

Title: Research Talk 19 - Regge sum rules and stringy amplitudes

Speakers:

Collection: Strings 2023

Date: July 27, 2023 - 11:00 AM

URL: <https://pirsa.org/23070040>

Regge sum rules and stringy amplitudes

Alexander Zhiboedov (CERN)
Strings 2023

based on ongoing work with Kelian Häring

One of the characteristic features of stringy scattering is the Regge behavior at high energies

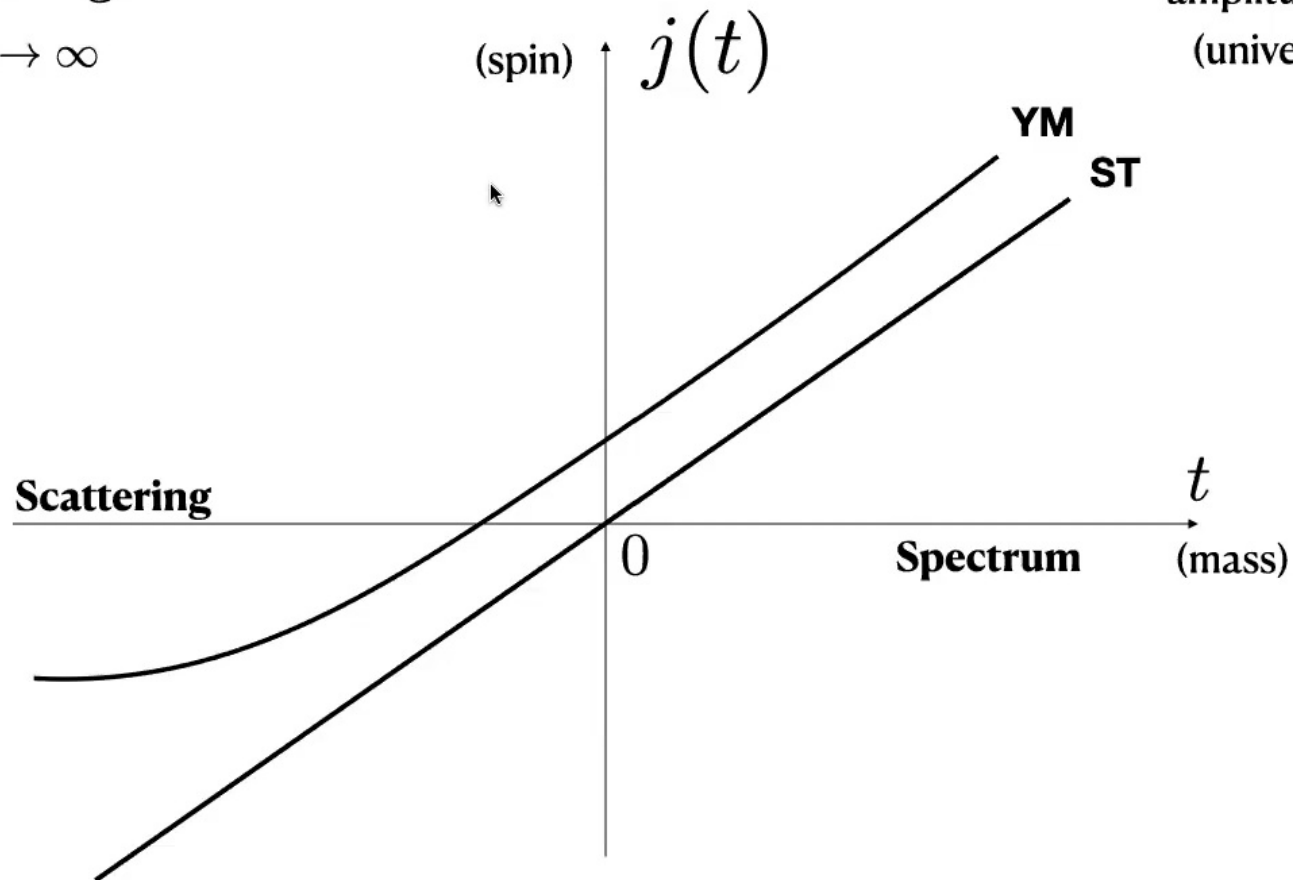
$$\text{(tree level)} \quad T(s, t) \sim s^{j(t)} \quad \begin{array}{l} s \rightarrow \infty \\ t \text{ fixed} \end{array}$$

$j(t)$ **leading Regge trajectory**

$$T \sim s^{j(t)}$$

$$s \rightarrow \infty$$

amplitude is BIG
(universal, IR)

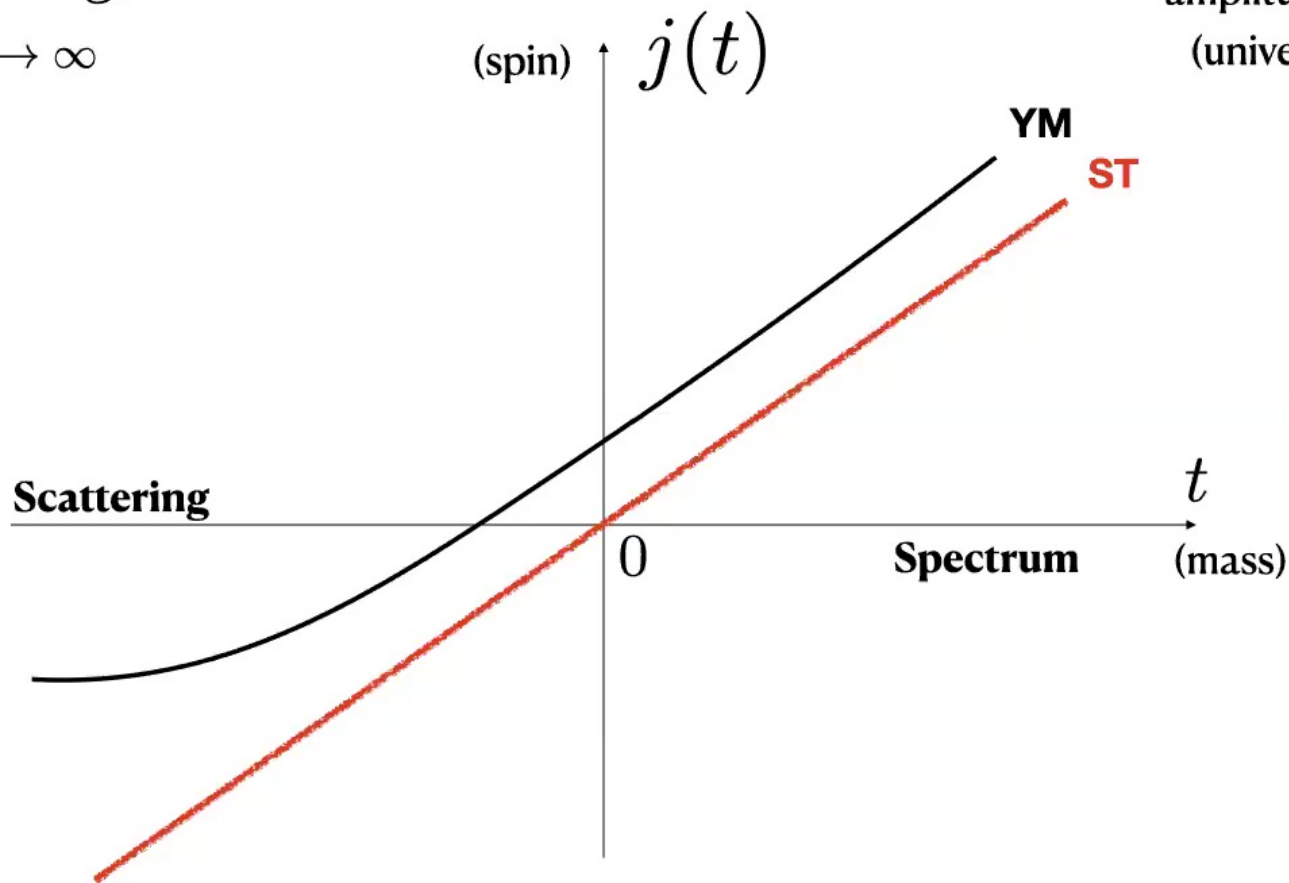


amplitude is SMALL
(non-universal, UV)

$$T \sim s^{j(t)}$$

$$s \rightarrow \infty$$

amplitude is BIG
(universal, IR)



amplitude is SMALL
(non-universal, UV)

$$S = \frac{1}{M_P^2} \int d^4x \sqrt{-g} \left(R + \beta_{R^3} R^3 + \beta_{R^4} C^2 + \tilde{\beta}_{R^4} \tilde{C}^2 + \dots \right)$$

Causality requires massive higher spin particles.

$$\beta_i \sim \frac{1}{m_{\text{HS}}^{\#}}$$

mass of higher spin particles

[Camanho, Edelstein, Maldacena, AZ, Arkani-Hamed, T-C Huang, Y-t Huang, Chiang, Rodina, Weng, Tolley, Wang, Zhou, Caron-Huot, van Duong, Sinha, Zahed, Mazac, Rastelli, Simmons-Duffin, Y-Z Li, Bellazzini, Riemann, Riva, Carrillo Gonzales, de Rham, Jaitly, Pozsgay, Tokareva, Henriksson, McPeak, Russo, Vichi...]

