Title: Research Talk 14 - Irreversibility, QNEC, and defects

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Abstract: In this talk we will analyze renormalization group flows on d-dimensional planar defects, embedded in a D-dimensional conformal field theory. This general setup includes the case of quantum field theory with no defects ($D=d$), as well as defects of different dimensionality that are of interest in high energy and condensed matter physics. Using methods from quantum information theory, we will establish the irreversibility of renormalization group flows for defect dimensions $d \le 4$, and for all D. The main ingredients in the proof are strong subadditivity of the entanglement entropy, the Markov property of the conformal vacuum, and the quantum null energy condition.


Irreversibility, QNEC,
and defects
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Irreversibility of the renormalization group is a key property of nonperturbative QFT.
Goal: find renormalization group charges that partially characterize conformal field theories, and that decrease under the RG.

- Two directions where irreversibility has been developed:

1) Irreversibility theorems in unitary relativistic QFT have been established in d=2, 3 and 4 space-time dimensions (C, F, A)
2) Irreversibility for defect RG flows, embedded in CFTs. Started with the g-theorem.

- Framework for generalizing this:

- D-dimensional CFT coupled to d-dimensional planar static defect
- relevant interactions on defect trigger RG flow between UV and IR fixed points. Bulk remains conformal
- QFTs without defect: special case D=d

Overview of irreversibility inequalities

So far, several proofs for different (D, d). Those based on properties of energy-momentum tensor are:

$d \diagdown D$	2	3	4	5	...
1	reflection positivity for stress tensor	reflection positivity for stress tensor	reflection positivity for stress tensor	reflection positivity for stress tensor	reflection positivity for stress tensor
2	reflection positivity for stress tensor	reflection positivity of dilaton	reflection positivity of dilaton	reflection positivity of dilaton	reflection positivity of dilaton
3		No proof	No proof	No proof	No proof
4			unitarity dilaton scattering	unitarity dilaton scattering	unitarity dilaton scattering

D=d=2: [Zamolodchikov]'s original C-thm; D=d=4: [Komargodski, Schwimmer] (dilaton)
Later on: d=2, any D [Jensen, OBannon], d=4, any D: [Wang]. Dilaton methods
d=1, D=2: [Friedan, Konechny].
And recently: d=1, any D: [Cuomo, Komargodski, Raviv-Moshe]. Generalized to d=2, any D by [Sachar, Sinha, Smolkin].

In parallel, irreversibility theorems have also been obtained using methods from quantum information theory:

$d \setminus D$	2	3	4	5	$\cdots$
1	positivity of relative entropy	positivity of relative entropy	positivity of relative entropy	positivity of relative entropy	positivity of relative entropy
2	SSA or positivity of relative entropy	SSA + QNEC or positivity of relative entropy	SSA + QNEC or positivity of relative entropy	SSA + QNEC or positivity of relative entropy	SSA + QNEC or positivity of relative entropy
3		SSA	SSA + QNEC	SSA + QNEC	SSA + QNEC
4			SSA	SSA + QNEC	SSA + QNEC

D=d=2 entropic C-thm by [Casini, Huerta];
extended by [Casini, Huerta] to D=d=3 (not available via correlators/dilaton)
D=d=4 by [Casini, Teste, GT]. And unifies d=2,3,4 (Markov prop)
d=1, any D [Casini, Salazar, GT]; d=2, D=3 [Casini, Salazar, GT]

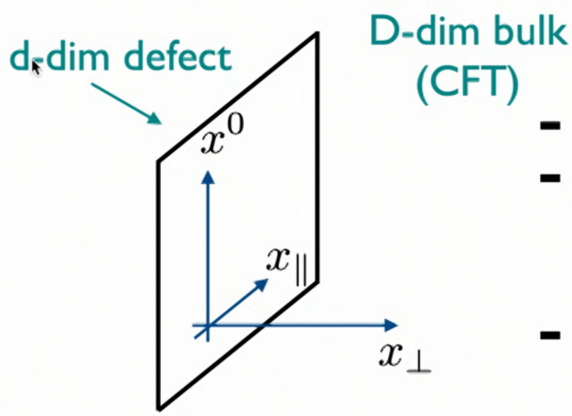
… This covers almost 40 years of developments! But a simple and general understanding is still lacking.

