Title: Research Talk 8 - The Virasoro Minimal String

Speakers: Lorenz Eberhardt

Collection: Strings 2023

Date: July 25, 2023 - 11:30 AM

URL: https://pirsa.org/23070034

Abstract: I will explain a new 2d gravity/matrix model duality. The bulk theory can be defined as a minimal string theory whose matter sector is given by timelike Liouville theory instead of a minimal model. The theory admits a dual description in terms of a double-scaled matrix integral whose leading density of states is given by the universal Cardy density of states in a 2d CFT of central charge c, thus motivating us to call the bulk theory the Virasoro minimal string. The duality can be derived by exploiting a certain relation to 3d gravity compactified on a circle. For large central charge, it reduces to the duality between JT-gravity and the corresponding double-scaled matrix integral. Based on work in collaboration with Scott Collier, Beatrix Mühlmann and Victor Rodriguez.

2d Gravity/Matrix integral duality

Gravity

(2,p) Minimal String

↓ Roughly $p \to \infty$

JT gravity

2d Gravity/Matrix integral duality

Gravity

(2,p) Minimal String

↓ Roughly $p \to \infty$

JT gravity

⟷

Matrix integral

$$\rho_0(E) = \sinh\left(\frac{p}{2}\operatorname{arccosh}(1+E)\right)$$

Brezin, Kazakov '90,
Gross, Migdal '90,
Douglas, Shenker '90, ...

↓

$$\rho_0(E) = \sinh(\sqrt{E})$$

Saad, Shenker, Stanford '19

2d Gravity/Matrix integral duality

Gravity

(2,p) Minimal String

↓ Roughly $p \to \infty$

JT gravity

↑ $b \to 0$

Virasoro Minimal String

⟷

Matrix integral

$$\rho_0(E) = \sinh\left(\frac{p}{2}\,\mathrm{arccosh}(1+E)\right)$$

↓

Brezin, Kazakov '90,
Gross, Migdal '90,
Douglas, Shenker '90, ...