[Rastelli (review), Strings 2022]

[Caron-Huot, Mizera (discussion) Strings 2023]

Sharper bounds from causality+unitarity+crossing.

$$S = \frac{1}{M_P^2} \int d^4x \sqrt{-g} \left(R + \beta_{R^3} R^3 + \beta_{R^4} C^2 + \tilde{\beta}_{R^4} \tilde{C}^2 + \dots \right)$$

Causality requires massive higher spin particles.

$$\beta_i \sim \frac{1}{m_{\text{HS}}^{\#}}$$

mass of higher spin particles

[Camanho, Edelstein, Maldacena, AZ, Arkani-Hamed, T-C Huang, Y-t Huang, Chiang, Rodina, Weng, Tolley, Wang, Zhou, Caron-Huot, van Duong, Sinha, Zahed, Mazac, Rastelli, Simmons-Duffin, Y-Z Li, Bellazzini, Riemann, Riva, Carrillo Gonzales, de Rham, Jaitly, Pozsgay, Tokareva, Henriksson, McPeak, Russo, Vichi...]

[Rastelli (review), Strings 2022]

[Caron-Huot, Mizera (discussion) Strings 2023]

Sharper bounds from causality+unitarity+crossing.

Wilson coefficients control the low-energy expansion of the amplitude

$$T(s, t) = \dots + \sum_{k,j} a_{k,j} s^{k-j} t^j$$

What happens to the bounds if in addition
we put in the leading Regge trajectory?

(How does the UV softness constrain the IR?)

What happens to the bounds if in addition
we put in the leading Regge trajectory?

(How does the UV softness constrain the IR?)

[D. Gross: “Go multiparticle!”]

What happens to the bounds if in addition
we put in the leading Regge trajectory?

(How does the UV softness constrain the IR?)

[D. Gross: "Go multiparticle!"]

[D. Gross: "Don't follow your elders"]

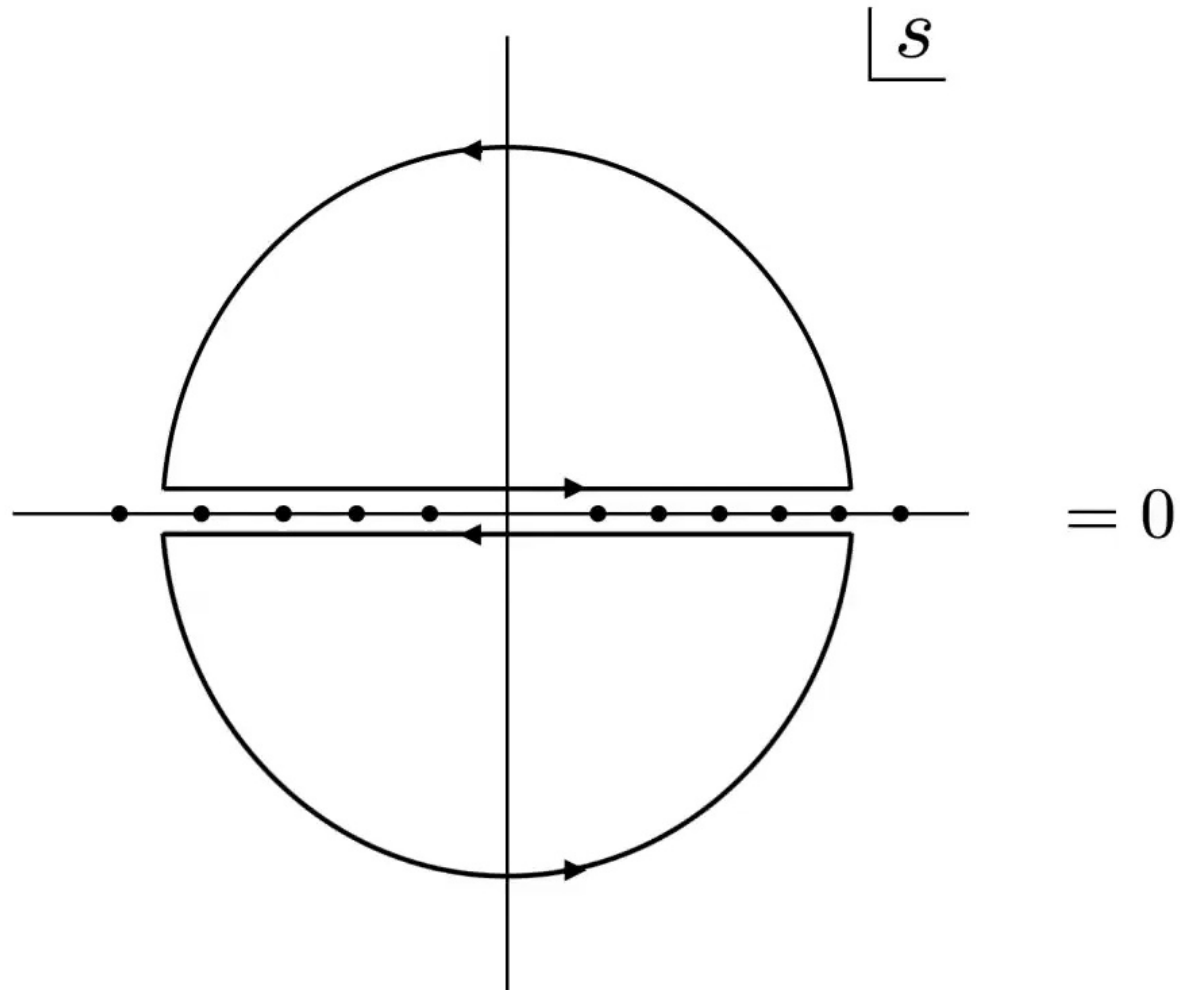
Plan:

0. Regge sum rules

I. Open strings

II. High energy, fixed angle scattering

III. Closed strings



$$\oint_C ds' (s')^n T(s', t) = 0$$

- **Regge sum rules (RSR):** $t < 0$

$$\oint_{\mathcal{C}_\infty} ds' (s')^n T(s', t) = 0 \quad \curvearrowright \quad \sim \int \frac{ds}{s} s^{j(t)+1+n}$$

$$\int_0^\infty ds' (s')^n T_s(s', t) = 0, \quad j(t) < -1 - n$$

(scattering is soft)

discontinuity in s

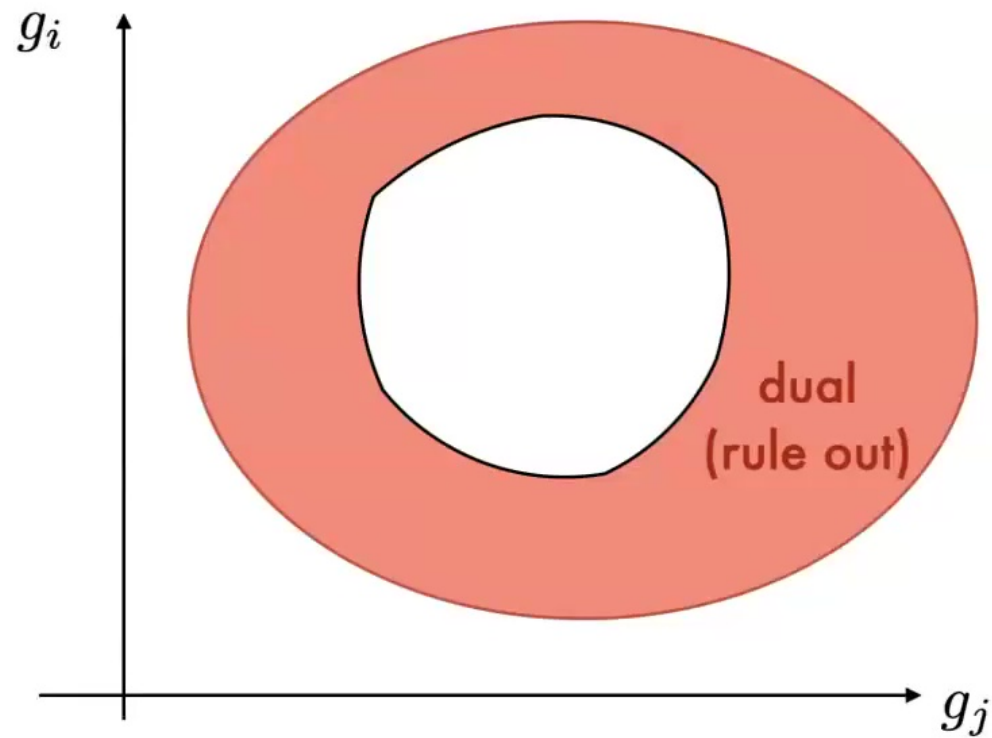
- **Finite energy sum rules (FESR):** $t > 0$

$$\int_0^s ds' T_s(s', t) \sim s^{j(t)+1}$$

(no accumulation point in the spectrum+HS particles)