In this talk: we prove an inequality for second derivatives of the Relative Entropy. This establishes the remaining irreversibility thms, and unifies all known theorems for (D, d). [Casini, Salazar, GT, 2023]

A. Key ingredients

QFT setup

D-dimensional CFT with a d-dimensional planar defect



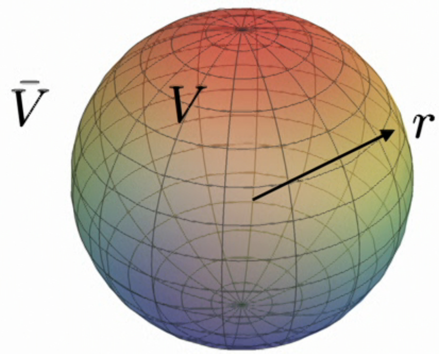
- the defect is conformal in the UV
- turn on relevant deformations on the defect

$$S = S_{UV,CFT} + \int d^d x \, g \mathcal{O}$$

- triggers an RG flow, which ends on a different IR conformal defect
- the bulk does not flow

Particular case: d=D, no bulk. Gives QFT RG flow without defects

Entanglement entropy



density matrix $\rho_r = \mathrm{tr}_{\bar{V}} |0\rangle\langle 0|$

von Neumann entropy $S(r) = -\mathrm{tr}_V (\rho_r \log \rho_r)$

Can probe RG flow by

$$mr \ll 1 \; (\mathrm{UV}) \;\Rightarrow\; mr \gg 1 \; (\mathrm{IR})$$

- Structure near a fixed point, first without defects (D=d):

$$S(r) = S_{local} + S_{non-local}$$

$$S_{local}(r) = \mu_{d-2} r^{d-2} + \mu_{d-4} r^{d-4} + \ldots$$ UV divergent

$$S_{non-local}(r) = \begin{cases} (-1)^{\frac{d}{2}-1} 4A \log \frac{r}{\epsilon} & d \text{ even} \\ (-1)^{\frac{d-1}{2}} F & d \text{ odd} \end{cases}$$ universal

RG irreversibility: decrease of A or F; proved for $d = 2, 3, 4$

The universal terms can also be isolated from the partition function on the sphere (free energy in de Sitter), or from the Weyl anomaly if d even.

- For d-dim conformal defect in a D-dim CFT, there are additional terms:

$$S(r) = \mu_{D-2} r^{D-2} + \mu_{D-4} r^{D-4} + \ldots + \tilde{\mu}_{d-2} r^{d-2} + \tilde{\mu}_{d-4} r^{d-4} + \ldots$$
$$+ \begin{cases} (-1)^{\frac{D-2}{2}} 4A \log \frac{r}{\epsilon} & D \text{ even} \\ (-1)^{\frac{D-1}{2}} F & D \text{ odd} \end{cases} + \begin{cases} (-1)^{\frac{d-2}{2}} 4\tilde{A} \log \frac{r}{\epsilon} & d \text{ even} \\ (-1)^{\frac{d-1}{2}} \tilde{F} & d \text{ odd} \end{cases}$$

For defect RG flows, quantities with tildes flow. One can try to prove irreversibility theorems for the universal $\tilde{F}, \tilde{A}$, but in general this does not work (exception: D=d+1).

[Jensen, O'Bannon]
[Kobayashi, Nishioka, Sato, Watanabe]

One reason: the defect contributes nonzero energy. The universal term in the EE no longer coincides with the one in the free energy. We should look at the quantum-information analog of free energy.

Relative entropy

For two density matrices σ and ρ, the relative entropy is

$$S_{rel}(\rho|\sigma) = \mathrm{tr}(\rho \log \rho - \rho \log \sigma)$$

Introducing the modular Hamiltonian $H = -\log \sigma$

$$S_{rel}(\rho|\sigma) = \mathrm{tr}(\rho \log \rho - \rho \log \sigma + \sigma \log \sigma - \sigma \log \sigma)$$

$$= \langle H \rangle_\rho - \langle H \rangle_\sigma - (S_\rho - S_\sigma)$$

$$= \Delta \langle H \rangle - \Delta S$$ ⟵ difference of "free energies"

For irreversibility in QFT, we will compare two density matrices:

$\sigma =$ vacuum density matrix for UV fixed point

$\rho =$ vacuum density matrix for QFT w/relevant deformations

The modular Hamiltonian for a CFT on a sphere is

$$H_\sigma = \int_\Sigma d^{D-1}x \, \eta^\mu \xi^\nu \, T_{\mu\nu}$$

-conf. transf. of Rindler Hamiltonian
- valid also with conformal defects

η^μ : unit normal to Cauchy surface Σ

$$\xi^\nu = \frac{\pi}{R}(R^2 - (x^0)^2 - \vec{x}^2, -2x^0 x^i)$$ Killing vector, R: radius of sphere

Note: $\langle H \rangle_\rho$ depends on choice of Cauchy surface. Reason: states evolve with different action.

