Title: Challenge Talk 3 - Symmetry/Topological-Order correspondence -- from string theory to condensed matter physics

Speakers: Xiao-Gang Wen

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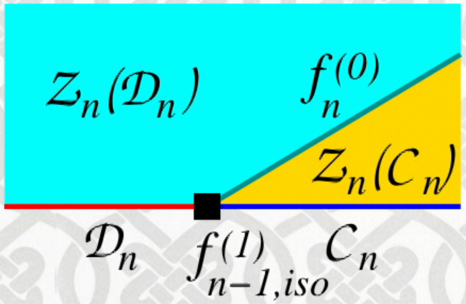
# Symmetry/Topological-Order (Symm/TO) correspondence


Xiao-Gang Wen (MIT)


**From string theory to condensed matter physics**

2023/07, Strings 2023, PI

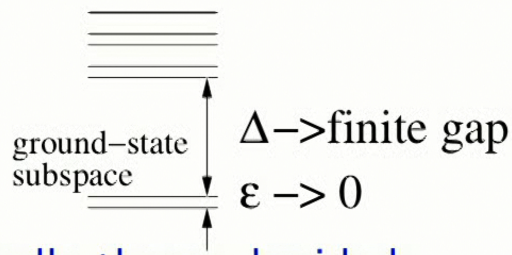



Kong Wen Zheng arXiv:1502.01690
Ji Wen arXiv:1905.13279
Ji Wen arXiv:1912.13492
Kong Lan Wen Zhang Zheng arXiv:2005.14178

Simons Collaboration on
Ultra-Quantum Matter

# Three kinds of quantum phases

All quantum systems discussed here have **lattice UV completion** which defines **condensed matter systems**



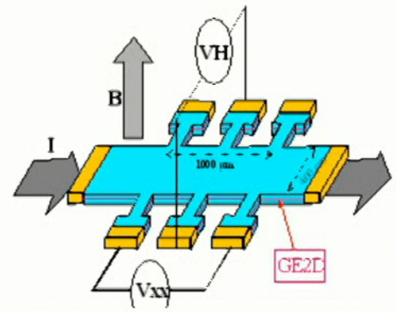

- **Gapped** $\to$ no low energy excitations
  All excitations has energy gap.
  Band insulators, FQH states
  General theory: topological order, moduli bundle theory, braided fusion higher category
- **Gapless (finite)** $\to$ finite low energy modes
  Finite low energy modes: Dirac/Weyl semimetal, superfluid, critical point at continuous phase transition
  General theory: quantum field theory, conformal field theory, ???
- **Gapless (infinite)** $\to$ infinite low energy modes
  Infinite low energy modes: Fermi metal, Bose metal, *etc*
  (Low energy effective theory is beyond quantum field theory)
  General theory: Landau Fermi liquid, ???

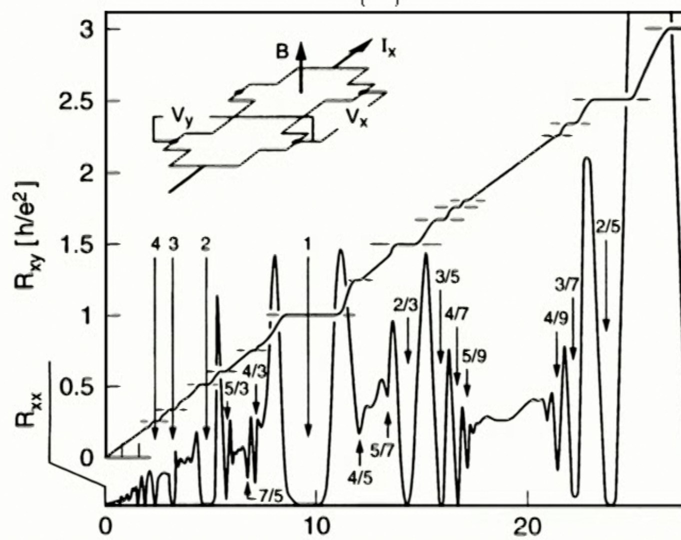
# Topological orders in quantum Hall effect

*For a long time, we thought that Landau symmetry breaking classify all phases of matter*

- Quantum Hall states $R_{xy} = V_y/I_x = \frac{m}{n}\frac{2\pi\hbar}{e^2}$

von Klitzing Dorda Pepper, PRL **45** 494 (1980)
Tsui Stormer Gossard, PRL **48** 1559 (1982)



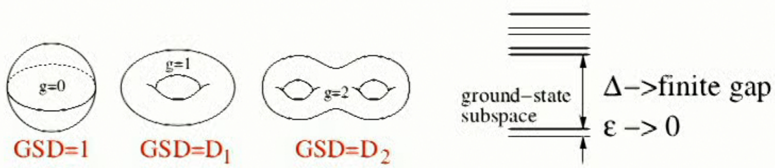

- FQH states have different phases even when there is no symm. and no symm. breaking.

