

Title: Research Talk 10 - A tensor model for approximate CFTs

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# A tensor model for approximate conformal field theories

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Strings 2023

Perimeter Institute

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Appearing soon with Alex Belin, Jan de Boer, Pranjali Nayak, Julian Sonner



- Introduction
- Approximate CFTs
- Random tensor model
- 3d gravity

# Ensembles for pseudo-random microdata

- Features of high energy microstates of chaotic quantum systems are pseudo-random.
- One can encode their statistical properties in an ensemble, such that typical elements preserve EFT observables.
- For quantum mechanics, this gives a random matrix model for the matrix elements of simple operators between the high energy microstates, for example  $H_{ij}, O_{ij}$ .
- For CFTs, there are also the three point functions of high dimension operators, so it is a random tensor model,  $C_{ijk}$ .

# Holographic matrix ensembles

- The [Saad Shenker Stanford](#) matrix model for Jackiw-Teitelboim gravity reinterprets a minimal string theory matrix model as the Wigner ensemble for the holographically dual Hamiltonian. The 't Hooft expansion of  $\langle \text{Tr}(e^{-\beta_1 H}) \text{Tr}(e^{-\beta_2 H}) \dots \rangle$  gives the sum over 2d topologies, as dense Feynman diagrams fill out the surfaces in a double scaled limit.
- The lack of euclidean factorization is puzzling from the perspective of exact 2d holography.
- How does it work in higher dimensional versions of this?

[Mertens Turiaci; Collier Maloney Maxfield Tsiaras; Cotler Jensen; Belin de Boer Liska; Belin de Boer Nayak Sonner; Anous Belin de Boer Liska; Chandra Collier Hartman Maloney ...]

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# CFTs

- The microdata of a CFT consists of the dilation operator acting on the Hilbert space graded by spin, and the structure constants,  $C$ . These give a matrix and a tensor.
- The operators best characterized by an ensemble are in the dense part of the spectrum. For simplicity, could consider a large  $c$  2d CFT, with no operators below the black hole threshold beside the identity Virasoro module.
- If there were light operators, these would define matrices  $C_{LHH}$
- At small  $c$ , the discussion applies to the high dimension region of the spectrum.

# OPE statistics from 3d gravity

- The Cardy formula gives the asymptotics of the spectral density

$$\rho \approx \exp \left[ 2\pi \left( \frac{c}{6} \left( h - \frac{c}{24} \right) \right) \right] \exp \left[ 2\pi \left( \frac{c}{6} \left( \bar{h} - \frac{c}{24} \right) \right) \right]$$

- The leading order variance of the structure constants is given by

$|C_{ijk}|^2 \approx C_0(h_i, h_j, h_k) C_0(\bar{h}_i, \bar{h}_j, \bar{h}_k)$  , where  $C_0$  is the Liouville three point function given by the DOZZ formula, normalized by the spectral density.

- The associated Gaussian ensemble reproduces certain handlebody and euclidean wormhole topologies in 3d gravity.

[Chandra Collier Hartman Maloney]

# Higher $d$ has more constraints

- An important point is that unlike QM, where any Hamiltonian is allowed, CFTs have many consistency constraints. 4 point crossing encodes locality, the vanishing of operator commutators outside the lightcone. In 2d, there is also torus 1 point function modular invariance.
- Therefore, we do not expect there to exist ensembles over exact CFT data, in contrast to QM Hamiltonians. No disordered exact duals are possible for higher dimensional gravity.
- We will define a notion of approximate CFT, which in an appropriate limit would localize to a true CFT.



## Which gravity EFTs have UV completions?

- In AdS, this turns into a problem of classifying CFTs: is there an exact CFT whose light operators behave like a given EFT's correlation functions in AdS?
- Around the AdS vacuum, perturbative solutions to the bootstrap exist for any EFT. There are some sharp “naturalness” bounds on the Wilson coefficients from dispersion relations.  
[Heemskerk Penedones Polchinski Sully; Caron-Huot Mazac Rastelli Simmons-Duffin]
- For black hole backgrounds, corresponding to thermal correlators, there are consistent ensembles for any EFT.  
[DLJ Kolchmeyer Mukhametzhanov Sonner]
- Major constraints must come from mutually locality of black hole microstate operators among themselves.

# Approximate CFTs

- Consists of a full collection of CFT data, subject to the conditions that crossing is obeyed up to a tolerance  $T$ , subject to bounds on
  - Total number of insertions
  - Dimension of external operators
  - Genus
  - Avoidance of extreme Lorentzian kinematics (Regge regime)
- These allow  $e^S$  moduli of deformation away from a true CFT.

