Title: Research Talk 23 - A holographic dual for Krylov complexity

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Abstract: There are several suggestions for an appropriate entry for quantum complexity in the holographic duality dictionary. The question is which notion of complexity and what is the precise duality. In this talk I endeavor to answer exactly this question for one, well studied, system. Krylov complexity has the signatures of quantum complexity at all time scales; it can be defined for operators or states. I will describe some of its features and show that in the setup of 2-dimensional JT gravity, Krylov complexity computed on the boundary has a well defined, precise geometrical meaning in the bulk.

A holographic dual for Krylov complexity
or
Measuring the wormhole with Krylov complexity

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Based on:

- E. Rabinovici, A. Sánchez-Garrido, RS and J. Sonner, "A bulk manifestation of Krylov complexity," arXiv:2305.04355[hep-th]
- E. Rabinovici, A. Sánchez-Garrido, RS and J. Sonner, "Krylov complexity from integrability to chaos," JHEP 07 (2022), 151
- E. Rabinovici, A. Sánchez-Garrido, RS and J. Sonner, "Krylov localization and suppression of complexity," JHEP 03 (2022), 211
- E. Rabinovici, A. Sánchez-Garrido, RS and J. Sonner, "Operator complexity: a journey to the edge of Krylov space," JHEP 06 (2021), 062
- J. L. F. Barbón, E. Rabinovici , RS and R. Sinha, "On The Evolution Of Operator Complexity Beyond Scrambling," JHEP 10 (2019), 264

Plan

- Why complexity? Which notion of complexity?
- Krylov complexity
- A holographic dual for Krylov complexity
- Open questions and future directions

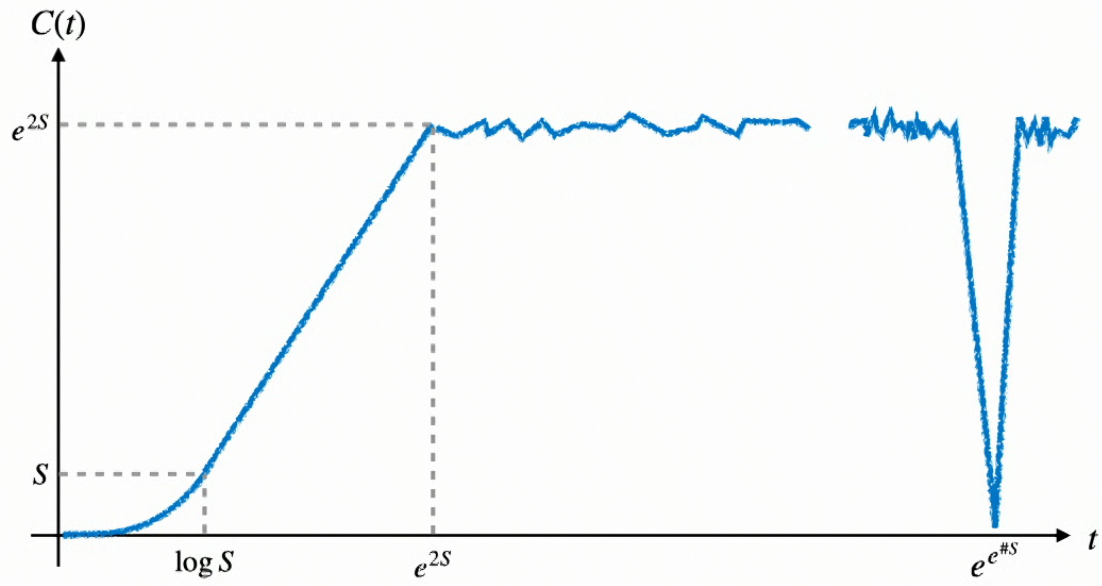
A puzzle in the holographic dictionary

- For a while now it has been recognized that new, possibly unknown, observables should be added to the holographic dictionary
- In particular, certain observables in the bulk seem to continue evolving in boundary time, long after known observables on the boundary cease to change

Question

What could be a corresponding observable on the boundary?