- FQH liquids must contain a new kind of order, named as **topological order**

# Characterize topological order quantitatively

- How to extract universal numbers (topological invariants) from complicated many-body wavefunction $\Psi(x_1, \cdots, x_{10^{20}})$

[1]E. Witten, Commun. Math. Phys. **121**, 351 (1989); **117**, 353 (1988).
[2]Y. Hosotani, Report No. IAS-HEP-89/8, 1989 (unpublished); G. V. Dunne, R. Jackiw, and C. A. Trungenberg, Report No. MIT-CTP-1711, 1989 (unpublished); S. Elitzur, G. Moore, A. Schwimmer, and N. Seiberg, Report No. IASSNS-HEP-89/20, 1989 (unpublished).
[3]V. Kalmeyer and R. Laughlin, Phys. Rev. Lett. **59**, 2095 (1988); X. G. Wen and A. Zee (unpublished); P. W. Anderson (unpublished); P. Wiegmann, in *Physics of Low Dimensional Systems*, edited by S. Lundqvist and N. K. Nilsson (World Scientific, Singapore, 1989).
[4]X. G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B **39**, 11 413 (1989); D. Khveshchenko and P. Wiegmann (unpublished).
[5]G. Baskaran and P. W. Anderson, Phys. Rev. B **37**, 580 (1988).




**Put the gapped system on space with various topologies, and measure the ground state degeneracy → topological order**

**Vacuum degeneracy of chiral spin states in compactified space**

X. G. Wen

*Institute for Theoretical Physics, University of California, Santa Barbara, California 93106*

(Received 10 May 1989)

A chiral spin state is not only characterized by the $T$ and $P$ order parameter $E_{123} = \mathbf{S}_1 \cdot (\mathbf{S}_2 \times \mathbf{S}_3)$, it is also characterized by an integer $k$. In this paper we show that this integer $k$ can be determined from the vacuum degeneracy of the chiral spin state on compactified spaces. On a Riemann surface with genus $g$ the vacuum degeneracy of the chiral spin state is found to be $2k^g$. Among those vacuum states, some $k^g$ states have $\langle E_{123}\rangle > 0$, while other $k^g$ states have $\langle E_{123}\rangle < 0$. The dependence of the vacuum degeneracy on the topology of the space reflects some sort of topological ordering in the chiral spin state. In general, the topological ordering in a system is classified by topological theories.

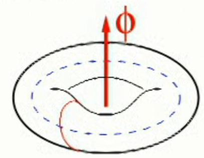
# Ground state degen. characterizes phase of matter

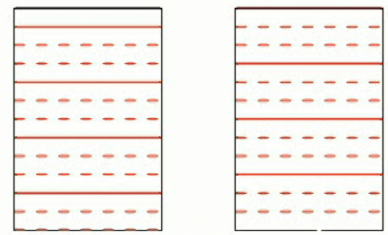
**Objection**: GSD on $S^2 \neq$ GSD on $T^2$ (coming from the motion of center mass). Ground state degeneracy is just a finite size effect. Ground state degeneracy does not reflect the thermodynamic phase of matter.

- **Robust topological ground state degeneracy**

- Inserting $2\pi$ flux pumps one quantum Hall ground state in magnetic field $B$ to another ground state.

- $k_x$ of the two ground states differ by $\Delta k_x \sim BL_y \to \infty|_{L_y \to \infty}$



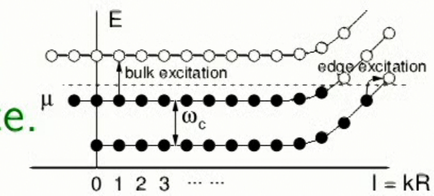
- Impurities can only cause momentum transfer $\delta k_x \sim \sqrt{B}$, and split ground state degeneracy by $\Delta E \sim \mathrm{e}^{-\# L_y \sqrt{B}}$

Wen Niu PRB 41, 9377 (90)

- Magnetic field $B \to$ UV-IR mixing and non-commutative geometry

# Even non-Abelian statistics can be realized

Let $\chi_n(z_i)$ be the many-body wave function of $n$ filled Landau level, which describes a gapped state.




- Products of gapped IQH wave functions $\chi_n$ are also gapped $\to$ new FQH states — Jain PRB **11** 7635 (90)
- $SU(m)_n$ state $\chi_1^k \chi_n^m$ via slave-particle — Wen PRL **66** 802 (1991)

$$\Psi_{SU(3)_2} = (\chi_2)^3,\ \nu = 2/3; \qquad \Psi_{SU(2)_2} = \chi_1(\chi_2)^2,\ \nu = 1/2;$$

$\to$ Effective $SU(3)_2$, $SU(2)_2$ Chern-Simons theory
$\to$ non-Abelian statistics (assume $\chi_1^k \chi_n^m$ is gapped, conjecture)

- Pfaffien state via CFT correlation — Moore-Read NPB **360** 362 (1991)

$$\Psi_{\text{Pfa}} = \mathcal{A}[\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots]\prod(z_i - z_j)^2 \mathrm{e}^{-\frac{1}{4}\sum|z_i|^2}, \qquad \nu = 1/2$$

Conformal block = multi-valueness of many-body wave function $\stackrel{\text{conjecture}}{\to}$ non-Abelian Berry phase $\to$ non-Abelian statistics

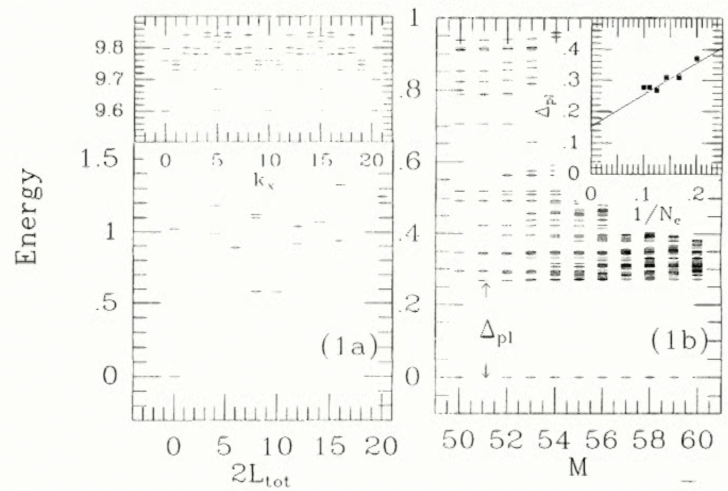
# Numerical confirmation of non-Abelian statistics

