j}, \Phi_j] = 0$$

Perturbation theory \rightarrow

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots$$

$$\Phi = \Phi_0 + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \dots$$

$$\Phi_P \in \mathcal{C} \Phi_0$$

f_0 known

$$\bar{\Phi}_1(\bar{x}) = \int U(\bar{x}, \bar{x}') f(\bar{x}') d^3x'$$

$$\bar{\Phi}_1(\bar{I}) = (2\pi)^3 \sum_{\bar{I}'} \int d\bar{\omega} \frac{|\Psi_{\bar{I}\bar{I}'}(\bar{I}, \bar{I}')|}{\gamma + i\epsilon} \bar{\Phi}_{1\bar{I}'}(\bar{I}')$$

$$\int d\bar{\omega} \int d\bar{x} U(\bar{x}, \bar{x}') e^{-i(\bar{\omega} - \bar{\omega}')t}$$

$$\bar{\Phi}_1(\bar{I}) = (2\pi)^3 \sum_{\bar{I}'} \int d\bar{\omega} \frac{\partial H_0}{\partial \bar{I}} \frac{|\Psi_{\bar{I}\bar{I}'}(\bar{I}, \bar{I}')|}{\gamma + i\epsilon} \bar{\Phi}_{1\bar{I}'}(\bar{I}')$$

$$+ (2\pi)^3 \sum_{\bar{I}'} \int d\bar{\omega} \frac{\Psi_{\bar{I}\bar{I}'}(\bar{I}, \bar{I}')}{\gamma + i\epsilon} f_{1\bar{I}'}(\bar{I}, \bar{I}')$$

- $\frac{\partial f_1}{\partial t} + [f_1, H_0] + [f_0, \Phi_P] + [f_0, \Phi] = 0$ $\nabla^2 \Phi_1 = 4\pi G \int f_1 d^3v$
- $\frac{\partial f_2}{\partial t} + [f_2, H_0] + [f_1, \Phi_P] + [f_1, \Phi] + [f_0, \Phi_2] = 0$ $\nabla^2 \Phi_2 = 4\pi G \int f_2 d^3v$

Action-angle space

$$H_0 = H_0(\bar{I})$$

$$f_0 = f_0(\bar{I})$$

$$f_1(\bar{x}, \bar{\omega}) = \sum_{\bar{I}'} f_{1\bar{I}'}(\bar{I}')$$

$$\Phi_1 = \sum_{\bar{I}'} \bar{\Phi}_{1\bar{I}'}(\bar{I}') e^{i\bar{\omega} t}$$

$$\Phi_P = \dots$$

$f_{1\bar{I}'} \sim e^{-i\bar{\omega} t}$ Phase-mixing

$$\int d\bar{x} \Psi^{*p} \rho^r = -4\pi G \delta_{pq}$$

$$\Phi = \sum_P A_P \Psi^P$$

$$f = \sum_P A_P \rho^P$$

$$\Psi_{\bar{I}\bar{I}'}(\bar{I}, \bar{I}') = - \sum_P \Psi_{\bar{I}\bar{I}'}^P(\bar{I}) \Psi_{\bar{I}\bar{I}'}^{*P}(\bar{I}')$$

$$\Psi_{\bar{I}\bar{I}'} = \int d\bar{\omega} e^{-i\bar{\omega} t} \Psi_P$$

$$\tilde{A} = (I - M)^{-1} \tilde{S}$$

$$M_{pp'} = -\frac{(2\pi)^3}{4\pi G} \sum_{\bar{I}} \int d\bar{\omega} \frac{\partial H_0}{\partial \bar{I}} \frac{\Psi_P^* \Psi_{p'}}{\gamma + i\epsilon}$$

$$S_p = -(2\pi)^3 \sum_{\bar{I}} \int d\bar{\omega} \frac{f_{1\bar{I}}(\bar{I}, \bar{I}')}{\gamma + i\epsilon} \Psi_P^*$$

No sig $\Phi_{1\bar{I}} \ll \Phi_{P\bar{I}}$

$$f_{1\bar{I}}(t) = f_{1\bar{I}}(0) e^{-i\bar{\omega} t} + i \bar{\omega} \frac{df_0}{d\bar{I}} \int_0^t d\tau e^{-i\bar{\omega} \tau} \bar{\Phi}_{1\bar{I}}(t-\tau)$$