$$\rho_0(E) = \sinh(\sqrt{E})$$

Saad, Shenker, Stanford '19

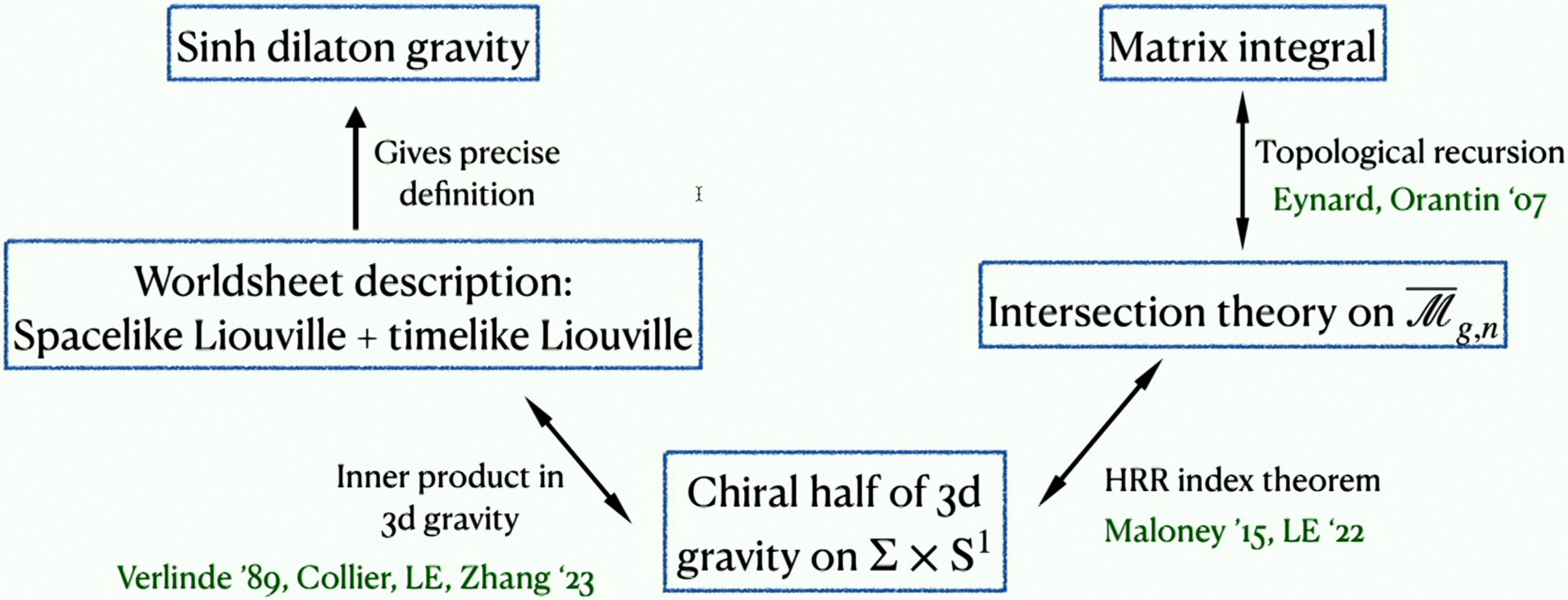
↑

$$\rho_0^{(b)}(E) = \frac{\sinh(b\sqrt{E})\sinh(b^{-1}\sqrt{E})}{\sqrt{E}}$$

Collier, LE, Mühlmann, Rodriguez

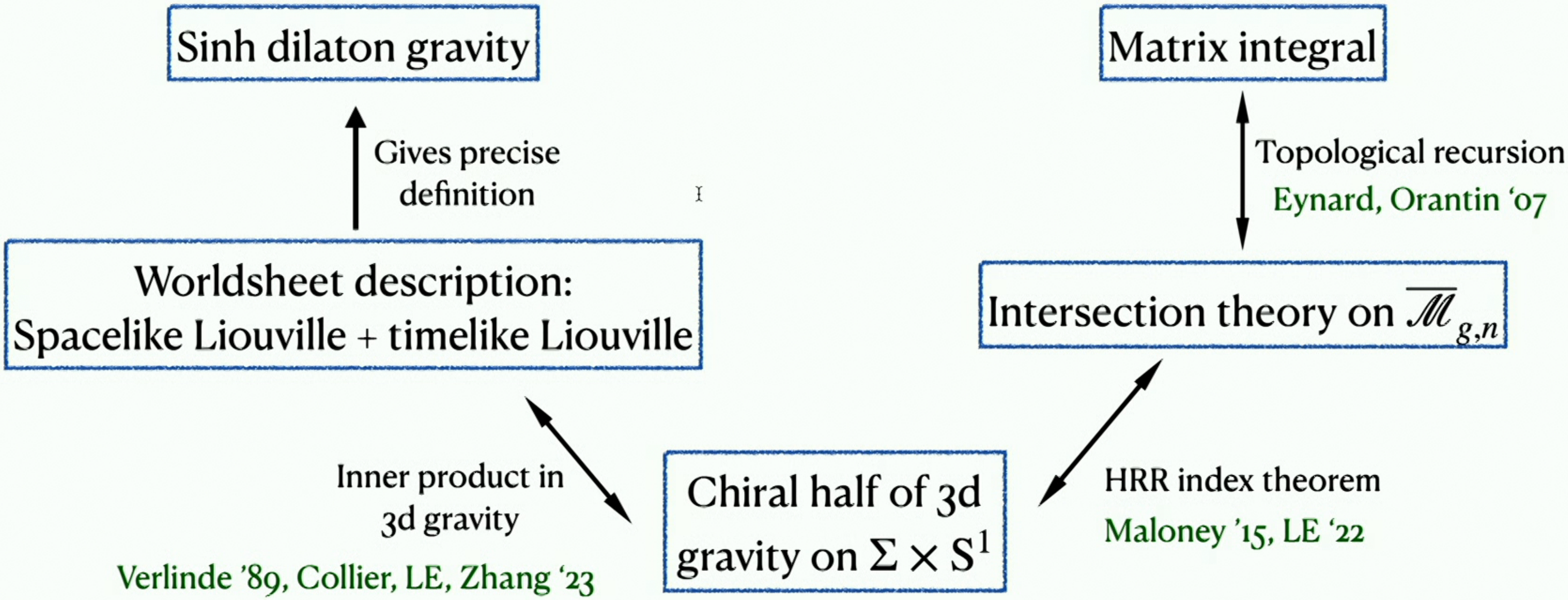
2

Cornerstones

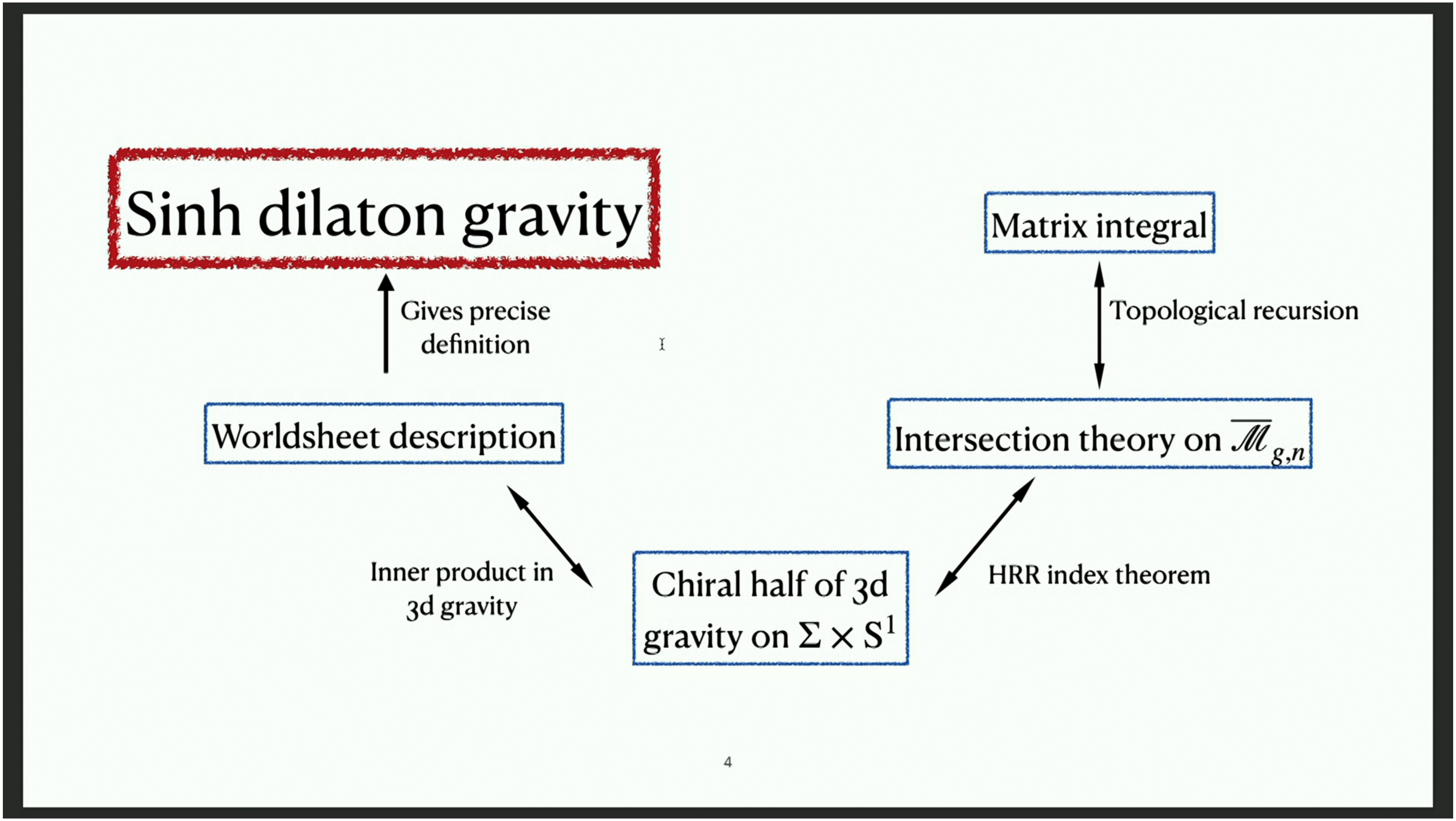


$\Rightarrow$ Gives a complete proof of the correspondence + many additional direct checks

Cornerstones



$\Rightarrow$ Gives a complete proof of the correspondence + many additional direct checks


Sinh dilaton gravity
Gives precise definition
Worldsheet description
Inner product in 3d gravity
Chiral half of 3d gravity on $\Sigma \times S^1$
HRR index theorem
Intersection theory on $\overline{\mathcal{M}}_{g,n}$
Topological recursion
Matrix integral
4

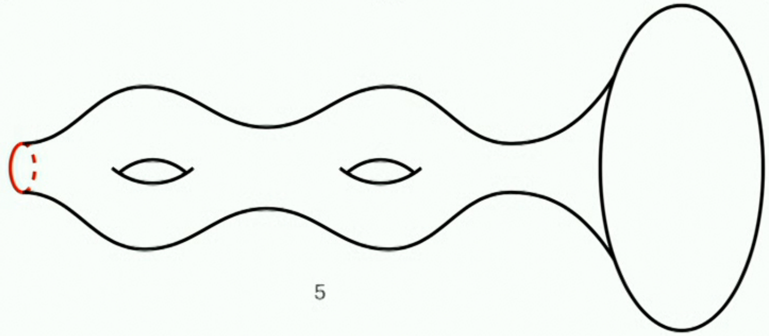
The Virasoro String as a 2d gravity theory

- The simplest way to define the bulk theory is as a dilaton gravity:

$$S = -\frac{1}{2}\int_{\Sigma} d^2x\sqrt{g}\,(\Phi\mathscr{R} + W(\Phi)) - \int_{\partial\Sigma}\sqrt{h}\Phi K + \text{Euler term}\;, \quad W(\Phi) = \frac{\sinh(2\pi b^2\Phi)}{\sin(\pi b^2)}$$

Mertens, Turiaci '20, Suzuki Takayanagi '21, Fan, Mertens '21

- $b \to 0$: JT-gravity
- Can consider the theory on surfaces with geodesic and/or asymptotic boundaries



5

Change of Variables

- Linear combination of the Weyl factor and the dilaton rewrites the action as a sum

$$\text{Spacelike Liouville theory } c = 1 + 6(b + b^{-1})^2$$
$$+$$
$$\text{Timelike Liouville theory } \hat{c} = 1 - 6(b - b^{-1})^2$$

Seiberg, Stanford,
Mertens, Turiaci '20

- This defines a critical string theory and can be taken as a more well-defined definition of the theory
- We will **NOT** take the second factor to be a minimal model!