Application of TQFT/CFT correspondence. Witten, CMP **121** 352 (89)

- Edge state of Abelian FQH state (classified by $K$-matrices) always has an integral central charge $c \in \mathbb{N}$, Wen Zee PRB **46** 2290 (92)
- If edge states are described by a fractional central charge → The bulk must be a non-Abelian state. Wen PRL **70** 355 (93)
- For $\nu = 1/2$ state with a three-body interaction, the edge spectrum is given by (for 8 electrons on 20 orbits):

| $L_{tot}$ : | 52 | 53 | 54 | 55 | 56 | 57 |
|---|---|---|---|---|---|---|
| NOS : | 1 | 1 | 3 | 5 | 10 | 15 |

Edge states are described by:
$1\frac{1}{2}$ chiral phonon modes $c = 1\frac{1}{2}$
$=1$ chiral phonon mode
$+ 1$ chiral Majorana fermion
$=3$ chiral Majorana fermions **The Pfaffien state is non-Abelian**



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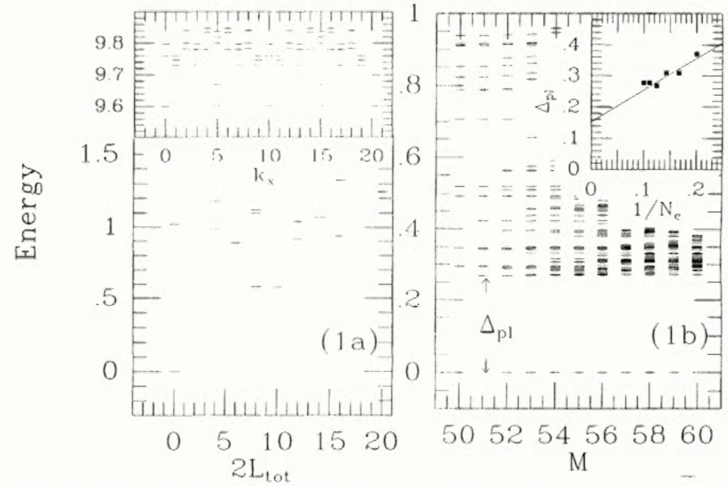


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$=1$ chiral phonon mode

$+\ 1$ chiral Majorana fermion

$=3$ chiral Majorana fermions **The Pfaffien state is non-Abelian**

# Topo. order & theory of long range entanglement

The microscopic mechanism of superconductivity: electron pairing

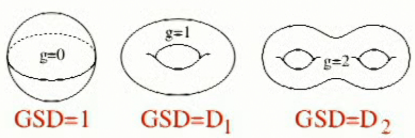
- The microscopic mechanism of topological order:
  **Topological order = pattern of long range entanglement**

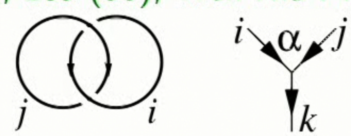
Wen, PRB **40** 7387 (89); IJMPB 4, 239 (90). Chen Gu Wen arXiv:1004.3835

*Symmetry breaking orders are described by group theory. What theory describes topological orders (long range entanglement)?*

- **Ground states**: Robust degenerate ground states form vector bundles on moduli spaces of gapped Hamiltonians → **moduli bundle theory** for topological orders.

Wen, IJMPB 4, 239 (90); Wen Niu PRB 41, 9377 (90)



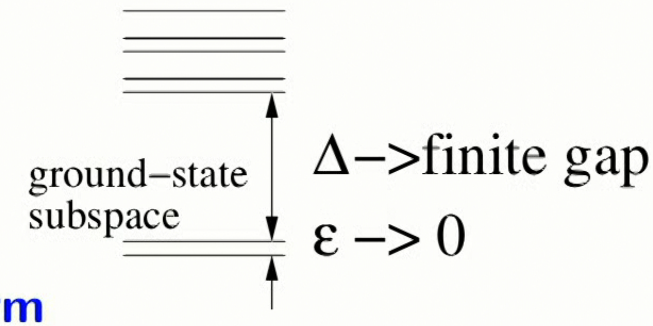




- **Excitations**: The anyons are described by their **fusion and braiding** → **modular tensor category theory** for topological orders

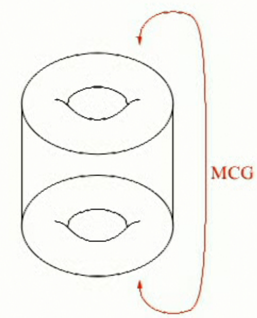
Moore Seiberg CMP **123** 177 (89). Witten, CMP **121** 352 (89)

# Moduli bundle theory of topological order

The important data is the **connections** of ground-state vector bundle on moduli space.