# From EFT to microstate

- One can show that this implies approximate crossing for almost all operators, including heavy operators, above the previous cutoff. This is done by performing an appropriate inversion transform on a higher genus or higher point function to pick out contributions of (bands of) heavy operators in some channel.
- It follows from Moore-Seiberg that it is equivalent to having 4 point crossing and torus 1 point modular invariance approximately satisfied for almost all operators.
- The CFT data purely for light operators can be taken to be fixed, or approximately fixed.

# Variance of crossing

- Although the expectation value of the crossing equation vanishes in the gaussian ensemble, individual instantiations strongly violate it and are not approximate CFTs. This is captured by a large variance, which must be cancelled by higher moments.

From genus 2  $\left( \text{Diagram 1} - \text{Diagram 2} \right)^2$

$$G_{1122}(x)|_s = \sum_k |C_{12k}|^2 |\mathcal{F}_{1221}(O_k|x)|^2, \quad G_{1122}(x)|_t = \sum_{k'} C_{11k'} C_{22k'} |\mathcal{F}_{1122}(O_{k'}|1-x)|^2$$

$$\overline{s-t} = 0, \text{ but } \overline{(s-t)^2} = \overline{s^2} + \overline{t^2} - 2\overline{st} \neq 0$$

- The s t cross term vanishes in the Gaussian ensemble. Therefore there must be a quartic moment to cancel the variance.

$$\overline{C_{ijk} C_{iml} C_{njl} C_{nmk}}|_c = \left\{ \begin{matrix} \mathcal{O}_k & \mathcal{O}_j & \mathcal{O}_i \\ \mathcal{O}_l & \mathcal{O}_m & \mathcal{O}_n \end{matrix} \right\} = \left| \mathbf{F}_{kl} \begin{bmatrix} n & j \\ m & i \end{bmatrix} C_0(h_i, h_j, h_k) C_0(h_k, h_n, h_m) \rho_0(h_l)^{-1} \right|^2$$

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# “Local” version of crossing

- The Ponsot-Teschner crossing kernel  $F$  is exactly the object which transforms s-channel to t-channel blocks.

$$\mathcal{F}_{ijmn}(O_k|x) = \int d[O_l] \mathbf{F}_{kl} \begin{bmatrix} n & j \\ m & i \end{bmatrix} \mathcal{F}_{imjn}(O_l|1-x)$$

- Expand the crossing equation in s-channel principal series

$$\sum_q \left( C_{i_1 i_2 q} C_{i_3 i_4 q} \delta^{(2)}(P_s - P_q) - C_{i_1 i_4 q} C_{i_2 i_3 q} \left| \mathbf{F}_{P_q P_s} \begin{bmatrix} P_3 & P_4 \\ P_2 & P_1 \end{bmatrix} \right|^2 \right) = 0,$$

- For Virasoro, the principal series are the above threshold physical weights. For other operators (including Id), the  $\delta$  involves contour manipulation. In approximate CFTs, the restriction on kinematics implies the  $\delta$  is smeared.

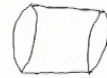
# Tensor model

- Instead of specifying moments, write an explicit ensemble:

$$\mathcal{Z} = \int D[L_0, \bar{L}_0] D[C] e^{-a V[L_0, \bar{L}_0, C]}$$

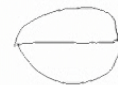
- The maximum ignorance ensemble consistent with crossing has  $V$  given by the sum of squares of the constraints.

$$V_4 = 2 \sum_{i_1 \dots i_4} \sum_{p,q} \left( \frac{C_{i_1 i_2 p} C_{i_3 i_4 p} C_{i_1 i_2 q} C_{i_3 i_4 q}}{|\rho_0(p) C_0(12p) C_0(34p)|^2} \delta^{(2)}(P_p - P_q) - \frac{C_{i_1 i_2 p} C_{i_3 i_4 p} C_{i_1 i_4 q} C_{i_2 i_3 q}}{|C_0(12p) C_0(34p) C_0(23q) C_0(14q)|^2} \begin{Bmatrix} \mathcal{O}_q & \mathcal{O}_4 & \mathcal{O}_1 \\ \mathcal{O}_p & \mathcal{O}_2 & \mathcal{O}_3 \end{Bmatrix} \right)$$



- $p=Id, 1=2, 3=4$  gives the propagator.