$$\text{RSR} : \quad \langle (-1)^J c^J \rangle = 0 \quad c > 1$$

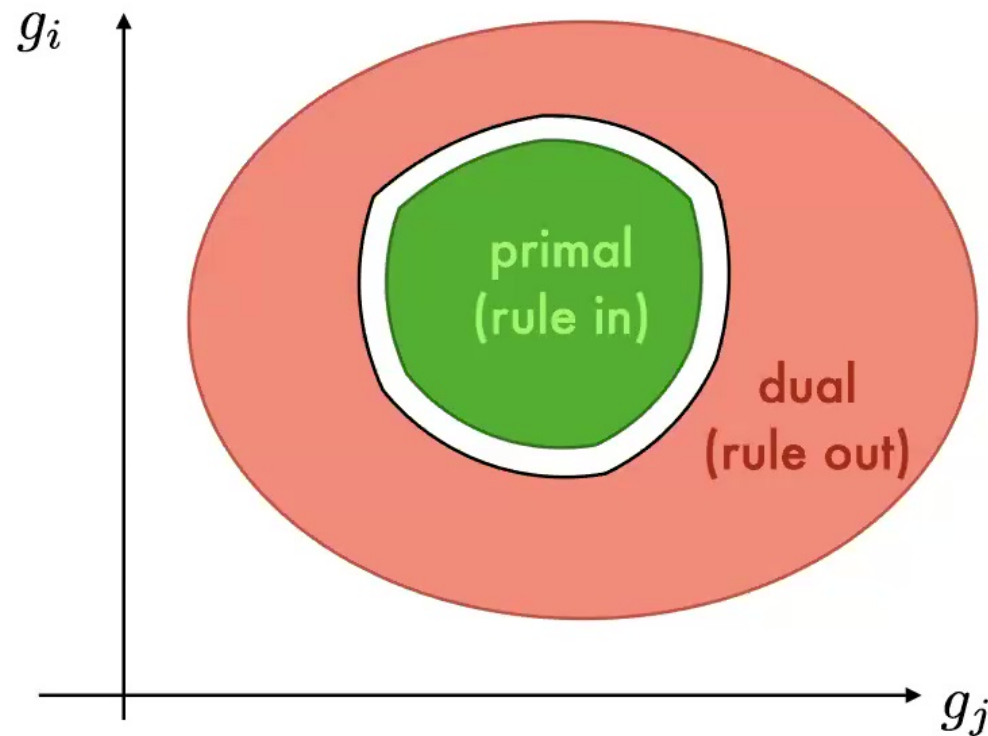
$$\langle \dots \rangle \equiv \int dm^2 \rho(m^2) \dots, \quad \rho(m^2) \geq 0.$$



Basic idea: Explore the allowed region by explicitly constructing amplitudes with desired properties.

$$\text{RSR} : \quad \langle (-1)^J c^J \rangle = 0 \quad c > 1$$

$$\langle \dots \rangle \equiv \int dm^2 \rho(m^2) \dots, \quad \rho(m^2) \geq 0.$$



To proceed we consider an **ansatz** for $T(s, t)$ with **manifest**:

[Veneziano, Mandelstam, Khuri, Matsuda, Altarelli, Rubinstein, Virasoro, Sivvers, Yellin, Gross, ...]

[Cheung, Remmen, Geiser, Lindwasser, Arkani-Hamed, Huang, Huang, ...]

1. Meromorphy

$$T(s, t) \sim \frac{1}{s - m^2}$$

2. Crossing symmetry

$$T(s, t) = T(t, s)$$

3. Polynomial residues

$$-\text{Res}_{s=m^2} T(s, \frac{s}{2}(z-1)) = \sum_J c_J P_J(z)$$

4. FESR+RSR

5*. All particles have

$$m^2 = n, \quad n \in \mathbb{Z}$$

We will then need to impose (numerically):

6. Unitarity

$$c_J \geq 0$$

For the open string case the ansatz is

$$T(s, t) = \sum_{i=0}^{\infty} \sum_{k=0}^i b_{ik} \frac{\Gamma(i-s)\Gamma(i-t)}{\Gamma(i+k-s-t)}$$

- All terms are linearly independent.

- For $t > 0$, FESR is automatic: $T_s(s, t) \sim s^t$

- For $t < 0$, RSR is easy: $\int_0^{\infty} ds' (s')^n T_s(s', t) = 0$

- Constraints on b_{ik} come from unitarity:

$$\text{Res}_{s=n} T(s, t) = \sum_{J=0}^n c_J P_J\left(1 + \frac{2t}{n}\right), \quad c_J \geq 0.$$

- The ansatz is complete (see back-up slide).

Comment 1: The truncated ansatz with $i_{\max} \geq 1$

$$T(s, t) = \sum_{i=0}^{i_{\max}} \sum_{k=0}^i b_{ik} \frac{\Gamma(i-s)\Gamma(i-t)}{\Gamma(i+k-s-t)}$$

satisfies RSR and violates unitarity (unless Veneziano).

Comment 2: The Cheung-Remmen ansatz

$$T(s, t) = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{r}{r+i} \frac{\Gamma(i-s)\Gamma(i-t)}{\Gamma(i-s-t)}, \quad r \geq -1/2.$$

violates RSR and satisfies unitarity.