General time-dependent profile of complexity



Different notions of quantum complexity

- **Circuit complexity:** The minimal number of local gates g_i needed to construct an operator U starting from an initial operator U_0, up to a tolerance parameter ϵ
- **Geometric complexity:** Choose a penalty metric on space of unitaries; define complexity as shortest path to reach U from U_0
- **Krylov complexity:** Defined using the system's Hamiltonian and initial state/operator

Can we find a precise bulk-boundary correspondence?

Krylov space

Consider an operator $\mathcal{O}$ evolving in time under a Hamiltonian H

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}e^{-iHt} = \mathcal{O} + it[H,\mathcal{O}] + \frac{(it)^2}{2!}[H,[H,\mathcal{O}]] + \dots \quad (1)$$

Krylov space: the space spanned by the operator's time evolution

$$\mathcal{K} = \mathrm{Span}\left\{\mathcal{O}, [H,\mathcal{O}], [H,[H,\mathcal{O}]], \dots\right\} \quad (2)$$

Define the *Liouvillian* super-operator: $\mathcal{L} \equiv [H, \quad]$

$$|\mathcal{O}(t)) = e^{i\mathcal{L}t}|\mathcal{O}) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\mathcal{L}^n|\mathcal{O}) \quad (3)$$

Krylov space dimension

- **Hilbert space dimension**: $\dim(\mathcal{H}) = D \sim e^S$
- **Operator space dimension**: $\dim(\mathcal{H}^2) = D^2 \sim e^{2S}$

Upper bound on Krylov space dimension:

$$K \leq D^2 - D + 1 \sim e^{2S} \tag{5}$$

[Rabinovici Sánchez-Garrido RS Sonner 2020]

The upper bound is saturated for

- A dense operator which has non-zero projection on every eigenstate of the Liouvillian, $|E_a\rangle\langle E_b|$
- No degeneracies in the spectrum of the Liouvillian $\omega_{ab} = E_a - E_b$ except for the D-fold degeneracy of zero frequencies $\omega_{aa} = E_a - E_a = 0$

Lanczos algorithm

Orthonormalize the set $\{|\mathcal{O}), \mathcal{L}|\mathcal{O}), \mathcal{L}^2|\mathcal{O}), \ldots\}$:

Input: $\mathcal{L}$ and $|\mathcal{O})$

1. Set $|\mathcal{O}_0) = |\mathcal{O})/\|\mathcal{O}\|$
2. $|\mathcal{A}_1) = \mathcal{L}|\mathcal{O}_0)$, compute $\|\mathcal{A}_1\|$ **if** $\|\mathcal{A}_1\| = 0$ **STOP**
   otherwise define $b_1 = \|\mathcal{A}_1\|$ and $|\mathcal{O}_1) = |\mathcal{A}_1)/b_1$
3. For $n > 1$:
   $|\mathcal{A}_n) = \mathcal{L}|\mathcal{O}_{n-1}) - b_{n-1}|\mathcal{O}_{n-2})$
   compute $\|\mathcal{A}_n\|$ **if** $\|\mathcal{A}_n\| = 0$ **STOP**
   otherwise define $b_n = \|\mathcal{A}_n\|$ and $|\mathcal{O}_n) = |\mathcal{A}_n)/b_n$

Output:

- **The Krylov chain:** $\{|\mathcal{O}_0), |\mathcal{O}_1), |\mathcal{O}_2), \ldots, |\mathcal{O}_{K-1})\}$
- **The Lanczos sequence:** $\{b_1, b_2, b_3, \ldots, b_{K-1}\}$

where K is the Krylov space dimension

Operator time evolution on the Krylov chain

- Time-evolving operator can be expanded in Krylov basis:

$$|\mathcal{O}(t)) = e^{i\mathcal{L}t}|\mathcal{O}_0) = \sum_{n=0}^{K-1} \phi_n(t)|\mathcal{O}_n) \tag{6}$$

where $\phi_n(t) = (\mathcal{O}_n|\mathcal{O}(t))$

- The Liouvillian is **tridiagonal** in the Krylov basis

$$\begin{pmatrix} 0 & b_1 & & & & \\ b_1 & 0 & b_2 & & & \\ & b_2 & 0 & b_3 & & \\ & & b_3 & 0 & \ddots & \\ & & & \ddots & \ddots & b_{K-1} \\ & & & & b_{K-1} & 0 \end{pmatrix} \tag{7}$$