- for space-like Σ, $\langle H \rangle_\rho \sim R^D$ dominates relative entropy
- for null Σ, $\langle H \rangle_\rho \sim R^{D-2}$ comparable to EE. We choose this limit.

It contributes a universal term, proportional to energy of defect. Then

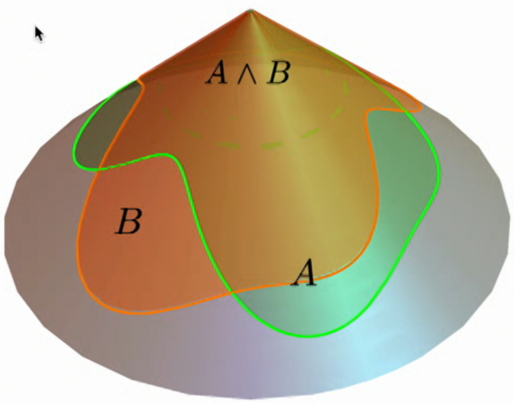
$$-\lim_{R\to\infty} S_{\rm rel}(R) = \Delta\mu'_{d-2}\, R^{d-2} + \Delta\mu'_{d-4}\, R^{d-4} + \ldots + \begin{cases} (-)^{\frac{d-2}{2}} 4\,\Delta A' \, \log(R/\epsilon) & d \text{ even} \\ (-)^{\frac{d-1}{2}} \Delta F' & d \text{ odd} \end{cases}$$

Strong subadditivity and Markov property

Key property of unitary quantum mechanics: strong subadditivity of the EE [Lieb, Ruskai, 1973]

$$S(A) + S(B) \geq S(A \cup B) + S(A \cap B)$$

For a CFT and for regions with boundary in null cone, Markov property: [Casini, Testé, GT, 2017]



$$S(A \cup B) + S(A \cap B) = S(A) + S(B)$$

$$\Leftrightarrow \quad \log \rho_{A \cup B} = \log \rho_A + \log \rho_B - \log \rho_{A \cap B}$$

✓This is called a quantum Markov state
✓mod Hamiltonian local on null surfaces
✓Tracing out a subsystem becomes a reversible process

Combining SSA w/Markov, we obtain strong superadditivity

$$S_{\rm rel}(A) + S_{\rm rel}(B) \leq S_{\rm rel}(A \cup B) + S_{\rm rel}(A \cap B)$$

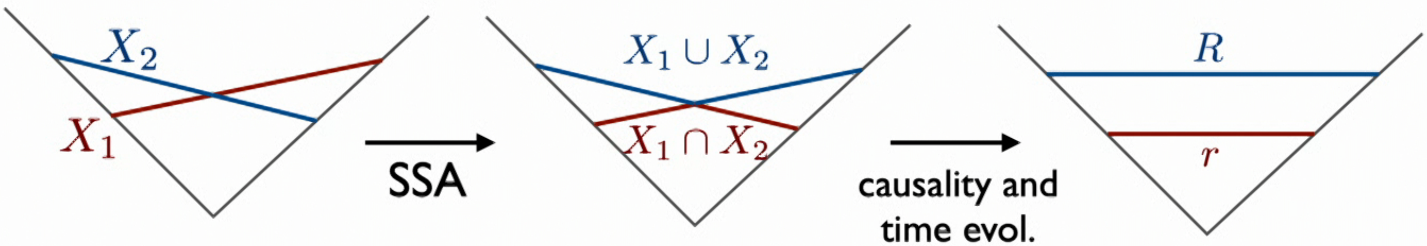
B. Proof of irreversibility inequality

We will explain the proof for a d=2 defect, in D-dim CFT.
General case: [Casini, Salazar, GT, 2023] and talk at IFQ next week

The entropic C-theorem

[Casini, Huerta, '04, '12]

Warm-up without defect, ie D=d=2



$$\Delta S(X_1) + \Delta S(X_2) \geq \Delta S(X_1 \cup X_2) + \Delta S(X_1 \cap X_2) = \Delta S(R) + \Delta S(r)$$