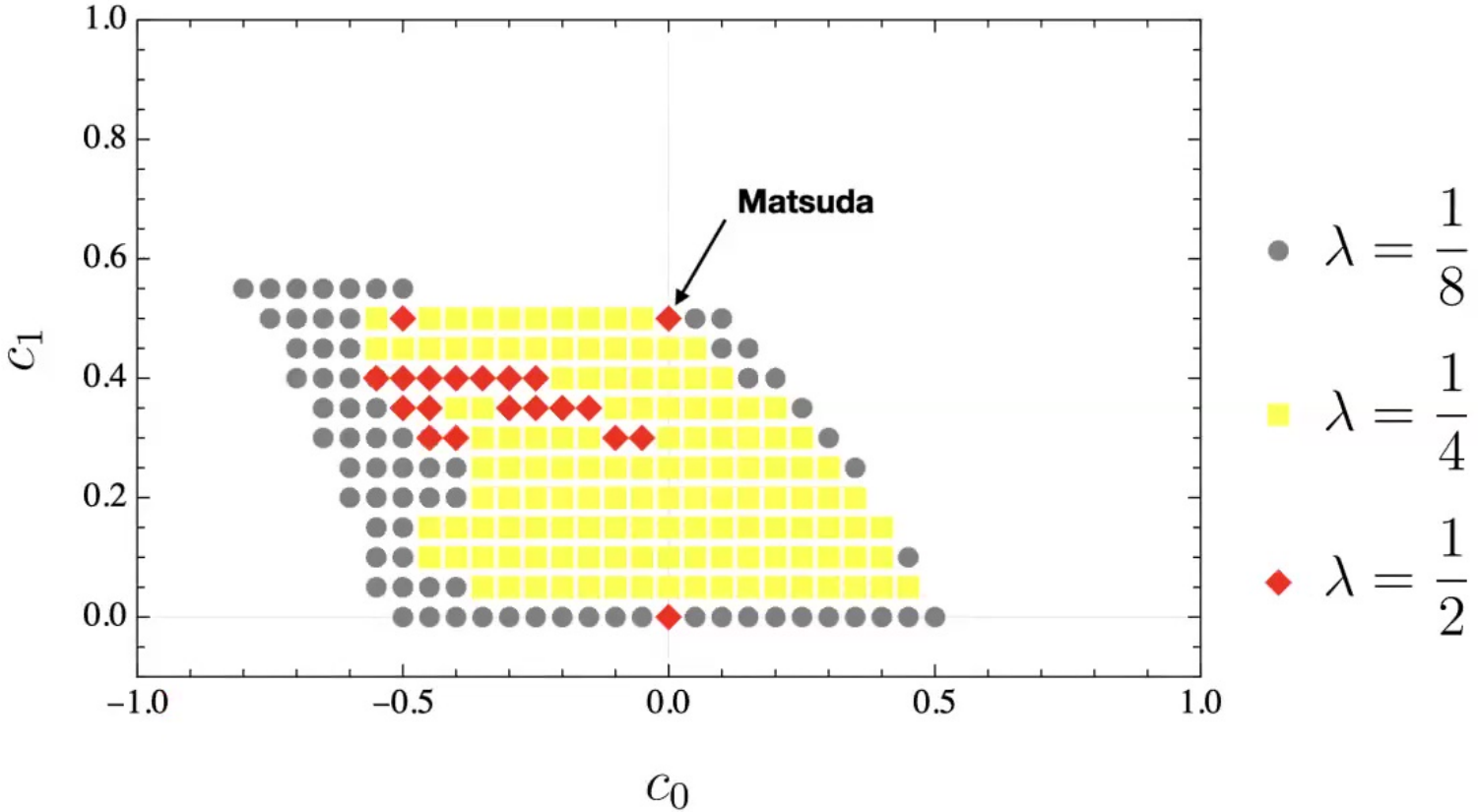
fixed J singularities

$$T_s(s, t) \sim s^t \quad T(s, t) \sim f(t)s^t + \frac{r}{(1+t)s} + \dots$$

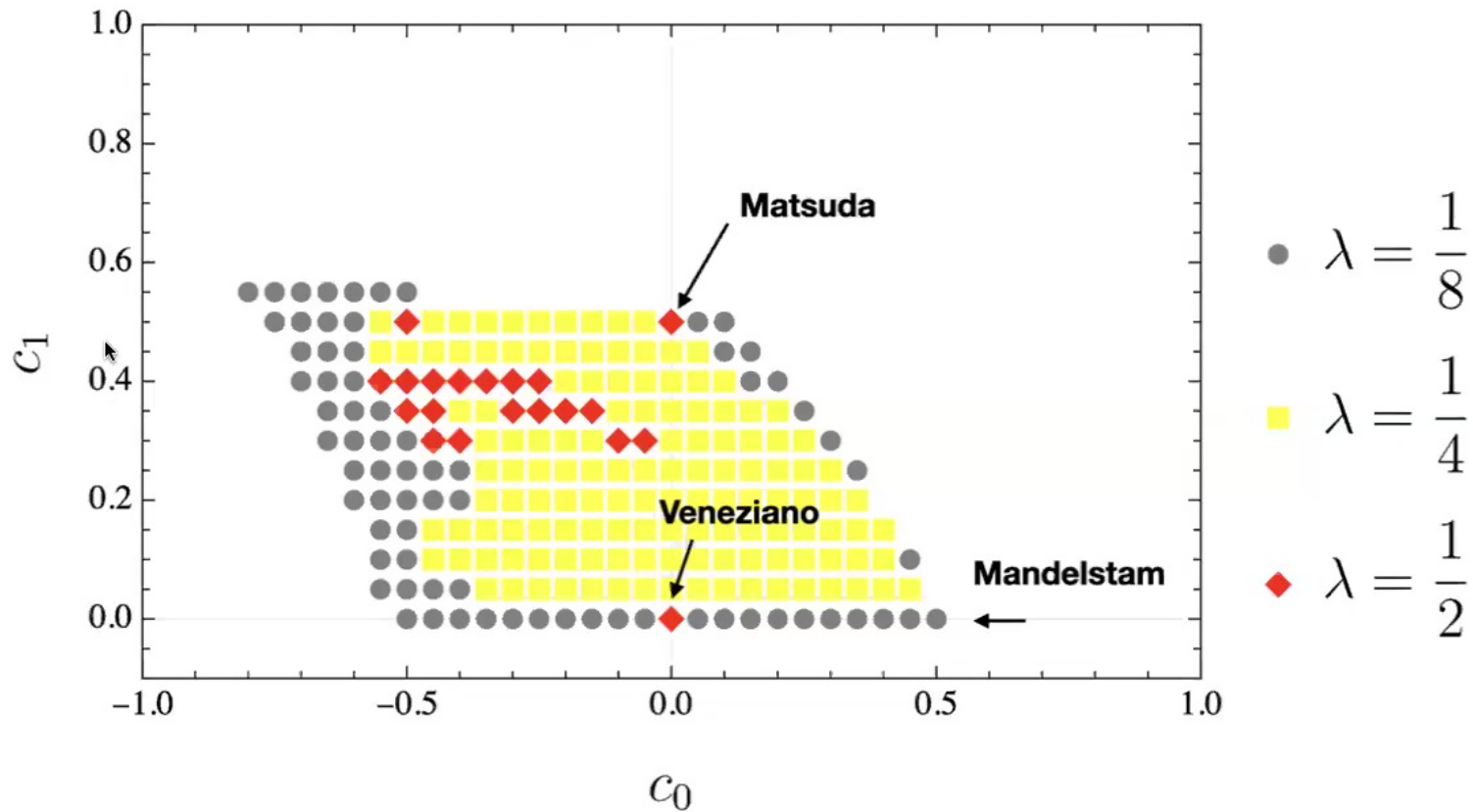
Comment 3: There exist amplitudes different from the Veneziano amplitude that satisfy RSR and satisfy unitarity

$$T_{c_0, c_1, \lambda}(s, t) = \int_0^1 dz z^{-s-1} (1-z)^{-t-1} (1-4\lambda(1-z)z)^{c_0+c_1(s+t)}, \quad 0 \leq \lambda < 1/2$$

Unitarity (numerically, $n_{\max} = 0, \dots, 50, \dots, 450$)



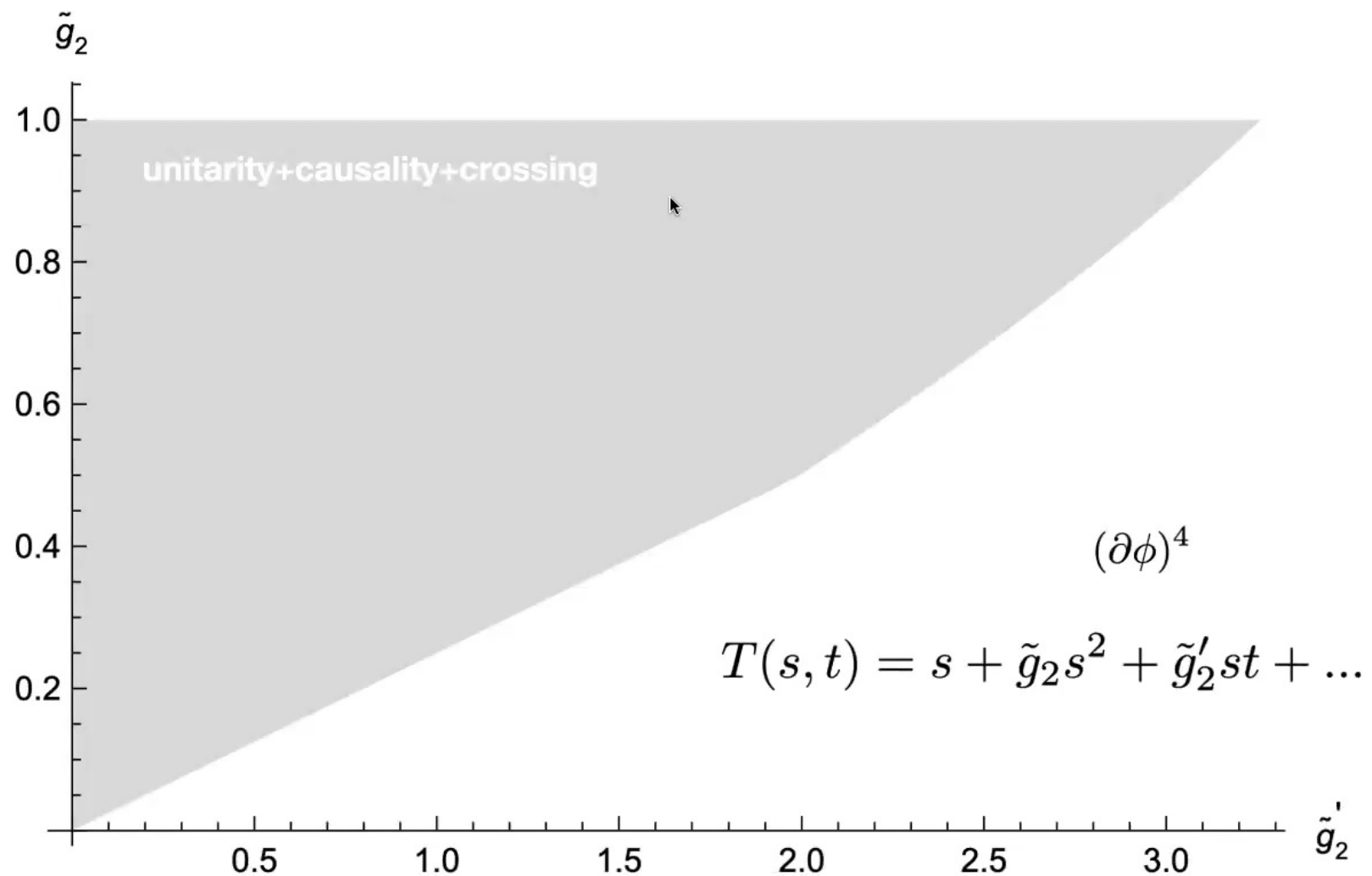
Unitarity (numerically, $n_{\max} = 0, \dots, 50, \dots, 450$)



We observe that unitarity implies $c_1 \geq 0$.