- Define **position** operator over Krylov basis

$$\hat{n} = \sum_{n=0}^{K-1} n\,|\mathcal{O}_n)(\mathcal{O}_n| \tag{8}$$

Krylov complexity:
a probe of operator time evolution at all time scales

K-complexity is the expectation value of position:

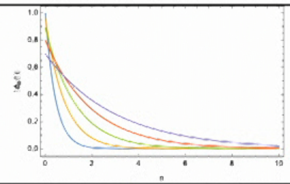
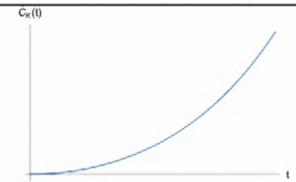
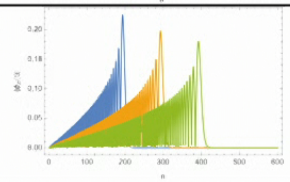
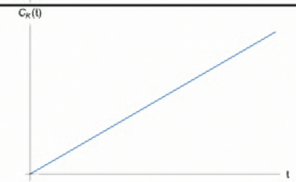
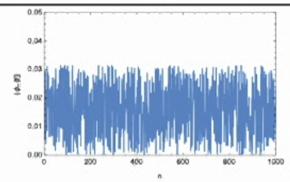
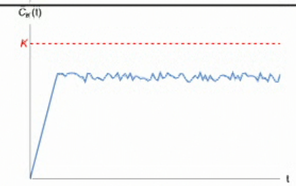
$$C_K(t) = \langle \hat{n}(t) \rangle = \sum_{n=0}^{K-1} n \, |\phi_n(t)|^2 \qquad (9)$$

[Parker Cao Avdoshkin Scaffidi Altman 2018]

- Bounded by Krylov space dimension, $0 \leq C_K(t) \leq K$
- Introduced in [Parker et al 2018] to study operator growth in the *thermodynamic limit*
- and as a measure of operator complexity at *all time scales* for *finite systems* in [Barbón Rabinovici RS Sinha 2019]

Dynamics of Krylov complexity

[Parker et al 2018] [Barbón Rabinovici RS Sinha 2019] [Rabinovici Sánchez-Garrido RS Sonner 2020]

n	b_n	wavefunction	K-complexity	time scales
$n \ll S$	$b_n \sim n$			$t \lesssim \log S$
$n > S$	$b_n \sim \Lambda S$			$t > \log S$
$n \sim e^{2S}$	"descent"			$t \gtrsim e^{2S}$

Numerical results: SYK_4 setup

- SYK_4 is a maximally chaotic many-body system. Consider complex SYK_4 with L fermions

$$H_{SYK} = \sum_{i,j,k,l}^{L} J_{ij,kl} c_i^\dagger c_j^\dagger c_k c_l \tag{10}$$

where $\{c_i, c_j^\dagger\} = \delta_{ij}$ and $\{c_i, c_j\} = 0 = \{c_i^\dagger, c_j^\dagger\}$