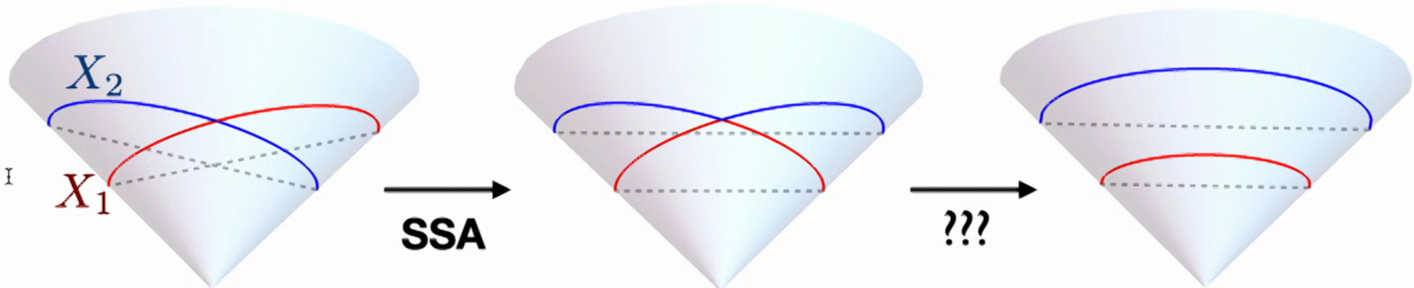
$2\Delta S(\sqrt{rR}) \geq \Delta S(R) + \Delta S(r)$. When $r \to R \Rightarrow (R\,\Delta S'(R))' \leq 0$

implies the decrease of the central charge $C_{IR} < C_{UV}$

d=2 defect

Let's try the same for a d=2 defect, in a D-dim CFT bulk.

The problem is that now the spheres extend into the bulk, and their causal union and intersection no longer give causal diamonds of spheres.



➡New ingredient: use QNEC in the bulk (conformal), to bound the entropy of union and intersection by entropies of diamonds (spheres)

As we will see, this requires the relative entropy:

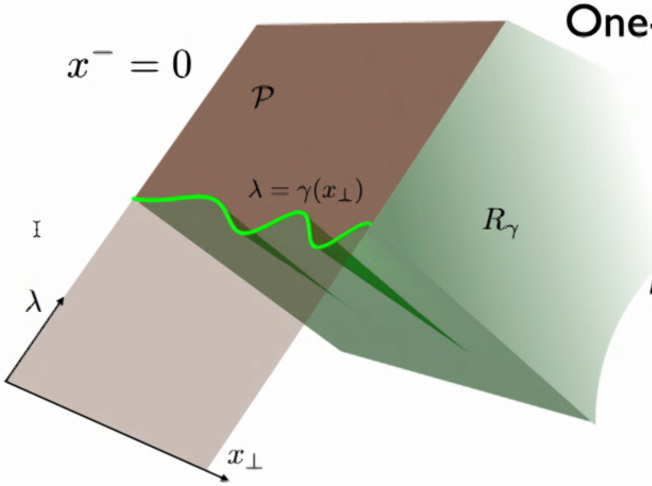
$$S_{\rm rel}(X_1) + S_{\rm rel}(X_2) \leq S_{\rm rel}(X_1 \cup X_2) + S_{\rm rel}(X_1 \cap X_2) \leq S_{\rm rel}(R) + S_{\rm rel}(r)$$

Then we would get defect irreversibility, $(RS'_{\rm rel}(R))' \geq 0$

QNEC

[Bousso, Fisher, Leichenauer, Wall]
[Balakrishnan, Faulkner, Khandker, Wang]
[Ceyhan, Faulkner]

Originated from quantum focusing conjecture, but is valid in general QFT. Simplest setup: deformations on null plane



One-parameter deformation of null boundary:

$$x^+ = \gamma_s(x_\perp) = \gamma(x_\perp) + a(x_\perp)s$$
$$(a(x_\perp) \geq 0)$$

Relative entropy between σ (vacuum) and ρ (excited state) is convex under null defs

$$\frac{d^2 S_{\rm rel}(\gamma_s)}{ds^2} \geq 0$$

QNEC is also valid on light-cone if we have a CFT. It is valid in the CFT bulk of the theory with defect.

$$\Rightarrow [S_{\rm rel}(R) - S_{\rm rel}(X_1 \cup X_2)] - [S_{\rm rel}(X_1 \cap X_2) - S_{\rm rel}(r)] \geq 0$$

and this establishes the irrev. inequality for d=2 and all D.

C. Conclusions

The result for general (d,D) is $RS''_{\rm rel}(R) - (d-3)S'_{\rm rel}(R) \geq 0$

[Casini, Salazar, GT, 2023]

- The result is independent of D; depends only on defect dim. d
- The theory on defect is nonlocal (it interacts w/bulk). It's remarkable that we get the same inequality as with no bulk (d=D)
- Relative entropy appears, required by QNEC.
- Implies irreversibility of defect RG flows for d=2,3,4 and all D.

Future directions:

1) Irreversibility for d>4? Requires new tools.
2) Theories with less symmetries? Interesting for condensed matter
3) Relation to non-entropic results?