- Non-Abelian Berry's phase along **contractable loops** in moduli space
  $\to$ a diagonal $U(1)$ factor acting on the degenerate ground states
  $\to$ **gravitational Chern-Simons term**
  $\to$ **chiral central charge** $c$ of edge state




- Non-Abelian Berry's phase along **non-contractable loops** in moduli space $\to$ $S, T$ unitary matrices acting on the degenerate ground states $\to$ **projective representation of mapping-class-group** (which is $SL(2, \mathbb{Z})$ for torus, generated by
  $s : (x, y) \to (-y, x), \quad t : (x, y) \to (x + y, y)$ )

Wen, PRB **40** 7387 (89); IJMPB 4, 239 (90).

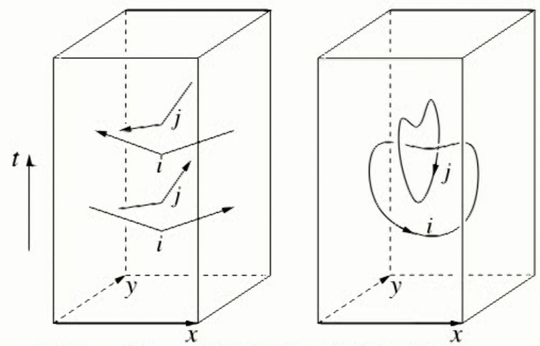
# Modular tensor category theory for anyons and 2+1D topological orders

- Excitation in 2+1D topological order → **Braided fusion category (modular tensor category)** → A theory for 2+1D topological orders for bosons.
  rational CFT → TQFT → MTC

Moore-Seiberg CMP **123** 177 (89); Witten, CMP **121** 352 (89)

- In higher dimensions, topological excitations can be **point-like**, **string-like**, *etc* , which can fuse and braid →
- Topological excitations are described by **non-degenerate braided fusion higher categories** → theory of topological order

- The ground state degeneracy *GSD* on torus and fractional statistics $\theta = \pi \frac{p}{q}$ of topological excitations are closely related $U_x U_y U_x^\dagger U_y^\dagger = e^{2\pi \frac{p}{q}}$: *GSD* is a multiple of $q$.




Wen Niu PRB **41** 9377 (90).

# Classify 2+1D bosonic topological orders (TOs)

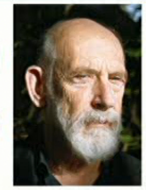
Using moduli bundle theory (*ie* $SL(2,\mathbb{Z})$ representations), plus input from modular tensor category, we can classify 2+1D bosonic topological orders (up to invertible $E(8)$ states):

| # of anyon types (rank) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| # of 2+1D TOs | 1 | 4 | 12 | 18 | 10 | 50 | 28 | 64 | 81 | 76 | 44 |
| # of Abelian TOs | 1 | 2 | 2 | 9 | 2 | 4 | 2 | 20 | 4 | 4 | 2 |
| # of non-Abelian TOs | 0 | 2 | 10 | 9 | 8 | 46 | 26 | 44 | 77 | 72 | 42 |
| # of prime TOs | 1 | 4 | 12 | 8 | 10 | 10 | 28 | 20 | 20 | 40 | 44 |

Rowell Stong Wang, arXiv:0712.1377: up to rank 4
Bruillard Ng Rowell Wang, arXiv:1507.05139: up to rank 5
Ng Rowell Wang Wen, arXiv:2203.14829: up to rank 6
Ng Rowell Wen, to appear: up to rank 11

- **This classifies all 2+1D gapped phases for bosonic systems without symmetry, with 11 topological excitations or less.**

# Topological holographic principle




**String holographic principle**: Susskind hep-th/9409089
boundary CFT = bulk AdS gravity Maldacena hep-th/9711200

- **Holographic principle of topological order**:
  Boundary determines bulk, but bulk does not determine boundary Kong Wen arXiv:1405.5858; Kong Wen Zheng arXiv:1502.01690
  The excitations in a topological order are described by a braided fusion category $\mathcal{M}$. The excitations on a gapped boundary of a topological order are described by a fusion category $\mathcal{F}$
  $\mathcal{F}$ **determines** $\mathcal{M}$: $\mathcal{Z}(\mathcal{F}) = \mathcal{M}$ ($\mathcal{Z}$ is generalized Drinfeld-center)

- String-operators that create pairs of boundary excitations form an algebra which is characterized by a braided fusion category $\mathcal{M}$.
  Chatterjee Wen arXiv:2205.06244

- **A generalization of anomaly in-flow**: Callan Harvey, NPB **250** 427 (1985)
  The theory described by fusion category $\mathcal{F}$ has a (non-invertible) gravitational anomaly (*ie* no UV completion) Kong Wen arXiv:1405.5858
  **(non-invertible) grav anomaly = bulk topological order** $\mathcal{M}$

# Classification of 3+1D bosonic topological orders (*ie* classification of 4D fully extended TQFTs)

An application of topological holographic principle

- 3+1D bosonic topological orders with only bosonic point-like excitations are classified by 3+1D Dijkgraaf-Witten theory of finite groups. Lan Kong Wen arXiv:1704.04221; Johnson-Freyd arXiv:2003.06663

- 3+1D fully extended TQFT's with only bosonic point-like excitations are classified by Dijkgraaf-Witten theories of finite groups.

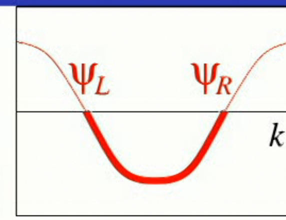
- A duality relation: 3+1D twisted higher gauge theories of finite higher group with only bosonic point-like excitations are equivalent to twisted 1-gauge theories of finite group.