- The coupling constants are taken from a Gaussian distribution with

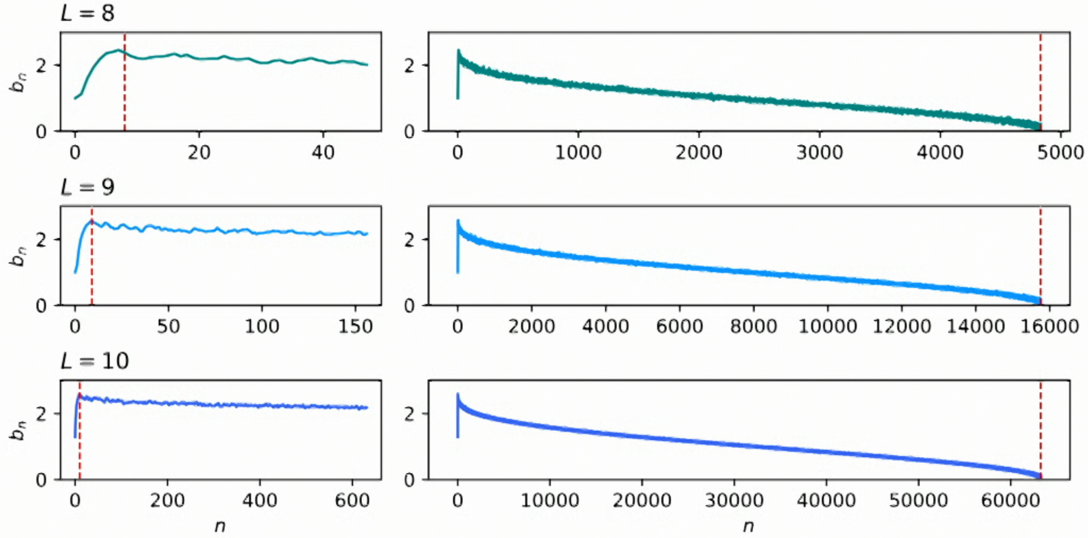
$$\overline{J_{ij,kl}} = 0 \quad \overline{|J_{ij,kl}|^2} = \frac{3! J^2}{L^3} \tag{11}$$

- Operator taken to be the hopping operator

$$\mathcal{O} = c_{L-1}^\dagger c_L + c_L^\dagger c_{L-1} \tag{12}$$

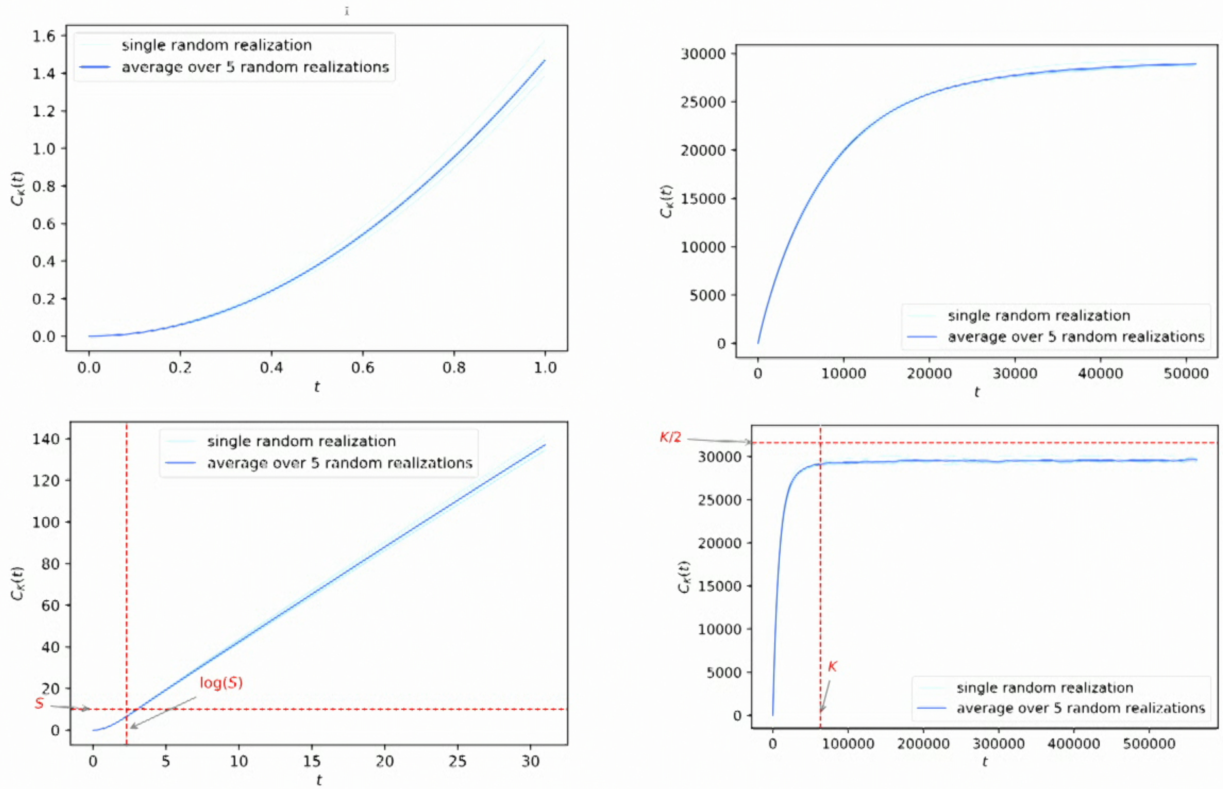
shown in [Sonner Vielma 2017] to satisfy ETH