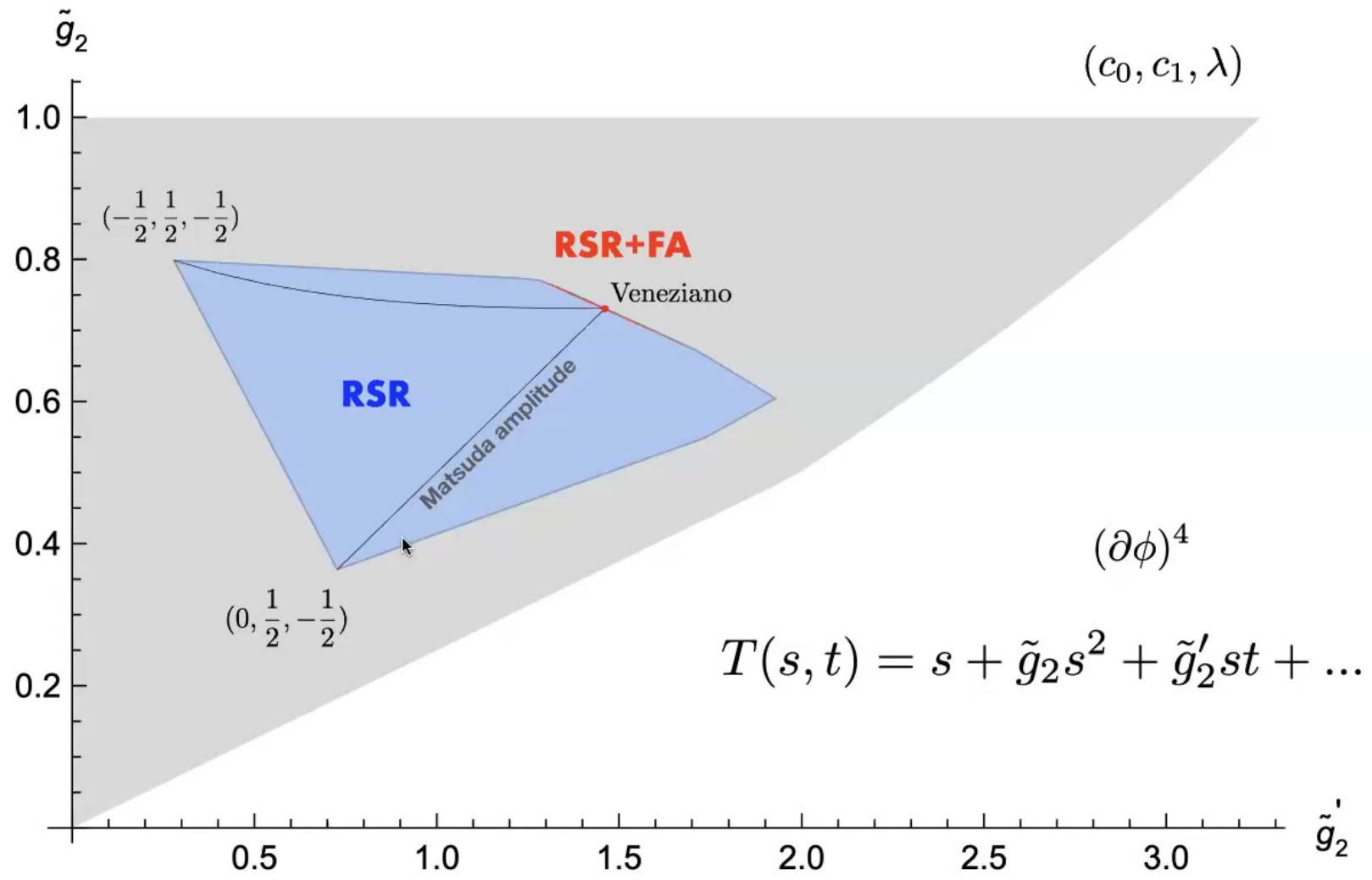
Bounds on Wilson coefficients

[Albert, Rastelli '22]



Bounds on Wilson coefficients

[Albert, Rastelli '22]



0. Regge sum rules

I. Open strings

II. High energy, fixed angle scattering

III. Closed strings

$$s \rightarrow \infty$$

$$s/t \text{ fixed}$$

Consider the semi-classical limit $s, t \gg 1$. Here we explicitly get

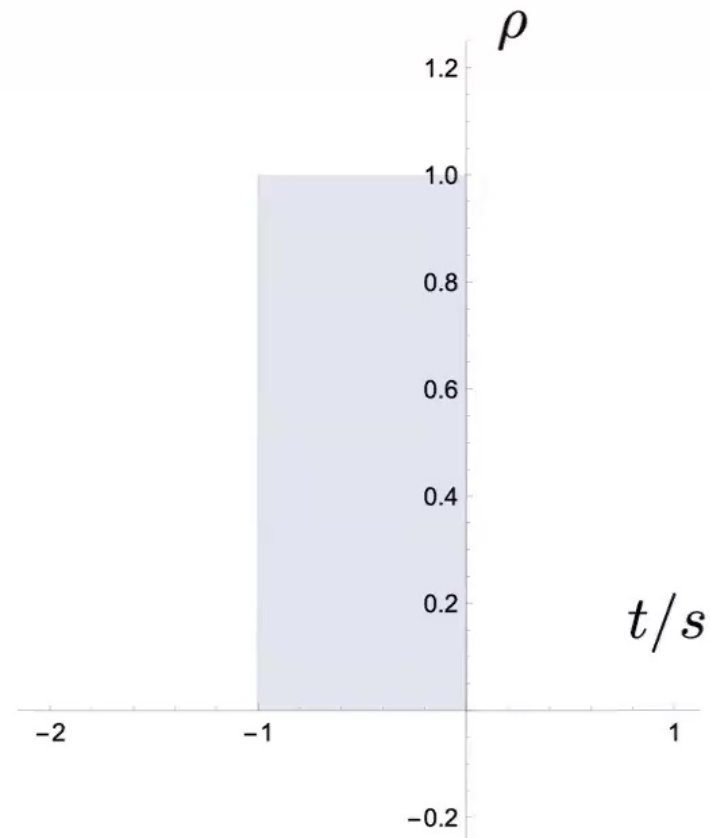
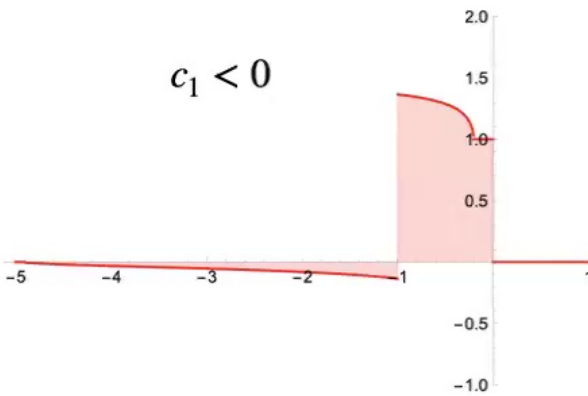
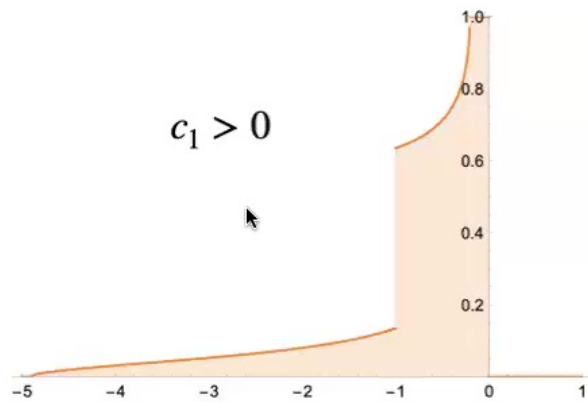
$$\begin{aligned} \log T(s, t) &= (s + t) \log(s + t) - s \log s - t \log t \\ &+ c_1 \left(t \log \frac{1}{2} \left(1 - \tilde{\lambda} \frac{s-t}{s+t} + \sqrt{1 - \tilde{\lambda}} \sqrt{1 - \tilde{\lambda} \frac{(s-t)^2}{(s+t)^2}} \right) + \{s \leftrightarrow t\} \right) \\ &0 \leq \tilde{\lambda} = 4\lambda(1 - \lambda) < 1 \end{aligned}$$

In this regime the amplitude is controlled by the zeros of positive sums of Legendre polynomials

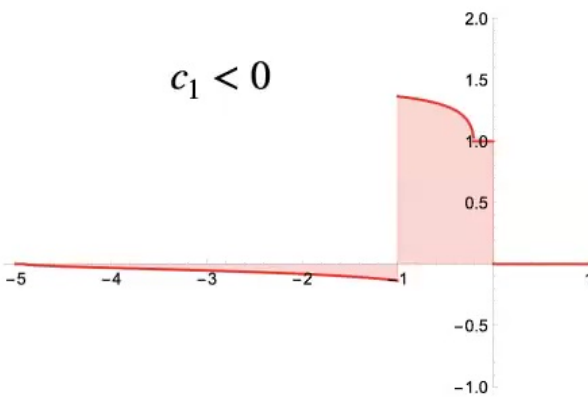
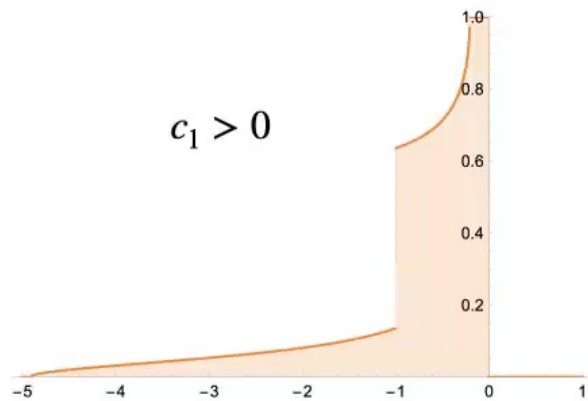
$$\log T(s, t) \sim \log \sum_J^s c_J(s) P_J \left(1 + \frac{2t}{s} \right)$$

[Caron-Huot, Komargodski, Sever, AZ '16]