$$V_2 = \sum_{ijk} \frac{C_{ijk}^2}{|C_0(ijk)|^2}$$



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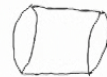
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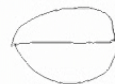
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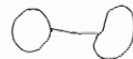
# Scaling limit

- The random tensor model is defined by truncating to a finite number of primaries, and then taking a triple scaled limit, in which the rank and overall coefficient  $a$  are taken to infinity, while the width of the smeared delta functions is taken to zero.
- In principle, in the strict limit, crossing is imposed exactly and one either finds a specific exact CFT or no solutions. What is interesting is to study the theory in the 't Hooft matrix + triple line tensor Feynman expansion. The expansion parameter is  $e^{-c}$
- The ultimate fate of the strict limit is a “doubly non-perturbative” question - sum over all topologies.

# Modular invariance

- Similarly, modular invariance of torus 1 point functions can be written in terms of the modular inversion kernel,  $S$ , and we take its square as part of the potential.

$$V_{S, \neq 1} = 2 \sum'_{i,j,k} |\rho_0(P_i) C_0(P_i, P_i, P_j)|^2 C_{iij} C_{kkj} \left( \delta^{(2)}(P_i - P_k) - |\mathbf{S}[\mathcal{O}_j]_{P_i P_k}|^2 \right)$$



- This is another quadratic term. With the Id inserted, one gets

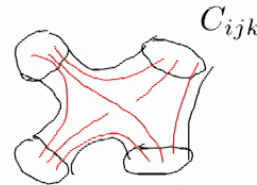
$$V_{S,1} = 64 \left( \sum'_{i,j} \left| \sinh(2\pi b P_i) \sinh\left(2\pi \frac{P_i}{b}\right) \right|^2 \delta^{(2)}(P_i - P_j) - \sum'_i \left| \sinh(2\pi b P_i) \sinh\left(2\pi \frac{P_i}{b}\right) \right|^2 - 8 \sum'_{i,j} \left| \sinh(2\pi b P_i) \sinh\left(2\pi \frac{P_i}{b}\right) \cos(4\pi P_i P_j) \right|^2 \right)$$

- T invariance imposes integrality of spin

$$V_T = \text{Tr} \sin^2(\pi(L_0 - \bar{L}_0))$$

# 3d gravity

- Conjecture: in the case with no fixed primaries aside from Id (thus a gap to the black hole threshold), then in  $e^{-c}$  expansion, the triple scaled limit of the tensor model is exactly pure 3d gravity, including the sum over all hyperbolic 3-manifolds!
- Building block of the tensor part is the 4 boundary wormhole associated to the  $6j$  vertex. The index loops are filled in with 't Hooft diagrams of the matrices – leading disk corresponds to BTZ filling of solid torus.



[Collier Eberhardt Zhang]

## 2 and 4 point resummed

- However, the bare propagator of the tensor model has a factor of  $-\frac{1}{a}$ . This is similar to imposing  $\langle x^2 \rangle = v$  by an integral

$$\int dx e^{-a(x^2/v-1)^2} \sim \int dx e^{-a(-2x^2/v+x^4/v^2)}$$

- Need to resum infinitely many diagrams. One can check Schwinger-Dyson equations



- Very similar to simplicial 3d gravity of Regge, ... Boulatov also involving  $6j$  symbol vertices. Here matrix part also plays important role. But there is a sum over many diagrams for each 3-manifold. Integrability seems to imply that they are all proportional! So also like Turaev-Viro, Collier-Eberhardt-Zhang.

# Relation to M5 branes

- An intriguing fact is that the reduction of the 2 M5 brane theory on  $S^3$  in the context of  $N=2$  sphere partition functions is exactly the integration cycle for  $SL(2,C)$  Chern-Simons theory that gives TTQFT on the 3-manifold.

[Dimofte Gaiotto Gukov; Cordova DLJ; Mikhaylov]

- There is probably a connection of the tensor model to the M5 brane similar to the matrix model/minimal string.
- A distinction again is that here the sum over all of the complicated 3-manifold topologies ends up imposing exact crossing, rigidifying the model.

# Summary

- Defined a notion of approximate CFT, which is CFT data that obeys crossing up to small corrections away from extreme kinematics.
- One can define ensembles of such data, which must be strongly non-Gaussian.
- Leads to a tensor model that is completely determined by conformal symmetries – maximum ignorance ensemble.
- Its expansion is related to 3d gravity, connecting simplicial gravity and VTQFT.