Non-perturbative "Descent"



K-complexity for SYK_4 with 10 complex fermions

[Rabinovici Sánchez-Garrido RS Sonner 2020]

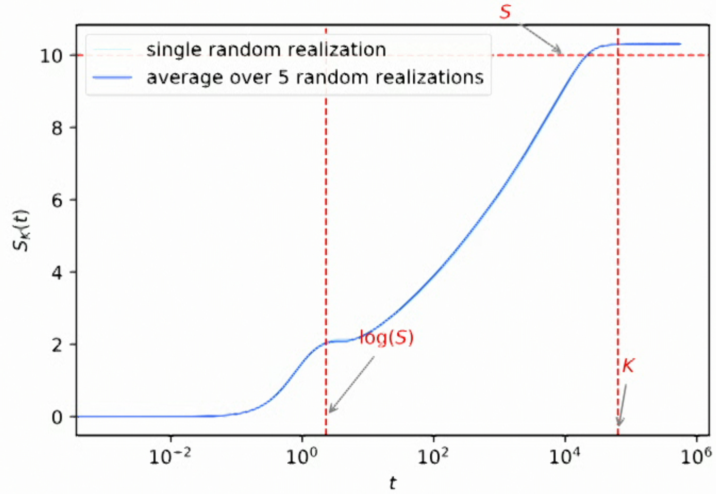


K-entropy

[Barbón Rabinovici RS Sinha 2019]

- K-entropy measures the amount of disorder in the wavefunction

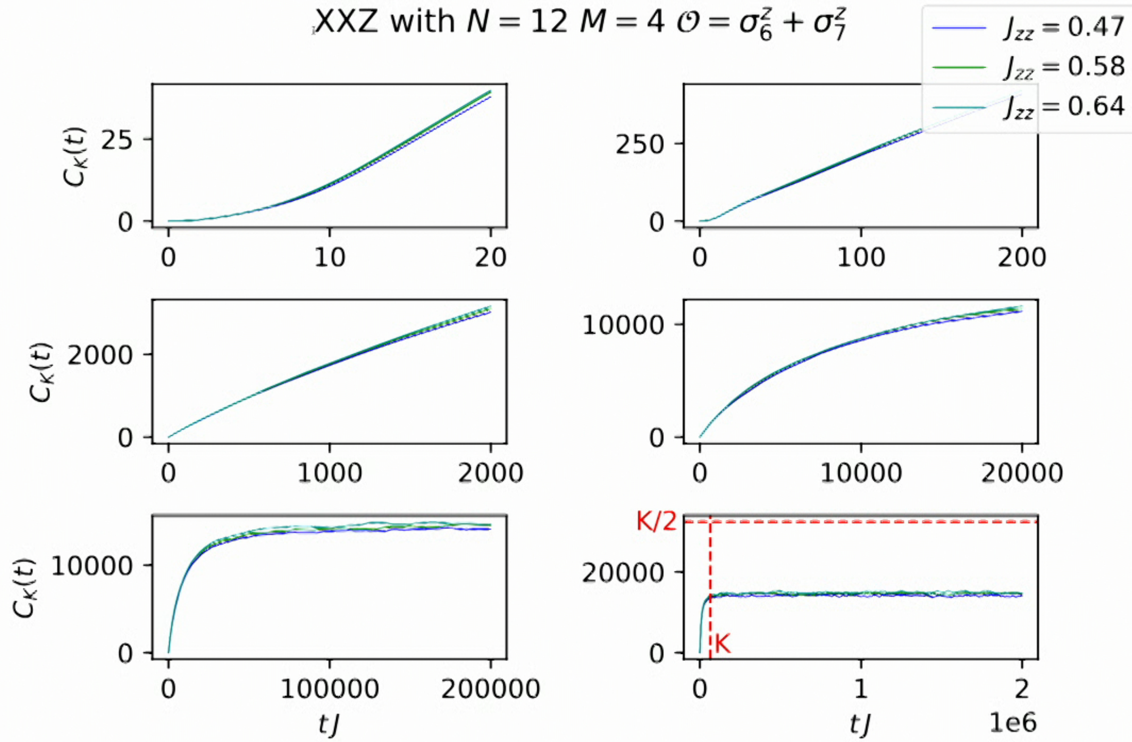
$$S_K(t) = -\sum_{n=0}^{K-1} |\phi_n(t)|^2 \log |\phi_n(t)|^2 \qquad (13)$$