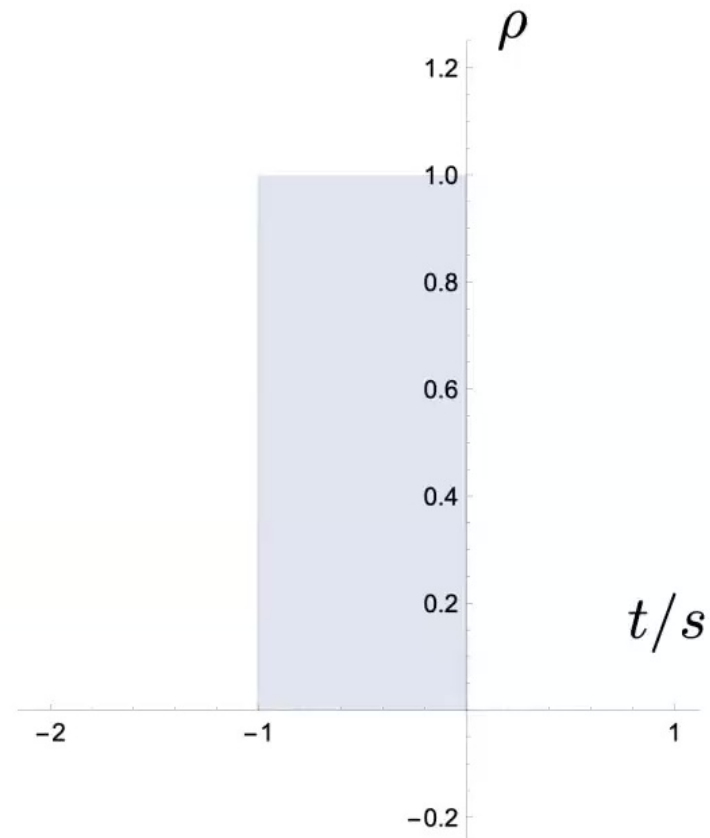
$$\log T(s, t) = s \int_{-\infty}^0 dz \rho(z) \log \left(1 - \frac{t}{sz} \right), \quad \rho(z) \geq 0$$



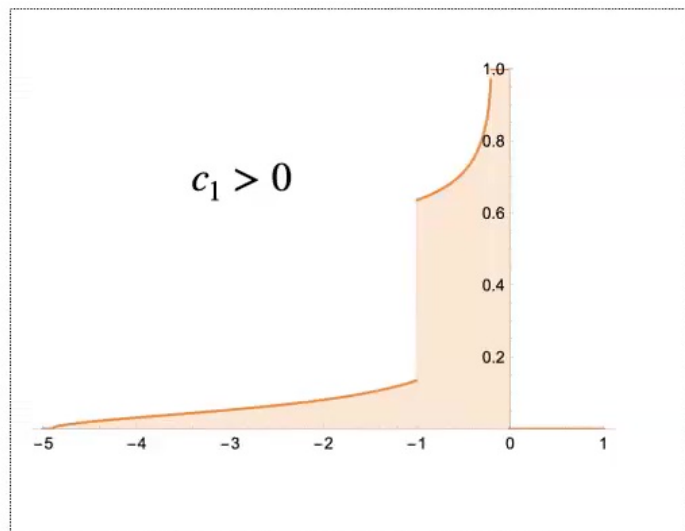
$$\log T(s, t) = s \int_{-\infty}^0 dz \rho(z) \log \left(1 - \frac{t}{sz} \right), \quad \rho(z) \geq 0$$



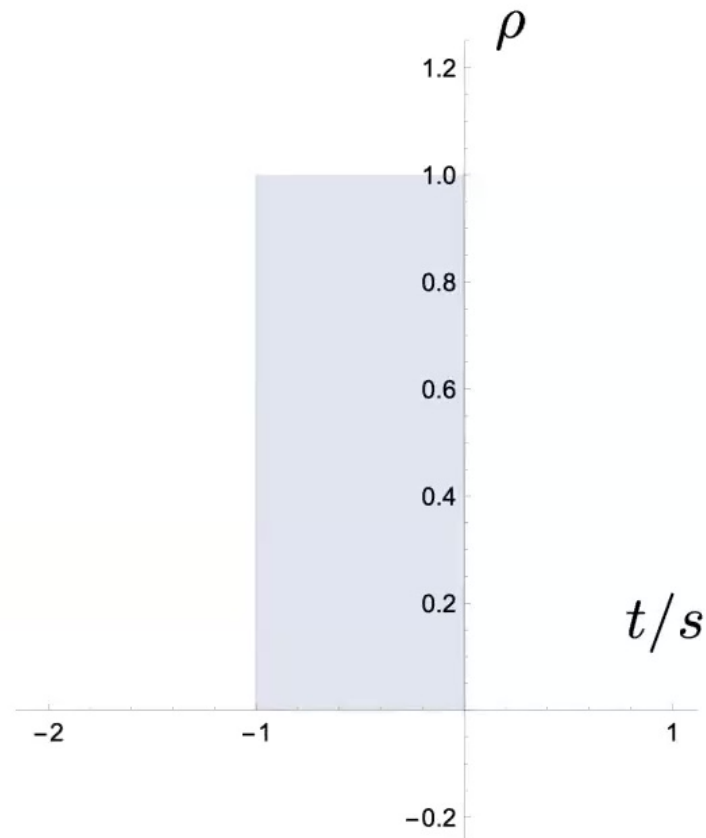
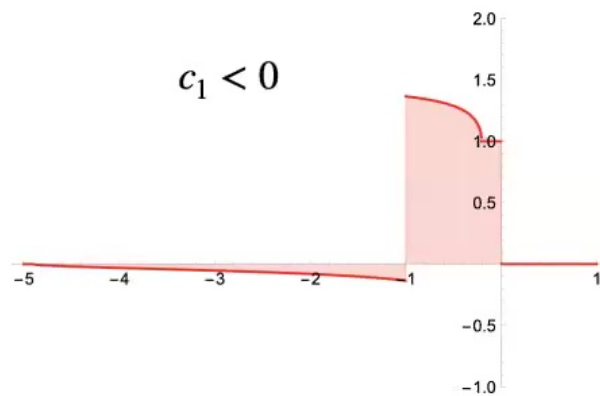
➔ $c_1 \geq 0$



$$\log T(s, t) = s \int_{-\infty}^0 dz \rho(z) \log \left(1 - \frac{t}{sz} \right), \quad \rho(z) \geq 0$$



➔ $c_1 \geq 0$



[Caron-Huot, Komargodski, Sever, AZ '16]

Maybe a simpler bootstrap target for $s, t \gg 1$ is

[M. Correia, AZ]

$$\log T(s, t) \leq (s + t) \log(s + t) - s \log s - t \log t$$

we proved it **assuming** the zeros are real and are supported along the negative axis.

Problem in 2d electrostatics: Find the bound on the electric field generated by the distribution of positive charges

$$\int_{-\infty}^0 dz \rho(z) = 1, \quad \rho(z) \geq 0 .$$

that satisfies crossing

(dual problem)

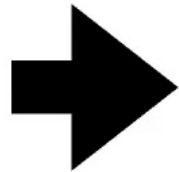
$$\int_{-\infty}^0 dz \rho(z) \left(\beta \log\left(1 - \frac{1}{\beta z}\right) + \log\left(1 - \frac{\beta}{z}\right) \right) = 0, \quad \beta > 0 .$$

Veneziano behavior is extremal

Given an **upper** bound on scattering for **complex** angles, one can use analyticity to derive a **lower** bound on scattering for **real** angles

$$\text{Regge} \leq (\text{Fixed real angle}) \times (\text{Fixed complex angle})$$

[Hadamard three-circle theorem]



$$(\text{Fixed real angle}) \geq \frac{\text{Regge}}{(\text{Fixed complex angle})}$$

[Cerulus, Martin '64]

[Tourkine, AZ '23]

[Buoninfante, Tokuda, Yamaguchi '23]

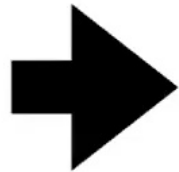
Stringy Cerulus-Martin bound:

$$\max_{|z| \leq z_0} |T(s, z)| \gtrsim e^{-s \log \frac{1 + \sqrt{z_0^2 - 1}}{z_0}}$$

Given an **upper** bound on scattering for **complex** angles, one can use analyticity to derive a **lower** bound on scattering for **real** angles

$$\text{Regge} \leq (\text{Fixed real angle}) \times (\text{Fixed complex angle})$$

[Hadamard three-circle theorem]



$$(\text{Fixed real angle}) \geq \frac{\text{Regge}}{(\text{Fixed complex angle})}$$

[Cerulus, Martin '64]

[Tourkine, AZ '23]

[Buoninfante, Tokuda, Yamaguchi '23]

Stringy Cerulus-Martin bound:

$$\max_{|z| \leq z_0} |T(s, z)| \gtrsim e^{-s} \log \frac{1 + \sqrt{z_0^2 - 1}}{z_0}$$

0. Regge sum rules

I. Open strings

II. High energy, fixed angle scattering

III. Closed strings

Consider the MHV amplitude for gravitons

$$T_{--++}(s, t, u) = (\langle 12 \rangle [34])^4 f(s|t, u) \stackrel{CMF}{=} s^4 f(s|t, u)$$

$$f(s|t, u) = -\frac{\Gamma(-s)\Gamma(-t)\Gamma(u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} + \sum_{c_s, c_t, d_s, d_t} \alpha_{c_s, c_t, d_s, d_t} \frac{\Gamma(c_s - s)\Gamma(c_t - t)\Gamma(c_t - u)}{\Gamma(d_s + s)\Gamma(d_t + t)\Gamma(d_t + u)}$$

The Regge limit is

$$f(s|t, u) \sim s^{-2+2t} \quad s \rightarrow \infty$$

$$f(s|t, u) \sim t^{2+2s} \quad t \rightarrow \infty$$

Unitarity has to be checked!

Observation: A truncated ansatz

$$f_{N_{\max}}(s|t, u) = -\frac{\Gamma(-s)\Gamma(-t)\Gamma(u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} + \sum_{c_s, c_t=0}^{N_{\max}} \sum_{d_s, d_t=1}^{2N_{\max}} \alpha_{c_s, c_t, d_s, d_t} \frac{\Gamma(c_s - s)\Gamma(c_t - t)\Gamma(c_t - u)}{\Gamma(d_s + s)\Gamma(d_t + t)\Gamma(d_t + u)}$$

is consistent with unitarity.