- 3+1D bosonic topological orders with both bosonic and fermionic point-like excitations are also classified.

Lan Wen arXiv:1801.08530; Johnson-Freyd arXiv:2003.06663

# Next step: a general theory for 'finite' gapless state

A gapless state has emergent (and exact) symmetry:




- Group-like symmetries Heisenberg, Wigner, 1926 $U(2) \to$
- Anomalous symmetries 't Hooft, 1980 $U_R(2) \times U_L(2)$
- Higher-form symmetries Nussinov Ortiz 09; Gaiotto Kapustin Seiberg Willett 14
- Higher-group symmetries Kapustin Thorngren 2013
- Algebraic higher symmetry Thorngren Wang 19; Kong Lan Wen Zhang Zheng 20

  algebraic (higher) symmetry = non-invertible (higher) symmetry
  = fusion (higher) category symmetry = ... ...

  Petkova Zuber 2000; Coquereaux Schieber 2001; ... for 1+1D CFT
- (Non-invertible) gravitational anomalies Kong Wen 2014; Ji Wen 2019

• Conjecture: **The maximal emergent (generalized) symmetry largely determine the gapless states.**

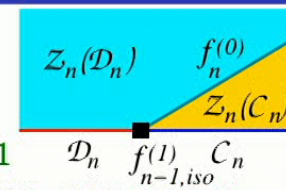
A classification of maximal emergent (generalized) symmetries → A classification of "finite" gapless states. Chatterjee Ji Wen arXiv:2212.14432

*What is the general theory for all those generalized symmetries, which are beyond group and higher group?*

# Symmetry/Topological-Order correspondence

A symmetry corresponds to:

- an **isomorphic decomposition** $\mathcal{D}_n \cong \mathcal{C}_n \boxtimes_{\mathcal{Z}_n(\mathcal{C}_n)} f_n^{(0)}$




  Kong Wen Zheng arXiv:1502.01690; Freed Moore Teleman arXiv: 2209.07471
- a **non-invertible gravitational anomaly** Ji Wen arXiv:1905.13279
- a **symmetry + dual symmetry + braiding** Ji Wen arXiv:1912.13492
  Conservation/fusion-ring of **symmetry charges** = symmetry
  Conservation/fusion-ring of **symmetry defects** = dual-symmetry
- a **gappable-boundary topological order** in one higher dimension
  Ji Wen arXiv:1912.13492; Kong Lan Wen Zhang Zheng arXiv:2005.14178
- a **Braided fusion higher category in trivial Witt class**
  Thorngren Wang arXiv:1912.02817; Kong Lan Wen Zhang Zheng arXiv:2005.14178.

  $\rightarrow$ a unified frame work to classify SSB, TO, SPT, SET phases.
- a **topological skeleton** in QFT Kong Zheng arXiv:2011.02859
- an **algebra of patch commutant operators.**
  Kong Zheng arXiv:2201.05726; Chatterjee Wen arXiv:2205.06244

# Symmetry $\sim$ non-invertible gravitational anomaly

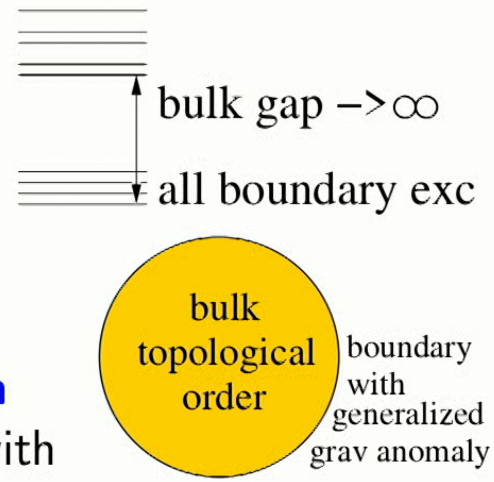
- A symmetry is generated by an unitary operators $U$ that commute with the Hamiltonian: $UH = HU$.
- We describe a symmetric system (with lattice UV completion) restricted in the symmetric sub-Hilbert space
$$U\mathcal{V}_{\text{symmetric}} = \mathcal{V}_{\text{symmetric}}.$$
Both system and the probing instruments respect the symmetry
- The symmetry transformation $U$ acts trivially within $\mathcal{V}_{\text{symmetric}}$.
*How to know there is a symmetry? How to identify the symmetry?*

- The total Hilbert space $\mathcal{V}_{\text{tot}}$ has a tensor product decomposition $\mathcal{V}_{\text{tot}} = \otimes_i \mathcal{V}_i$, where $i$ labels sites, due to the lattice UV completion.
- The symmetric sub-Hilbert space $\mathcal{V}_{\text{symmetric}}$ does not have a tensor product decomposition $\mathcal{V}_{\text{symmetric}} \neq \otimes_i \mathcal{V}_i$, indicating the presence of a symmetry.
- Lack of tensor product decomposition $\rightarrow$ gravitational anomaly.
$\rightarrow$ **symmetry $\cong$ non-invertible gravitational anomaly**

# Symmetry ≅ topological order in one higher dim

- **Gravitational anomaly = topo. order in one higher dim**

- The total boundary Hilbert space of a topologically ordered state has no tensor product decomposition. Yang etal arXiv:1309.4596 Kong Wen arXiv:1405.5858
  Lack of tensor product decomposition is described by boundary of topological order
  **Systems with a (generalized) symmetry (restricted within $\mathcal{V}_{\text{symmetric}}$) can be fully and exactly simulated by boundaries of a topological order**, called **symmetry-TO** (with lattice UV completion) or **symmetry TFT**.