$\delta(\bar{\omega} - m\Omega_P)$

Dynamical friction

Resonances

near-spiral

Perimeter-A

Action-angle $\frac{\partial f_i}{\partial t} = 0$

$$\frac{\partial f_i}{\partial t} + [f_i, H_0] + [f_i, \Phi_P] + \sum [f_{i-1}, \Phi_i] = 0$$

Perturbation theory \rightarrow

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots$$

$$\Phi = \Phi_0 + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \dots$$

$$\Phi_P \sim \epsilon \Phi_P'$$

f_0 known

$[\bar{I}] \leftarrow$ Jeans theorem

$$1. \frac{\partial f_1}{\partial t} + [f_1, H_0] + [f_0, \Phi_P] + [f_0, \Phi_1] = 0 \quad \nabla^2 \Phi_1 = 4\pi G \int f_1 d^3v$$

$$2. \frac{\partial f_2}{\partial t} + [f_2, H_0] + [f_1, \Phi_P] + [f_1, \Phi_1] + [f_0, \Phi_2] = 0 \quad \nabla^2 \Phi_2 = 4\pi G \int f_2 d^3v$$

Action-angle space

$$H_0 = H_0(\bar{I})$$

$$f_0 = f_0(\bar{I}, \bar{\omega}) = \sum f_{i,c}$$

$$\Phi_1 = \sum \Phi_{1,c}$$

$$\Phi_P = \dots$$

$$\frac{\partial f_{1,c}}{\partial t} + i\bar{\omega} f_{1,c} = i\bar{\omega} \frac{\partial f_0}{\partial \bar{I}} (\Phi_{P,c}^{(+)} + \Phi_{1,c}^{(-)})$$

$\rightarrow f_{1,c} \sim e^{-i\bar{\omega}t}$ Phase-mixing

No sig $\Phi_{1,c} \ll \Phi_{P,c}$

$$f_{1,c}(t) = f_{1,c}(0) e^{-i\bar{\omega}t} + i\bar{\omega} \frac{\partial f_0}{\partial \bar{I}} \int_0^t dt' e^{-i\bar{\omega}(t-t')} \Phi_{P,c}$$

Dynamical friction

$\rightarrow \delta(\bar{\omega} - M\Omega_P)$

$\rightarrow \omega_{d,sp,1} + \dots$

Perimeter-A

$$\Phi_{1,c}(\bar{x}) = \int U(\bar{x}, \bar{x}') \rho(\bar{x}') d^3x'$$

$$\Phi_{1,c}(\bar{I}) = (2\pi)^3 \sum_i \int d\bar{I}' \frac{|\Psi_{1,c}(\bar{I}, \bar{I}')|}{\gamma + i\epsilon} \Phi_{1,c}(\bar{I}')$$

$$\int d\bar{\omega} \int d\bar{G} U(\bar{x}, \bar{x}') \frac{e^{i\bar{\omega}t}}{e^{i\bar{\omega}t}}$$

$$\Phi_{1,c}^{(+)} = (2\pi)^3 \sum_i \int d\bar{I}' \frac{\partial \rho_0}{\partial \bar{I}} \frac{|\Psi_{1,c}(\bar{I}, \bar{I}')|}{\gamma + i\epsilon} \Phi_{1,c}(\bar{I}')$$

$$+ (2\pi)^3 \sum_i \int d\bar{I}' \frac{\Psi_{1,c}(\bar{I}', 0) \rho_{1,c}(\bar{I}', 0)}{\gamma + i\epsilon}$$

$$\int d\bar{x} \Psi^{*p} \rho^r = -4\pi G \delta_{pp}$$

$$\Phi = \sum_P A_P \Psi_P$$

$$f = \sum_P A_P \rho^P$$

$$\Psi_{1,c}(\bar{I}, \bar{I}') = - \sum_P \Psi_P^*(\bar{I}) \Psi_P^{*p}(\bar{I}')$$