It automatically satisfies crossing, FESR, and RSR.

The number of independent parameters in the ansatz is

$$3N_{\max}^2 + N_{\max} - 2$$

We can therefore explore bounds on Wilson coefficients numerically (as a function of N_{\max}).

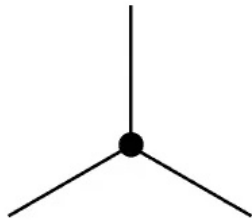
[Paulos, Penedones, Toledo, van Rees, Vieira '17]

$$N_{\max} = 20 : 1218 \text{ terms}$$

The low-energy expansion of the amplitude takes the form

$$f(s|t, u) = \frac{1}{stu} + |\beta_{R^3}|^2 \frac{tu}{s} - |\beta_\phi|^2 \frac{1}{s} + \sum_{k \geq j \geq 0} a_{k,j} s^{k-j} t^j$$

First, we look at the correction to the graviton three-point function



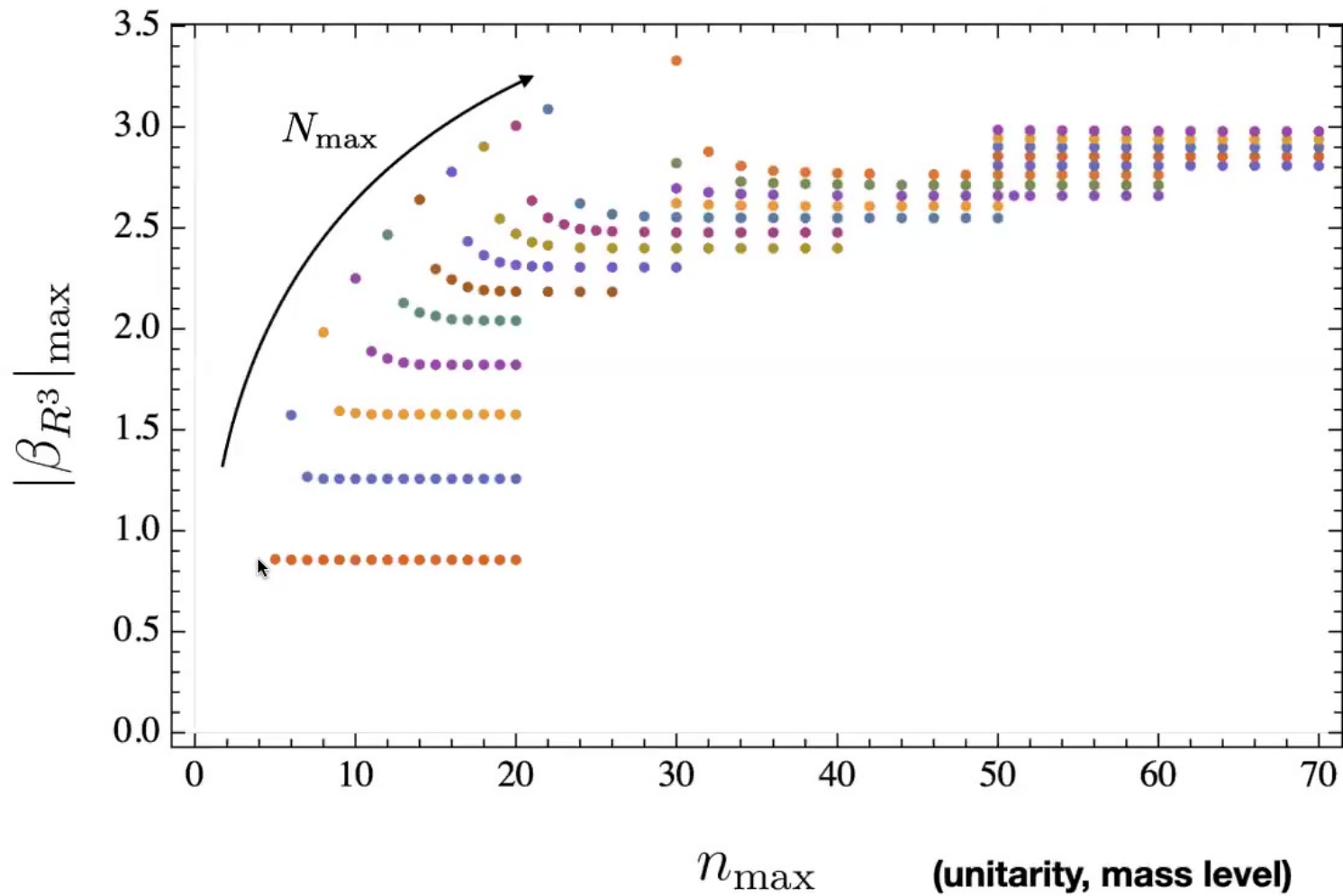
$$|\beta_{R^3}| \lesssim \frac{\log m_{\text{HS}} L_{\text{IR}}}{m_{\text{HS}}^4}$$

[Camanho, Edelstein, Maldacena, AZ '14]

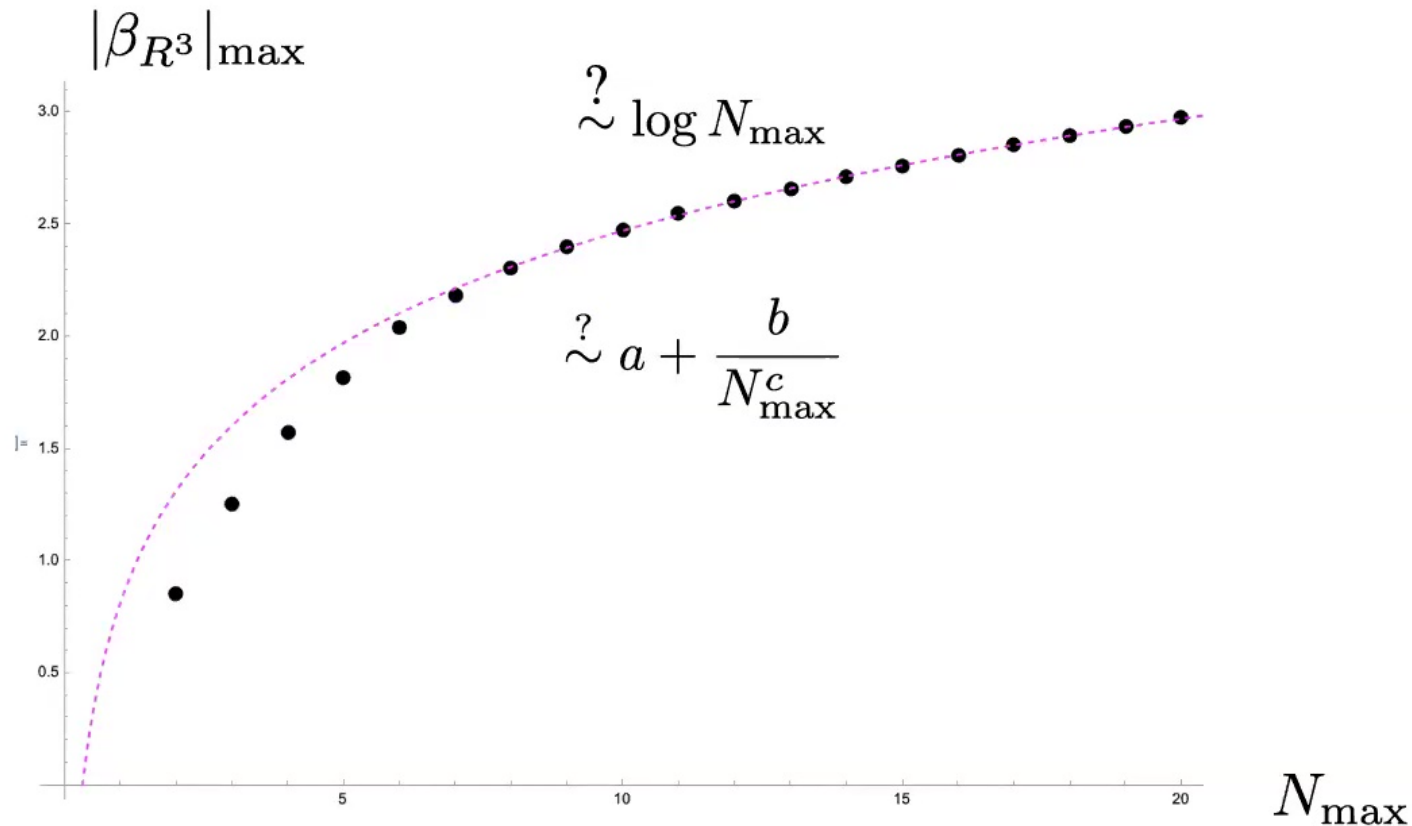
$$|\beta_{R^3}| \leq \# \frac{\log m_{\text{HS}} R_{\text{AdS}}}{m_{\text{HS}}^4}$$

[Caron-Huot, Mazac, Rastelli, Simmons-Duffin '21]

Bound on three-point coupling (causality+RSR)



Bound on three-point coupling (causality+RSR)