Ji Wen arXiv:1912.13492; Kong Lan Wen Zhang Zheng arXiv:2005.14178
Apruzzi Bonetti Etxebarria Hosseini Schafer-Nameki arXiv:2112.02092

- Symmetry-TO or symmetry TFT was originally called **categorical symmetry** in Ji Wen arXiv:1912.13492; Kong etal arXiv:2005.14178
  → **Symm/TO correspondence**

# Classify 1+1D symmetries (up to holo-equivalence)

Not every topological order describes a generalized symmetry.

- Only topological orders with gappable boundary (*ie* in trivial Witt class) correspond to (generalized) symmetries.

  Kong Lan Wen Zhang Zheng arXiv:2005.14178; Freed Moore Teleman arXiv:2209.07471

We refer to gappable-boundary topological order (TO) in one higher dimension as **symmetry-TO** (with lattice UV completion).

**Finite symmetries (up to holo-equivalence) are one-to-one classified by symmetry-TOs in one higher dimension**

- We can use 2+1D symmetry-TOs (instead of groups) to classify 1+1D finite (generalized) symmetries (up to holo-equivalence):

| # of symm charges/defects (rank) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| # of 2+1D TOs | 1 | 4 | 12 | 18 | 10 | 50 | 28 | 64 | 81 | 76 | 44 |
| # of symm classes (symm-TOs) | 1 | 0 | 0 | 3 | 0 | 0 | 0 | 6 | 6 | $\leq 3$ | 0 |
| # of (anomalous) group-symmetries | $1_{\mathbb{Z}_1}$ | 0 | 0 | $2_{\mathbb{Z}_2^\omega}$ | 0 | 0 | 0 | $6_{S_3^\omega}$ | $3_{\mathbb{Z}_3^\omega}$ | 0 | 0 |

- At rank-4: $\mathbb{Z}_2$ symm, anomalous $\mathbb{Z}_2$ symm, double-Fibonacci symm

# Local fusion category & isomorphic decomposition

An anomaly-free ordineray symmetry is decribed by a group

- An anomaly-free generalized (*ie* non-invertible higher) symmetry (*ie* algebraic higher symmetry) in $n+1$D is decribed by

- a **local fusion $n$-category** $\mathcal{R}_{\text{charge}}$ that describes symmetry charges (excitations over trivial symmetric ground state), or by
- a **local fusion $n$-category** $\widetilde{\mathcal{R}}_{\text{defect}}$ that describes symmetry defects.

Thorngren Wang arXiv:1912.02817 (1+1D); Kong Lan Wen Zhang Zheng arXiv:2005.14178

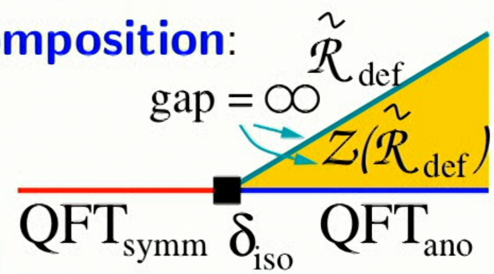
- **generalized symmetry = isomorphic decomposition**:

$\delta_{\text{iso}}$ : $QFT_{\text{symm}} \cong QFT_{\text{ano}} \boxtimes_{\mathcal{Z}(\widetilde{\mathcal{R}}_{\text{def}})} \widetilde{\mathcal{R}}_{\text{def}}$

Kong Wen Zheng arXiv:1502.01690

Kong Lan Wen Zhang Zheng arXiv:2005.14178

$\delta_{\text{iso}}$ : $Z(QFT_{\text{symm}}) = Z(QFT_{\text{ano}} \boxtimes_{\mathcal{Z}(\widetilde{\mathcal{R}}_{\text{def}})} \widetilde{\mathcal{R}}_{\text{def}})$




- **A similar but different theory**: A generalized (potentially anomalous) symmetry $= (\rho, \sigma = \mathcal{Z}(\rho)) =$ fusion $n$-categroy $\rho$ (**no local condition**).

Freed Moore Teleman arXiv: 2209.07471

# Classify gapped/gapless phases of symm systems

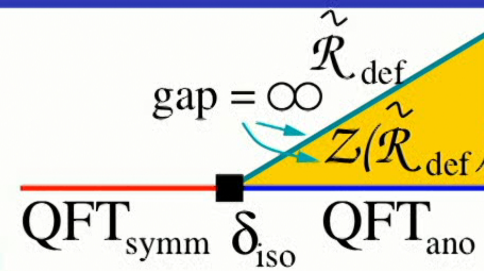
**via Symm/TO correspondence**:

$\delta_{\text{iso}}: \ QFT_{\text{symm}} \cong QFT_{\text{ano}} \boxtimes_{\mathcal{Z}(\widetilde{\mathcal{R}}_{\text{def}})} \widetilde{\mathcal{R}}_{\text{def}}$

Kong Wen Zheng arXiv:1502.01690

Kong Lan Wen Zhang Zheng arXiv:2005.14178

$\delta_{\text{iso}}: \ Z(QFT_{\text{symm}}) = Z(QFT_{\text{ano}} \boxtimes_{\mathcal{Z}(\widetilde{\mathcal{R}}_{\text{def}})} \widetilde{\mathcal{R}}_{\text{def}})$