Timelike Liouville theory

- Well-defined, non-unitary, modular invariant and crossing symmetric CFT with $\hat{c} \leq 1$ and spectrum

$$\hat{\Delta} = \frac{\hat{c}-1}{24} + \hat{P}^2 \geq \frac{\hat{c}-1}{24}$$

Zamolodchikov '05, Kostov, Petkova '05,
Harlow, Maltz, Witten '11,
Ribault, Santachiara '15, Ribault, Tsiares WIP

- Correlation functions

$$\left\langle \prod_{j=1}^{n} \hat{V}_{\hat{P}_j}(z_j) \right\rangle_g$$

are analytic in $\hat{P}_i$ and can trivially be analytically continued to imaginary values of $\hat{P}_i$ (no OPE contour deformation necessary)

Bautista, Dabholkar, Erbin '19

8

The worldsheet theory

- Parametrize

$$c = 1 + 6(b + b^{-1})^2 \,, \qquad \hat{c} = 1 - 6(b - b^{-1})^2$$

$$\Delta_i = \frac{c-1}{24} + P_i^2 \,, \qquad \hat{\Delta}_i = \frac{\hat{c}-1}{24} - P_i^2$$

so that $c + \hat{c} = 26$ and $\Delta_i + \hat{\Delta}_i = 1$.

- Can define the string theory correlators

$$\mathsf{V}^{(b)}_{g,n}(P_1, \ldots, P_n) = \int_{\mathcal{M}_{g,n}} \left\langle \prod_{j=1}^{n} V_{P_j}(z_j) \right\rangle_g \left\langle \prod_{j=1}^{n} \hat{V}_{\hat{P}_j = iP_j}(z_j) \right\rangle_g \times \text{ghosts}$$

``quantum volumes''

$\Rightarrow$ Absolutely convergent integral!

9

Numerical evaluation

- One can directly put this definition on a computer for low g and n.
- Result (error $< 10^{-3}\,\%$):

$$\mathsf{V}^{(b)}_{0,4}(P_1, P_2, P_3, P_4) = \frac{c-13}{24} + P_1^2 + P_2^2 + P_3^2 + P_4^2$$

$$\mathsf{V}^{(b)}_{1,1}(P_1) = \frac{1}{24}\left(\frac{c-13}{24} + P_1^2\right)$$

See Rodriguez '23 for the case $c = 25$

- Surprisingly simple!
- $b \to 0,\ P_i = \dfrac{\ell_i}{4\pi b}$: Weil-Petersson volumes of moduli space, up to normalization of the measure

Chiral half of 3d gravity on $\Sigma \times S^1$

- The key to understand the dual matrix model is the relation to 3d gravity (more precisely the Virasoro TQFT).
- Inner product on the Hilbert space of half of 3d gravity:

Verlinde '89 Collier, LE, Zhang '23

$$\left\langle \mathcal{F}^{(1)}_{g,n} \,\middle|\, \mathcal{F}^{(2)}_{g,n} \right\rangle = \int_{\mathcal{T}_{g,n}} \overline{\mathcal{F}}^{(1)}_{g,n} \mathcal{F}^{(2)}_{g,n} \left\langle \prod_{j=1}^{n} \hat{V}_{iP_j}(z_j) \right\rangle \times \text{ghosts} = \frac{\prod_j \delta(P_j^{(1)} - P_j^{(2)})}{\prod C_b(P_i^{(1)}, P_j^{(1)}, P_k^{(1)}) \prod \rho_0^{(b)}(P_j^{(1)})}$$

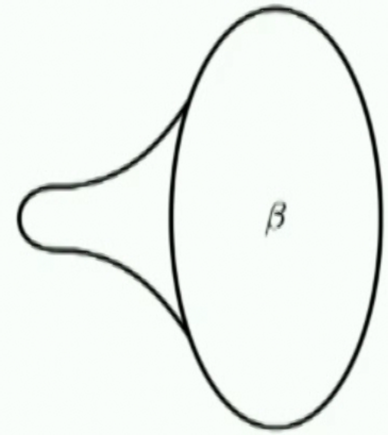
Teichmüller space DOZZ 3-point function Liouville 2-point function normalization

$$\Longrightarrow \mathsf{V}^{(b)}_{g,n}(P_1, \ldots, P_n) = Z_{\text{chiral 3d gravity}}(\Sigma_{g,n} \times S^1)$$

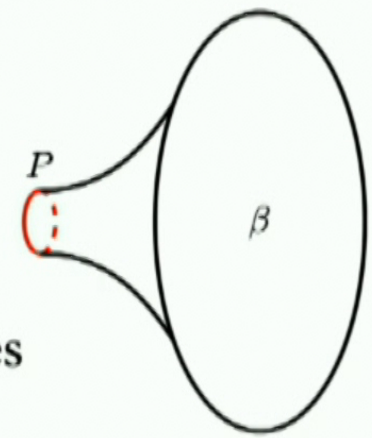
The disk and trumpet

- From 3d gravity:

$$\mathscr{Z}^{(b)}_{\text{disk}}(\beta) = \mathrm{e}^{\frac{\pi^2 c}{6\beta}} \prod_{n=2}^{\infty} \frac{1}{1-\mathrm{e}^{-\frac{4\pi^2 n}{\beta}}} = \frac{1}{\eta\left(\frac{\beta i}{2\pi}\right)} \underbrace{\sqrt{\frac{2\pi}{\beta}}\left(\mathrm{e}^{\frac{\pi^2(b+b^{-1})^2}{\beta}} - \mathrm{e}^{\frac{\pi^2(b-b^{-1})^2}{\beta}}\right)}_{Z^{(b)}_{\text{disk}}(\beta)}$$



$$\mathscr{Z}^{(b)}_{\text{trumpet}}(\beta, P) = \mathrm{e}^{-\frac{4\pi^2}{\beta}\left(P^2-\frac{1}{24}\right)} \prod_{n=1}^{\infty} \frac{1}{1-\mathrm{e}^{-\frac{4\pi^2 n}{\beta}}} = \frac{1}{\eta\left(\frac{\beta i}{2\pi}\right)} \underbrace{\sqrt{\frac{2\pi}{\beta}}\,\mathrm{e}^{-\frac{4\pi^2 P^2}{\beta}}}_{Z^{(b)}_{\text{trumpet}}(\beta,P)}$$