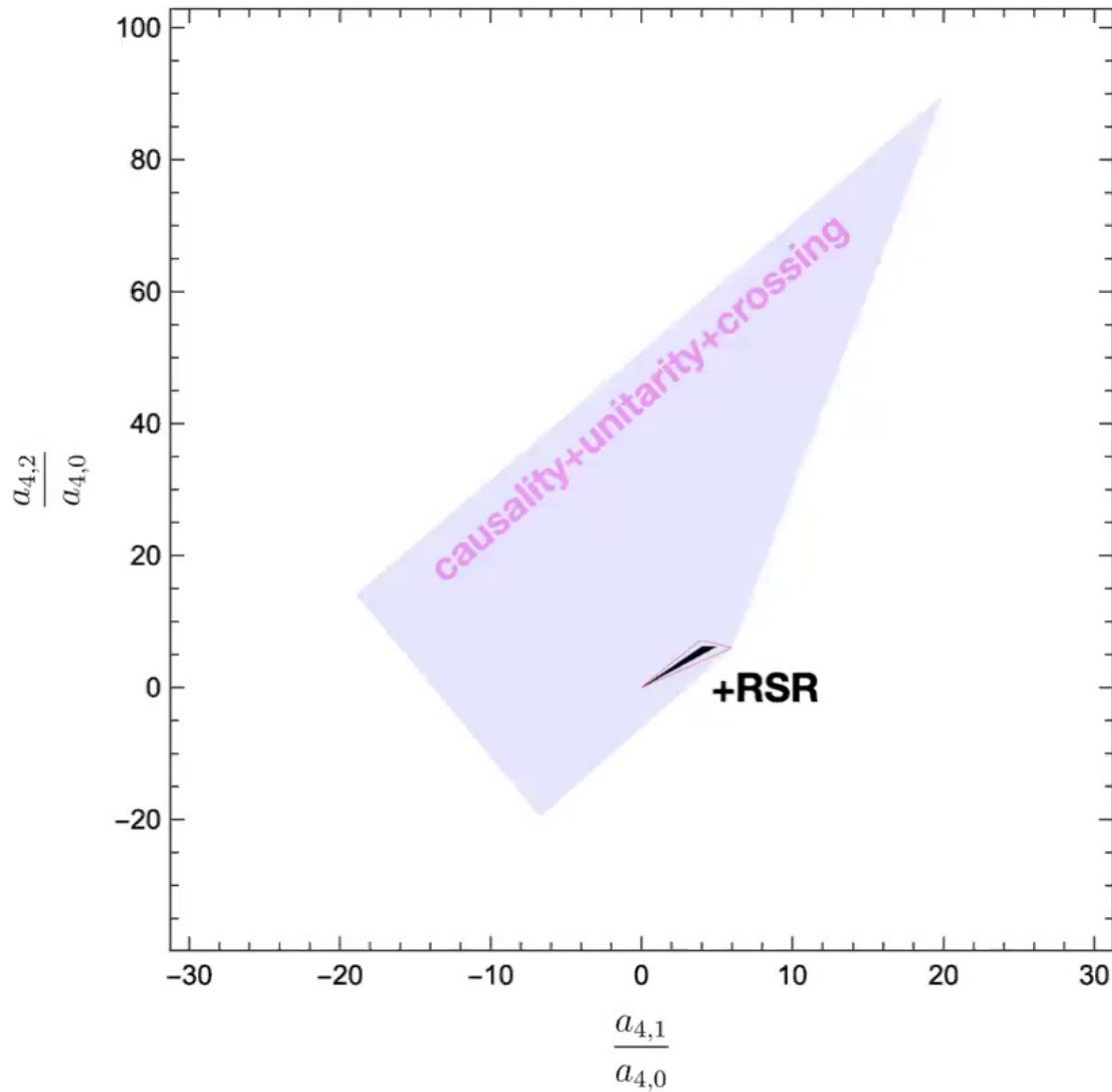


We loose the bound: $|\beta_{R^3}| \lesssim \log N_{\max}$

The extremal solution is non-unitary in $d > 4$.

[Caron-Huot, Li, Parra-Martinez, Simmons-Duffin '22]

Bound on Wilson coefficients



[Arkani-Hamed, Huang, Huang '20]
[Bern, Kosmopolous, AZ '21]

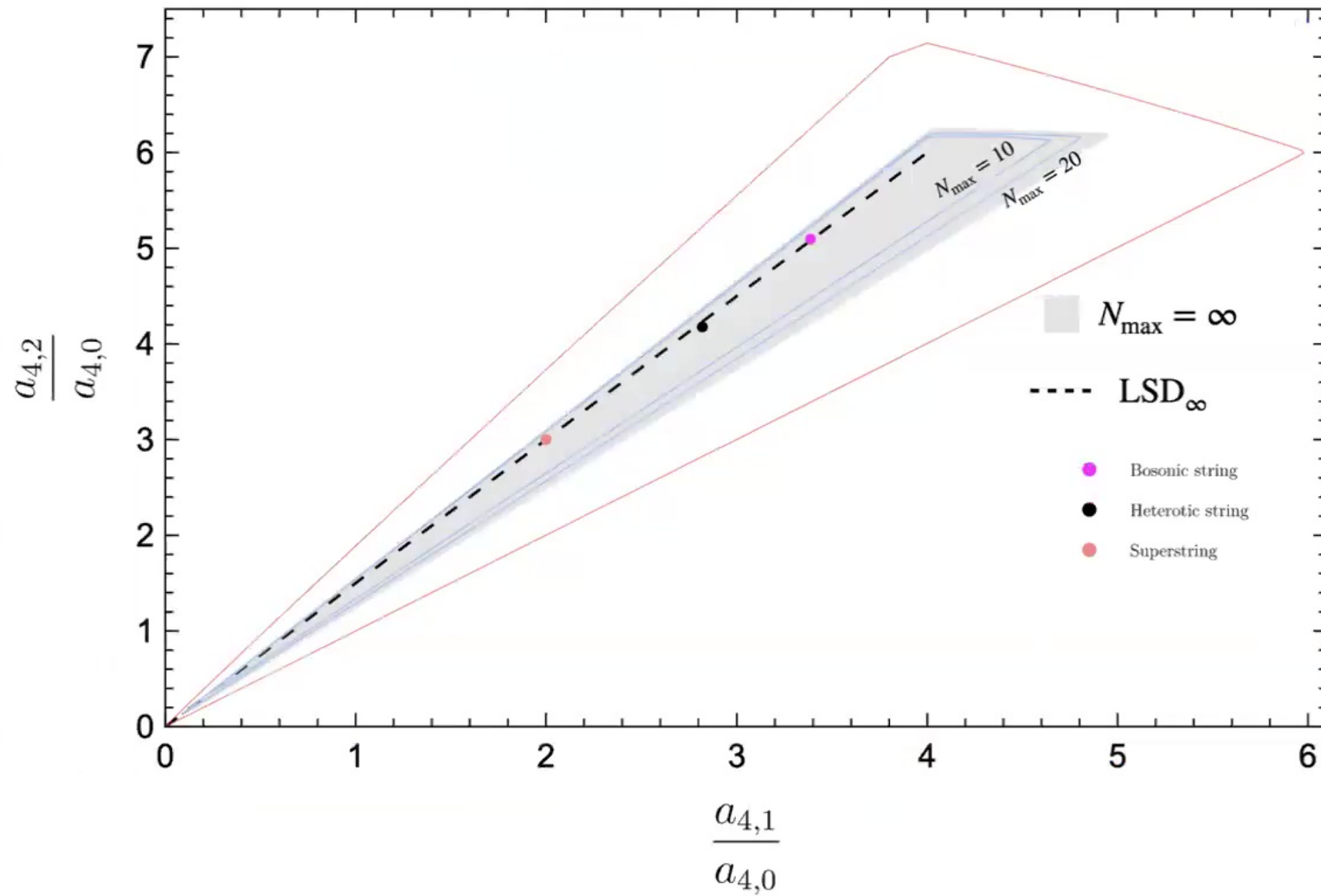
**+extra parametric
inhomogeneous constraint
("the pancake effect")**

$$\frac{a_{4,0}}{a_{0,0}} \geq 10^{-2}$$

[Caron-Huot, Li, Parra-Martinez,
Simmons-Duffin '22]

[Chiang, Y-t Huang, Li, Rodina, Wang '22]

Bound on Wilson coefficients



How does the UV softness constrain the IR?

Take away:

1. The Veneziano amplitude admits unitary deformations with linear Regge trajectory (**soft in the UV**).
2. The high energy, fixed angle behavior of such amplitudes is not unique (**both for real and complex angles**).
3. We probed allowed regions on Wilson coefficients for amplitudes that satisfy RSR and observed that they got smaller.
4. Our results are incomplete (equidistant spectrum+tree-level). A playground for future studies.

Future directions:

- More general $j(t)$ (discretized AdS) [Veneziano, Yankielowicz, Onofri '17]
- Apply to scattering of gauge bosons [Bachu, Hillman '22]
[Arkani-Hamed, Strings 2016]
- Relax the assumption on the equidistant spectrum (is it equivalent to FESR?)
- Multi-particle unitarity (or 1-loop) [Gross '69]
[Mizera, Eberhardt '23]
- Physical origin of the deformations
- Higher d
- Similar sum rules hold in QG (is there a good ansatz in that case?) [Guerrieri, Murali, Penedones, Vieira '22]
- FESR+RSR+FA $\overset{?}{\rightarrow}$ LSD

thank you!