- Gapped liquid phases are gapped boundaries of $\mathcal{Z}(\widetilde{\mathcal{R}}_{\text{def}})$(symm-TO)

- Includes spontaneous symmetry breaking orders, symmetry protected topological (SPT) orders, symmetry enriched topological (SET) orders for systems with algebraic higher symmetry $\widetilde{\mathcal{R}}_{\text{def}}$

- Gapless liquid phases are gapless boundaries of $\mathcal{Z}(\widetilde{\mathcal{R}}_{\text{def}})$ (symm-TO)
- SPT phases protected by algebraic higher symmetry $\widetilde{\mathcal{R}}_{\text{def}}$ are classified by the automorphisms $\alpha$ of the corresponding symmetry-TO $\mathcal{Z}(\widetilde{\mathcal{R}}_{\text{def}})$, that leave $\widetilde{\mathcal{R}}_{\text{def}}$ invariant.
- Anomalous algebraic higher symmetries are classified by $(\widetilde{\mathcal{R}}_{\text{def}}, \widetilde{\alpha})$, where $\widetilde{\alpha} \in \text{Auto}(\mathcal{Z}(\Sigma\widetilde{\mathcal{R}}_{\text{def}}))$ that leave $\Sigma\widetilde{\mathcal{R}}_{\text{def}}$ invariant.

# Local fusion category & isomorphic decomposition

An anomaly-free ordineray symmetry is decribed by a group

- An anomaly-free generalized (*ie* non-invertible higher) symmetry (*ie* algebraic higher symmetry) in $n+1$D is decribed by

- a **local fusion $n$-category** $\mathcal{R}_{\text{charge}}$ that describes symmetry charges (excitations over trivial symmetric ground state), or by

- a **local fusion $n$-category** $\widetilde{\mathcal{R}}_{\text{defect}}$ that describes symmetry defects.

Thorngren Wang arXiv:1912.02817 (1+1D); Kong Lan Wen Zhang Zheng arXiv:2005.14178

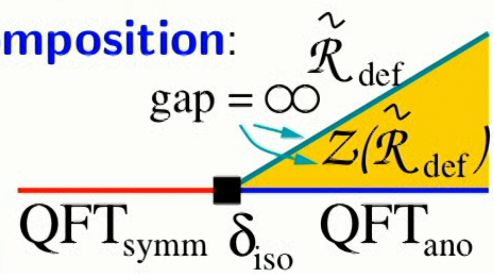
- **generalized symmetry = isomorphic decomposition**:

$\delta_{\text{iso}}: \ QFT_{\text{symm}} \cong QFT_{\text{ano}} \boxtimes_{\mathcal{Z}(\widetilde{\mathcal{R}}_{\text{def}})} \widetilde{\mathcal{R}}_{\text{def}}$

Kong Wen Zheng arXiv:1502.01690

Kong Lan Wen Zhang Zheng arXiv:2005.14178

$\delta_{\text{iso}}: \ Z(QFT_{\text{symm}}) = Z(QFT_{\text{ano}} \boxtimes_{\mathcal{Z}(\widetilde{\mathcal{R}}_{\text{def}})} \widetilde{\mathcal{R}}_{\text{def}})$




- **A similar but different theory**: A generalized (potentially anomalous) symmetry $= (\rho, \sigma = \mathcal{Z}(\rho))$ = fusion $n$-categroy $\rho$ (**no local condition**).

Freed Moore Teleman arXiv: 2209.07471

# A general theory of duality (holo-equivalence)

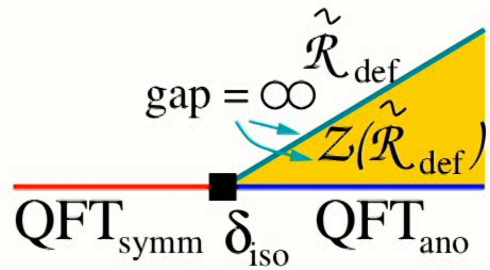
**via Symm/TO correspondence and isomorphic decomposition**:

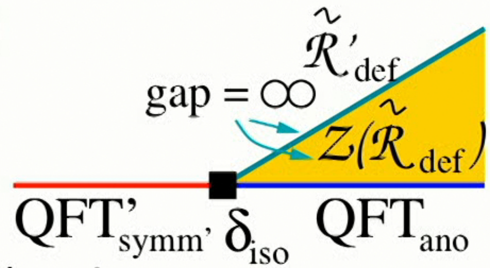
$\delta_{\text{iso}}: \ QFT_{\text{symm}} \cong QFT_{\text{ano}} \boxtimes_{\mathcal{Z}(\widetilde{\mathcal{R}}_{\text{def}})} \widetilde{\mathcal{R}}_{\text{def}}$

Kong Wen Zheng arXiv:1502.01690

Kong Lan Wen Zhang Zheng arXiv:2005.14178

$\delta_{\text{iso}}: \ Z(QFT_{\text{symm}}) = Z(QFT_{\text{ano}} \boxtimes_{\mathcal{Z}(\widetilde{\mathcal{R}}_{\text{def}})} \widetilde{\mathcal{R}}_{\text{def}})$




- Choose a different gapped boundary $\widetilde{\mathcal{R}}'_{\text{def}}$, without changing the bulk topological order $\mathcal{Z}(\widetilde{\mathcal{R}}_{\text{def}}) = \mathcal{Z}(\widetilde{\mathcal{R}}'_{\text{def}})$ and without changing the boundary $\text{QFT}_{\text{anom}}$ $\to$ the two quantum field theories, $\text{QFT}_{\text{symm}}$ and $\text{QFT'}_{\text{symm'}}$, are holo-equivalent, or are related by **duality** or **gauging** transformation. Bhardwaj Tachikawa arXiv:1704.02330

- $\text{QFT}_{\text{symm}}$ and $\text{QFT'}_{\text{symm'}}$ may have different generalized symmetries.