$$\Psi_{1,c} = \int d\bar{\omega} e^{-i\bar{\omega}t} \Psi_P$$

Laplace

$$\tilde{A} = (I - M)^{-1} \tilde{S}$$

$$M_{pp} = -\frac{(2\pi)^3}{4\pi G} \sum_i \int d\bar{I} \frac{e^{i\bar{\omega}t}}{\gamma + i\epsilon} \Psi_P^{*p} \Psi_P^{or}$$

$$S_p = -(2\pi)^3 \sum_i \int d\bar{I}' \frac{\rho_{1,c}(\bar{I}', 0)}{\gamma + i\epsilon} \Psi_P^{*p}$$

$$\det |I - M| = 0 \rightarrow e^{rt}$$

Action-angle $\frac{\partial f_0}{\partial \mathbb{I}} = 0$

$$\frac{\partial f_1}{\partial t} + [f_1, H_0] + [f_1, \Phi_p] + \sum [f_{l-j}, \Phi_j] = 0$$

Perturbation theory \rightarrow

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots$$

$$\Phi = \Phi_0 + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \dots$$

$$\Phi_p \in \mathcal{E}_p$$

f_0 : known

$$\nabla^2 \Phi_i = 4\pi G \int f_i d^3v$$

$[\mathbb{I}] \leftarrow$ Jeans theorem

Dynamical friction $\rightarrow \mathcal{E}(\bar{x}, m, p)$

$$Y = Y_R + i Y_I$$

$Y_R > 0$: Unstable
 $Y_R < 0$: Stable \rightarrow Landau damping

$\leq 0 \rightarrow Y_R < 0$

Resonances

$f_0 = f_0(H_0)$

$$\bar{c} \frac{\partial f_0}{\partial \mathbb{I}} = \bar{c} \bar{\omega} \frac{\partial p_0}{\partial H_0}$$

Phase-mixing \leftarrow

$$\Phi_1(\bar{x}) = \int U(\bar{x}, \bar{x}') f(\bar{x}') d^3x'$$

$$\frac{\partial \Phi_1}{\partial t} = (2\pi)^3 \sum_{\ell} \int d\mathbb{I}' \frac{\Psi_{\ell\ell'}(\bar{\mathbb{I}}, \bar{\mathbb{I}}')}{\gamma - i\ell'\bar{\omega}} \tilde{\Phi}_{\ell\ell'}(\bar{\mathbb{I}})$$

$$\tilde{\Phi}_{\ell\ell'}(\bar{\mathbb{I}}) = (2\pi)^3 \sum_{\ell''} \int d\mathbb{I}'' \frac{\Psi_{\ell\ell''}(\bar{\mathbb{I}}, \bar{\mathbb{I}}'')}{\gamma + i\ell''\bar{\omega}} f_{\ell\ell''}(\bar{\mathbb{I}}'')$$



$$\int d\bar{x} \Psi^{*p} p^r = -4\pi G \delta_{pq}$$

$$\Phi = \sum_p A_p \Psi^p$$

$$p = \sum_p A_p p^p$$

$$\Psi_{\ell\ell'}(\bar{\mathbb{I}}, \bar{\mathbb{I}}') = -\sum_p \Psi_{\ell\ell'}^p(\bar{\mathbb{I}}) \Psi_{\ell\ell'}^{*p}(\bar{\mathbb{I}}')$$

$$\Psi_{\ell} = \int d\bar{\omega} e^{i\bar{\omega}\bar{\mathbb{I}}} \Psi_p$$

Laplace $\tilde{A} = (I - M)^{-1} \tilde{S}$

$$M_{pq} = -\frac{(2\pi)^3}{4\pi G} \sum_{\ell'} \int d\mathbb{I}' \frac{\Psi_{\ell\ell'}^p(\bar{\mathbb{I}}, \bar{\mathbb{I}}') \Psi_{\ell\ell'}^q(\bar{\mathbb{I}}', \bar{\mathbb{I}})}{\gamma + i\ell'\bar{\omega}}$$

$$S_p = -(2\pi)^3 \sum_{\ell'} \int d\mathbb{I}' \frac{f_{\ell\ell'}(\bar{\mathbb{I}}')}{\gamma + i\ell'\bar{\omega}} \Psi_{\ell\ell'}^{*p}(\bar{\mathbb{I}}')$$

$(ZT) \rightarrow \det |I - M| = 0$