- The dual matrix model only captures the partition function of primaries

Quantization of $\mathscr{M}_{g,n}$

- The Hilbert space of chiral 3d gravity is the quantization of its phase space $\mathscr{M}_{g,n}$
- Partition functions on $\Sigma_{g,n} \times S^1$ compute the dimension of the 3d gravity Hilbert space and can also be computed from the Hirzebruch-Riemann-Roch index theorem:

$$\mathsf{V}^{(b)}_{g,n}(P_1, \ldots, P_n) = \int_{\overline{\mathscr{M}}_{g,n}} \mathrm{td}(\mathscr{M}_{g,n}) \; \mathrm{e}^{\frac{c}{48\pi^2}\omega_{\mathrm{WP}}\left(\ell_i^2 = \frac{96\pi^2}{c}(P_i^2 - \frac{1}{24})\right)}$$

Todd class

Curvature of the (projective) line bundle of conformal blocks

Friedan, Shenker '86

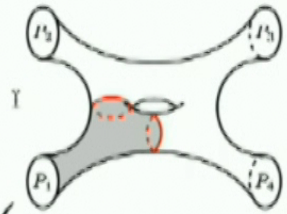
- Reduces the problem to intersection theory on $\overline{\mathscr{M}}_{g,n}$, which is mathematically well-developed

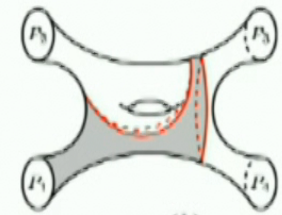
Mumford '83, Witten '90

Deformed Mirzakhani recursion

- Algebraic geometry and topological recursion gives a recursion for $\mathsf{V}^{(b)}_{g,n}(P_1,\ldots,P_n)$:

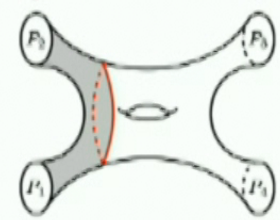
Mirzakhani '06, Eynard, Orantin '07, Eynard '11, LE '22





$$P_1\mathsf{V}^{(b)}_{g,n}(P_1,\mathbf{P}) = \int_0^\infty (2P\,\mathrm{d}P)\,(2P'\,\mathrm{d}P')\,H(P+P',P_1)\left(\mathsf{V}^{(b)}_{g-1,n+1}(P,P',\mathbf{P}) + \sum_{h=0}^{g}\sum_{I\sqcup J=\{2,\ldots,n\}}\mathsf{V}^{(b)}_{h,|I|+1}(P,\mathbf{P}_I)\mathsf{V}^{(b)}_{g-h,|J|+1}(P',\mathbf{P}_J)\right)$$

$$+\sum_{i=2}^{n}\int_0^\infty (2P\,\mathrm{d}P)\,\big(H(P,P_1+P_i)+H(P,P_1-P_i)\big)\mathsf{V}^{(b)}_{g,n-1}(P,\mathbf{P}\backslash P_i)$$



$$H(x,y)=\frac{y}{2}-\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}t\,\frac{\sin(4\pi t x)\sin(4\pi t y)}{\sinh(2\pi b t)\sinh(2\pi b^{-1}t)}$$

15

$V_{4,2}^{(b)}(P_1, P_2)$

16

Properties of the quantum volumes

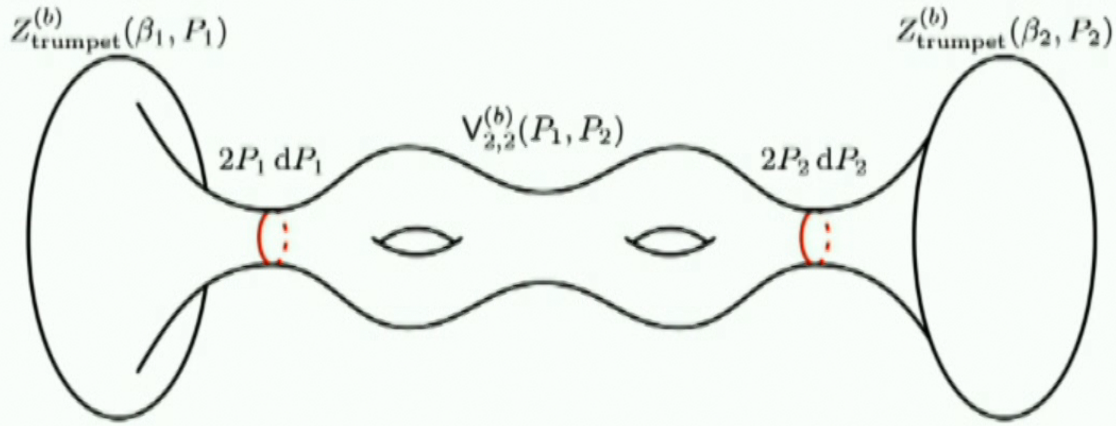
- This gives an effective algorithm to compute all $\mathsf{V}^{(b)}_{g,n}(P_1, \ldots, P_n)$.
- They are polynomial in $\mathbb{Q}[c, P_1^2, \ldots, P_n^2]$ of degree $3g - 3 + n$.
- They are invariant under switching the role of spacelike and timelike Liouville theory:

$$\mathsf{V}^{(ib)}_{g,n}(iP_1, \ldots, iP_n) = (-1)^{3g-3+n}\mathsf{V}^{(b)}_{g,n}(P_1, \ldots, P_n)$$

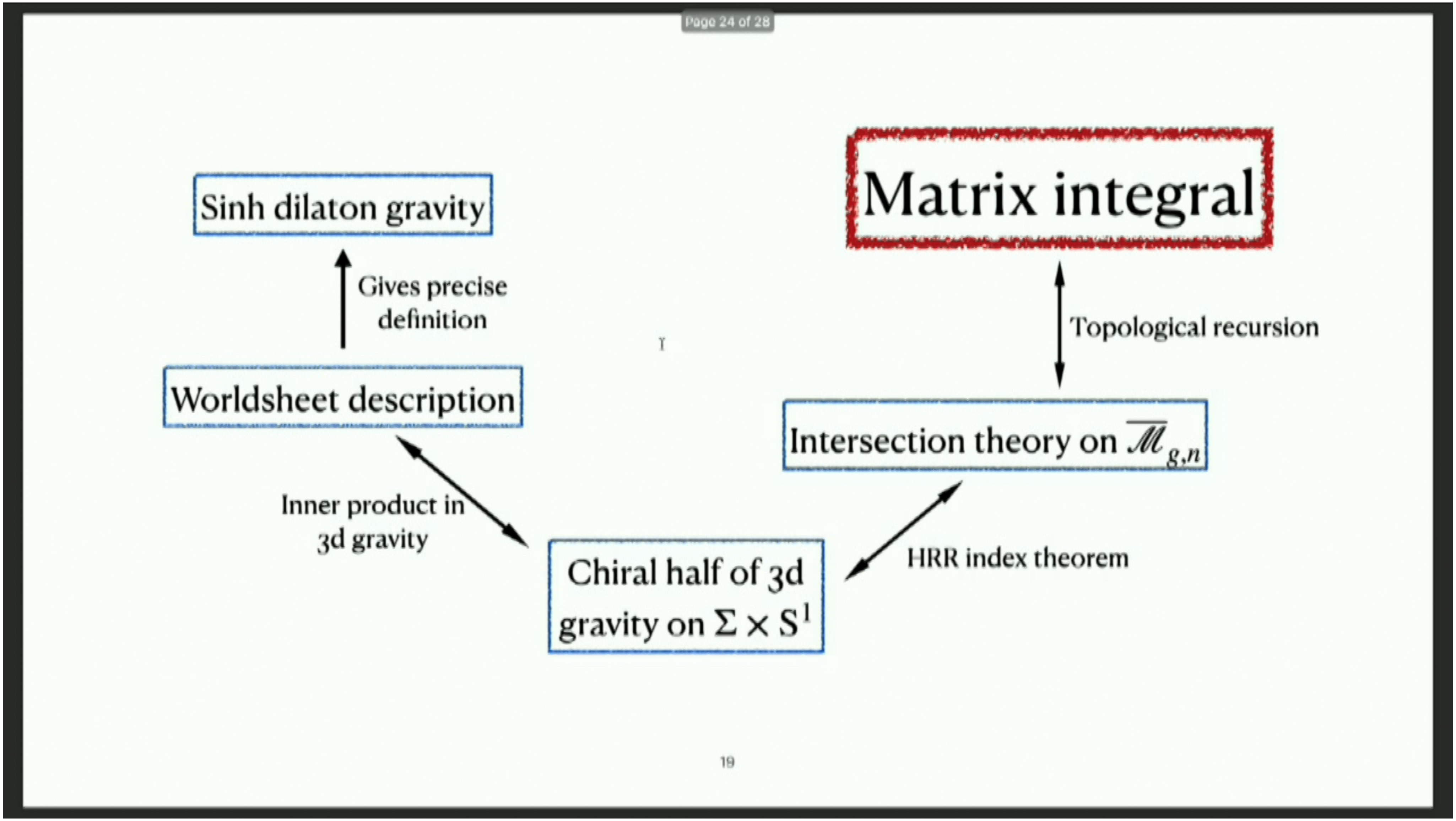
Adding asymptotic boundaries

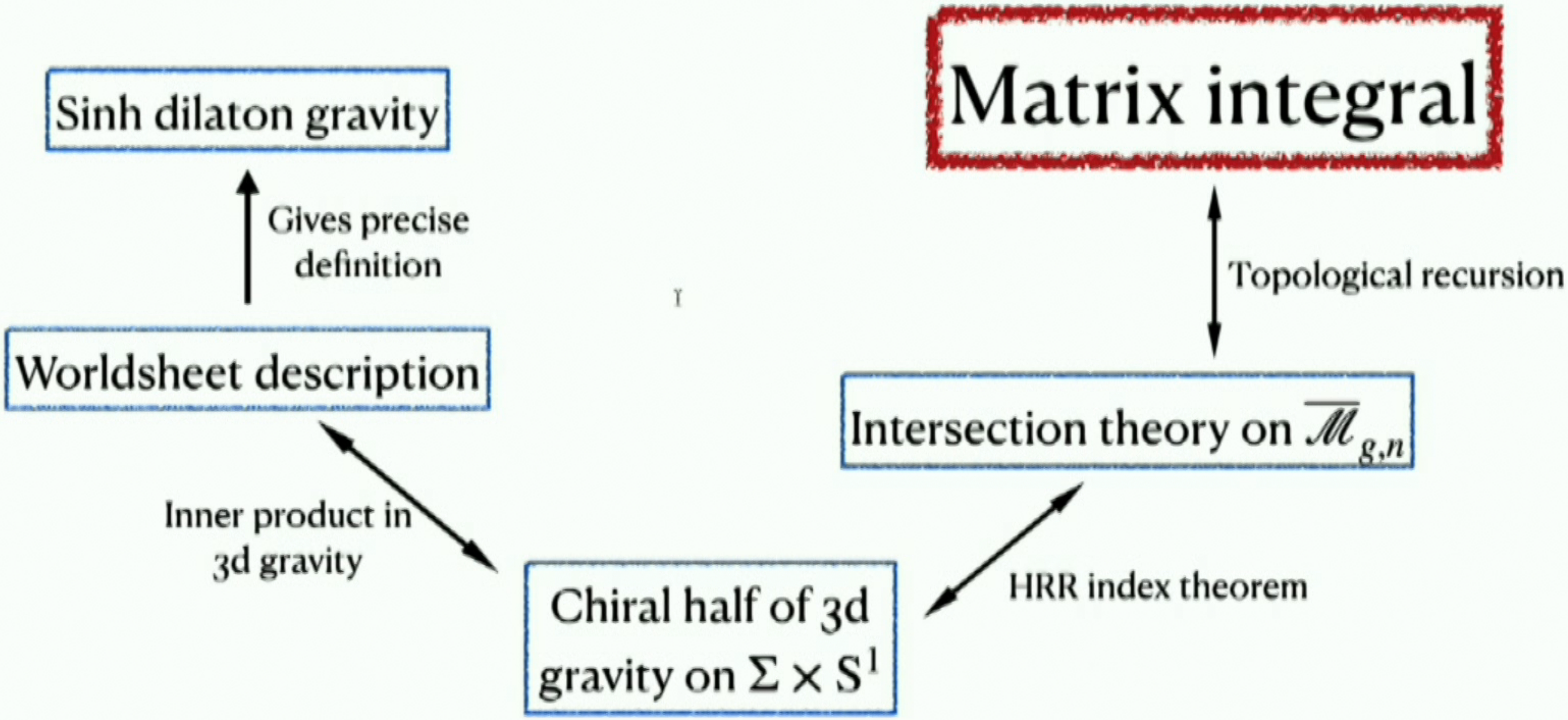
- Asymptotic boundaries can be added by gluing trumpets:

$$Z^{(b)}_{g,n}(\beta_1, \ldots, \beta_n) = \int_0^\infty \prod_{j=1}^{n} \left(2P_j\, \mathrm{d}P_j\, Z^{(b)}_{\text{trumpet}}(\beta_j, P_j)\right) \mathsf{V}^{(b)}_{g,n}(P_1, \ldots, P_n)$$



18


Page 24 of 28
Sinh dilaton gravity
Gives precise definition
Matrix integral
Topological recursion
Worldsheet description
Intersection theory on $\overline{\mathscr{M}}_{g,n}$
Inner product in 3d gravity
HRR index theorem
Chiral half of 3d gravity on $\Sigma \times S^1$
19



Sinh dilaton gravity
Gives precise definition
Worldsheet description
Inner product in 3d gravity
Chiral half of 3d gravity on $\Sigma \times S^1$
Matrix integral
Topological recursion
Intersection theory on $\overline{\mathcal{M}}_{g,n}$
HRR index theorem

19

The density of states

- The deformed Mirzakhani recursion is equivalent to the loop equations of a doubly-scaled matrix model
- Its leading density of states is the universal Cardy density in a CFT_2

$$\rho_0^{(b)}(E)\,\mathrm{d}E = 4\sqrt{2}\ \sinh(2\pi b P)\sinh(2\pi b^{-1} P)\,\mathrm{d}P\ , \qquad E = P^2$$

- This motivated us to call the corresponding bulk theory the Virasoro minimal string
- JT-limit $b \to 0$: $\rho_0^{(b)}(E) \to \sinh(2\pi\sqrt{E})$ after rescaling the energy