- Two generalized symmetries $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{R}}'$ are holo-equivalent, if they have the same bulk (*ie* the same symmetry-TO) $\mathcal{Z}(\widetilde{\mathcal{R}}) = \mathcal{Z}(\widetilde{\mathcal{R}}')$.

- 1+1D $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry with mixed anomaly $\cong \mathbb{Z}_4$ symmetry

# Gapped/gapless phases of symmetric systems are 'classified' by condensible algebras of symmetry-TO

- For **1+1D systems** with (generalized) symmetry, their gapped states and gapless states can be "classfied" by **condensible algebras** $\mathcal{A} = \mathbf{1} \oplus a \oplus b...$ (*ie* the sets of anyons that can condense together) in the corresponding symmetry-TO (in one higher dimension):

- **The maximal (Langrangian) condensible algebras of the 2+1D symmetry-TO classify (1-to-1) gapped phases.**
- **The non-maximal (non-Langrangian) condensible algebras of the 2+1D symmetry-TO label (1-to-many) gapless phases (1+1D CFTs).**

This is because the gappled/gapless boundaries of 2+1D topological orders $\mathcal{M}$ are "classified" by the condensible algebras $\mathcal{A}$ of $\mathcal{M}$.

## Classify 1+1D gapped phases for systems w/ $\mathbb{Z}_2^a \times \mathbb{Z}_2^b$ symm via Lagrangian condensable algebra

- The symmetry-TO for 1+1D $\mathbb{Z}_2^a \times \mathbb{Z}_2^b$ symmetry is 2+1D $\mathbb{Z}_2^a \times \mathbb{Z}_2^b$ gauge theory $\mathcal{G}au_{\mathbb{Z}_2^a \times \mathbb{Z}_2^b}$ , with excitations **generated by** $e_a, e_b, m_a, m_b$.
- Six Lagrangian condensible algebras: Chatterjee Wen arXiv:2205.06244

$$\mathbf{1} \oplus m_a \oplus m_b \oplus m_a m_b \rightarrow \mathbb{Z}_2^a\text{-symmetric-}\mathbb{Z}_2^b\text{-symmetric}$$

$$\mathbf{1} \oplus m_a \oplus e_b \oplus m_a e_b \rightarrow \mathbb{Z}_2^a\text{-symmetric-}\mathbb{Z}_2^b\text{-broken}$$

$$\mathbf{1} \oplus e_a \oplus m_b \oplus e_a m_b \rightarrow \mathbb{Z}_2^a\text{-broken-}\mathbb{Z}_2^b\text{-symmetric}$$

$$\mathbf{1} \oplus e_a \oplus e_b \oplus e_a e_b \rightarrow \mathbb{Z}_2^a\text{-broken-}\mathbb{Z}_2^b\text{-broken}$$

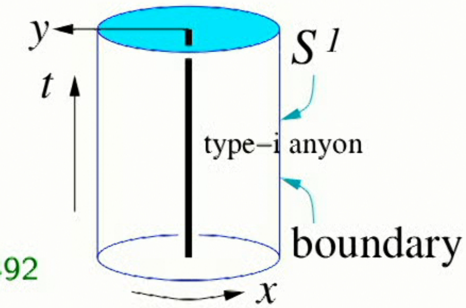
$$\mathbf{1} \oplus e_a e_b \oplus m_a m_b \oplus e_a m_a e_b m_b \rightarrow \text{diagonal-}\mathbb{Z}_2\text{-symmetric}$$

$$\mathbf{1} \oplus e_a m_b \oplus m_a e_b \oplus e_a m_a e_b m_b \rightarrow \mathbb{Z}_2^a \times \mathbb{Z}_2^b \text{ SPT phase}$$

# Q: How symmetry-TO determines gapless states? A: Via modular covariant partition function

A symmetry is described by its symmetry-TO. Its gapless states are simulated by the boundaries of the symmetry-TO.

- Boundary of 2+1D symmetry-TO has a **vector-valued partition function**, whose component $Z_i(\tau, \bar{\tau})$ is labeled by the anyon types $i$ of the 2+1D bulk topological order.
  Chen *etal* arXiv:1903.12334; Ji Wen arXiv:1905.13279, 1912.13492
  Kong Zheng arXiv:1905.04924, arXiv:1912.01760




- $Z_i(\tau, \bar{\tau})$ is not modular invariant but **modular covariant**:

$$T^{\mathcal{M}}: \ Z_i(\tau+1) = T^{\mathcal{M}}_{ij} Z_j(\tau), \qquad S^{\mathcal{M}}: \ Z_i(-1/\tau) = S^{\mathcal{M}}_{ij} Z_j(\tau).$$

where $S^{\mathcal{M}}$, $T^{\mathcal{M}}$-matrix characterize the 2+1D bulk topological order $\mathcal{M}$ (*ie* the symmetry-TO).

Ji Wen arXiv:1905.13279, 1912.13492; Lin Shao arXiv:2101.08343

- **CFT (gapless liquid phase) is a number theoretical problem.**