[Rabinovici Sánchez-Garrido RS Sonner 2020]

Suppression of K-complexity in XXZ

- Anderson localization on the Krylov chain



Krylov space dimension $K = 64771$

[Rabinovici Sánchez-Garrido RS Sonner 2021]

Krylov complexity for states

- Given a Hamiltonian H and an initial state $|\Omega\rangle$, the state evolves in time unitarily

$$|\Omega(t)\rangle = e^{-iHt}|\Omega\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} H^n |\Omega\rangle \tag{14}$$

- The Lanczos algorithm provides the **Krylov basis** and **Lanczos coefficients**
- The Hamiltonian is tridiagonal in the Krylov basis

$$H = \begin{pmatrix} a_1 & b_1 & & & & \\ b_1 & a_2 & b_2 & & & \\ & b_2 & a_3 & b_3 & & \\ & & b_3 & a_4 & \ddots & \\ & & & \ddots & \ddots & b_{K-1} \\ & & & & b_{K-1} & a_K \end{pmatrix} \tag{15}$$

A holographic dual for Krylov complexity

The boundary: DSSYK

[Berkooz Narayan Simon 2018][Berkooz Isachenkov Narovlansky Torrents 2018]

- Start with SYK with N fermions and p-body interactions

$$H_{SYK} = i^{p/2} \sum_{1 \le i_1 \le i_2 \le \cdots \le i_p \le N} J_{i_1 i_2 \ldots i_p} \, \psi_{i_1} \psi_{i_2} \ldots \psi_{i_p} \qquad (16)$$

where $\overline{J_{i_1 i_2 \ldots i_p}} = 0$ and $\overline{J^2_{i_1 i_2 \ldots i_p}} = \frac{J^2}{\lambda} \binom{N}{p}^{-1}$

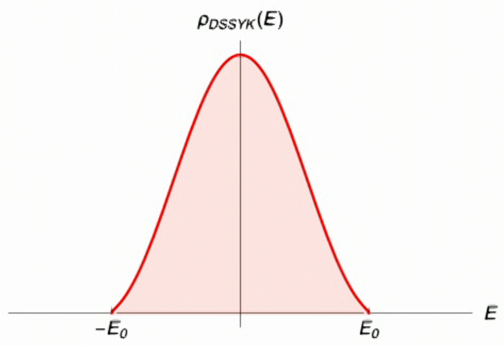
- Define the ratio parameter $\lambda \equiv \frac{2p^2}{N}$
- In the limit $N \to \infty$, $p \to \infty$ and λ fixed, the **ensemble averaged** effective Hamiltonian is

$$H = \frac{J}{\sqrt{\lambda(1-q)}} \begin{pmatrix} 0 & \sqrt{1-q} & & & \\ \sqrt{1-q} & 0 & \sqrt{1-q^2} & & \\ & \sqrt{1-q^2} & 0 & \sqrt{1-q^3} & \\ & & \sqrt{1-q^3} & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

where $q \equiv e^{-\lambda}$

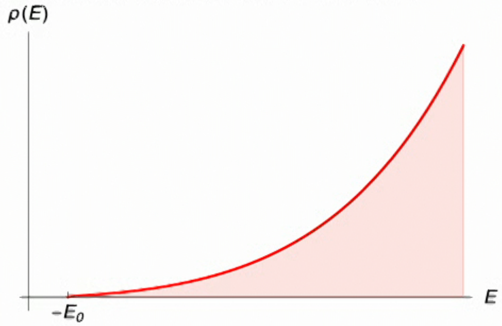
The triple-scaled limit of SYK [Lin 2022]