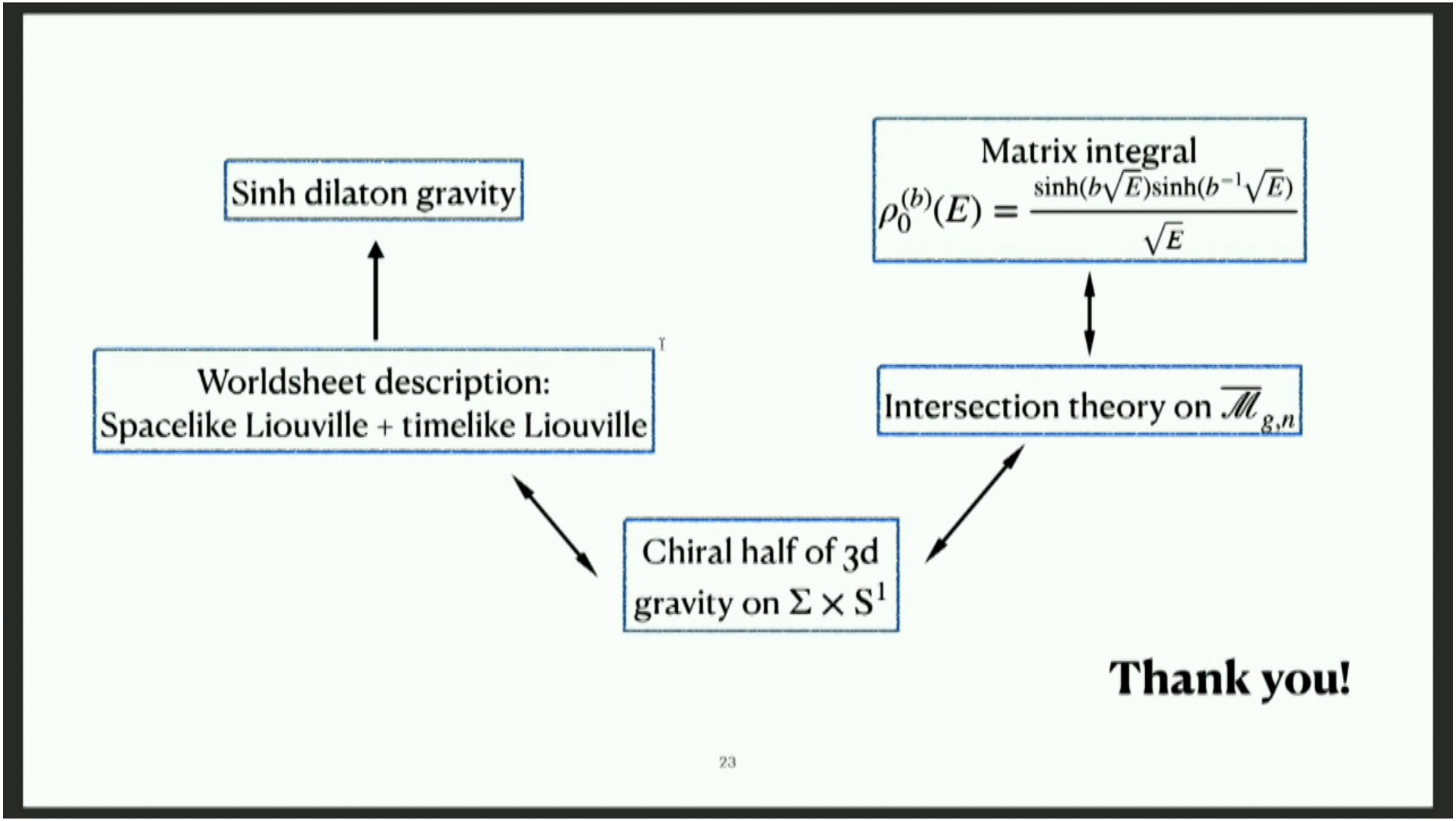
The full matrix integral

- Knowing the leading density of states $\rho_0^{(b)}(P)\,\mathrm{d}P$ determines the perturbative part of the double-scaled matrix integral completely.
- Our arguments **derive** this correspondence directly
- Much simpler than the $(2,p)$ minimal string:
  - Analytic in b and P_i
  - No contact terms

Non-perturbative effects

- The matrix integral also makes non-perturbative predictions
- Correspond to ZZ-instanton-like contributions from the worldsheet
- For $b = 1$, the theory is non-perturbatively stable
- For $b \neq 1$:
  - change the integration contour of the matrix integral or — Saad, Shenker, Stanford '19
  - Deform the model at low energies — Johnson '19
- Predicts large-g behavior of $\mathsf{V}^{(b)}_{g,0}$ for $b \neq 1$

$$\mathsf{V}^{(b)}_{g,0} \sim \frac{1}{2\sqrt{2}\pi^{\frac{5}{2}}} \left(4\sqrt{2}b\sin(\pi b^2)\right)^{2-2g}(1-b^4)^{2g-\frac{5}{2}}\,\Gamma\left(2g-\frac{5}{2}\right)$$


Sinh dilaton gravity
Matrix integral
$\rho_0^{(b)}(E) = \frac{\sinh(b\sqrt{E})\sinh(b^{-1}\sqrt{E})}{\sqrt{E}}$
Worldsheet description:
Spacelike Liouville + timelike Liouville
Intersection theory on $\overline{\mathcal{M}}_{g,n}$
Chiral half of 3d
gravity on $\Sigma \times S^1$
Thank you!
23