- In the triple-scaled limit we take $\lambda \to 0$



The triple-scaled limit of SYK [Lin 2022]

- In the triple-scaled limit we take $\lambda \to 0$ and zoom-in near the ground state of DSSYK



- The d.o.s. in the triple-scaled limit is $\rho(E) \propto \sinh(2\pi\sqrt{E})$
- The **Liouville** form of the triple-scaled Hamiltonian connects DSSYK with the Hamiltonian of JT gravity

K-complexity in the triple-scaled limit of SYK

[Rabinovici Sánchez-Garrido RS Sonner 2023]

- Recall that $l = \lambda n$ and hence $C_K(t) = \frac{\langle l(t) \rangle}{\lambda}$
- In the triple-scaled limit $C_K(t) = \frac{\langle \tilde{l}(t) \rangle}{\lambda}$
- Solving for the expectation value classically with $\tilde{l}(0) = x_0$ and $\dot{\tilde{l}}(0) = 0$ we find that

The bulk: JT gravity

- The action of JT gravity

$$S_{JT} = \int_{\mathcal{M}} d^2x\sqrt{-g}\Big[\Phi_0 R + \Phi(R+2)\Big] + 2\int_{\partial\mathcal{M}} dx\sqrt{\gamma}\Big[\Phi_0 K + \Phi(K-1)\Big] \tag{23}$$

- Boundary conditions

$$ds^2\Big|_{\partial\mathcal{M}} = -\frac{dt_b^2}{\epsilon^2}, \quad \Phi\Big|_{\partial\mathcal{M}} = \frac{\phi_b}{\epsilon} \tag{24}$$

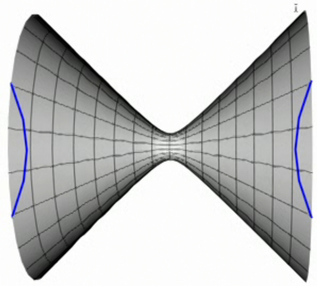
- Equations of motion

$$0 = R + 2 \Longrightarrow \mathrm{AdS}_2 \tag{25}$$

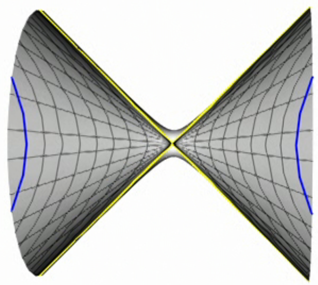
$$0 = (\nabla_\mu\nabla_\nu - g_{\mu\nu})\Phi \tag{26}$$

Wormhole (ERB) length

[Harlow Jafferis 2018]



(a) Global coordinates
$ds^2 = -(1+x^2)d\tau^2 + \frac{dx^2}{1+x^2}$
$\Phi(x,\tau) = \Phi_h\sqrt{1+x^2}\cos\tau$



(b) Schwarzschild coordinates
$ds^2 = -(r^2 - r_s^2)dt^2 + \frac{dr^2}{r^2-r_s^2}$
$\Phi(r,t) = \phi_b r$

Krylov complexity = wormhole length

[Rabinovici Sánchez-Garrido RS Sonner 2023]

K-complexity in triple-scaled SYK

$$\lambda C_K(t) = 2\log\left[\cosh\left(\sqrt{\lambda J E}\, t\right)\right] - \log\left(\frac{E}{4\lambda J}\right) \tag{28}$$

Summary, open questions and future directions

Result

Krylov basis in triple-scaled SYK = Wormhole length basis in JT gravity

Krylov complexity in triple-scaled SYK = Wormhole length in JT gravity

- Quantum corrections?
- Going away from the small λ limit
- Saturation of K-complexity?
- The Krylov basis with operators is richer, what is the precise geometric interpretation of operator insertions in DSSYK