Title: Challenge Talk 2 - Feynman's Last Blackboard: From Bethe Ansatz to Langlands Duality

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Abstract: Richard Feynman's last blackboard at Caltech contains a number of tantalizing inscriptions about Bethe Ansatz in quantum integrable models, which fascinated him in the last years of his life. After reviewing some basic examples, I will present a modern perspective on the subject, linking it to dualities in QFT and String Theory, as well as the Langlands duality in mathematics. I will then discuss some recent developments that lend support to Feynman's intuition that these ideas could be useful in the study of 4d gauge theory, and formulate some open questions.

Feynman's Last Blackboard: From Bethe Ansatz to Langlands Duality

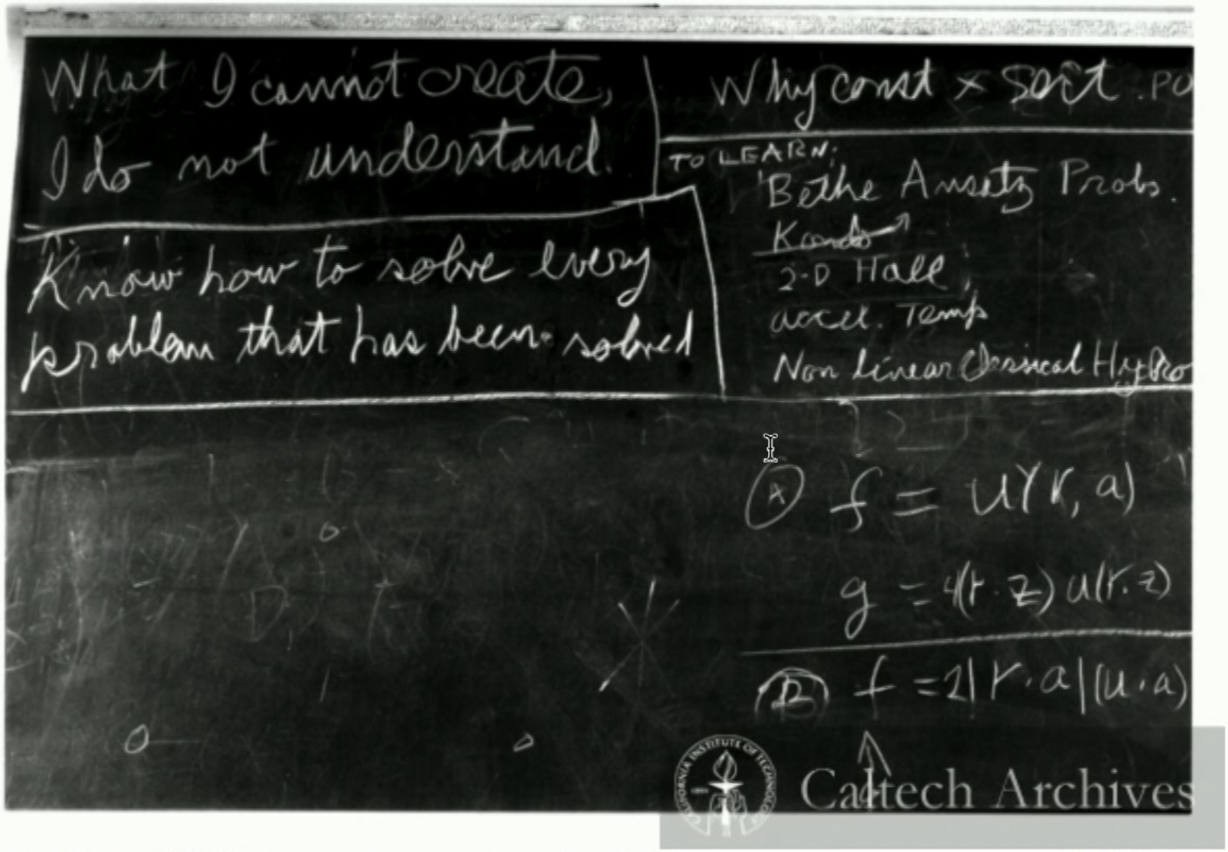
Edward Frenkel

Universty of California, Berkeley

July 25, 2023

Richard Feynman (1918–1988)

Richard Feynman's Last Blackboard at Caltech



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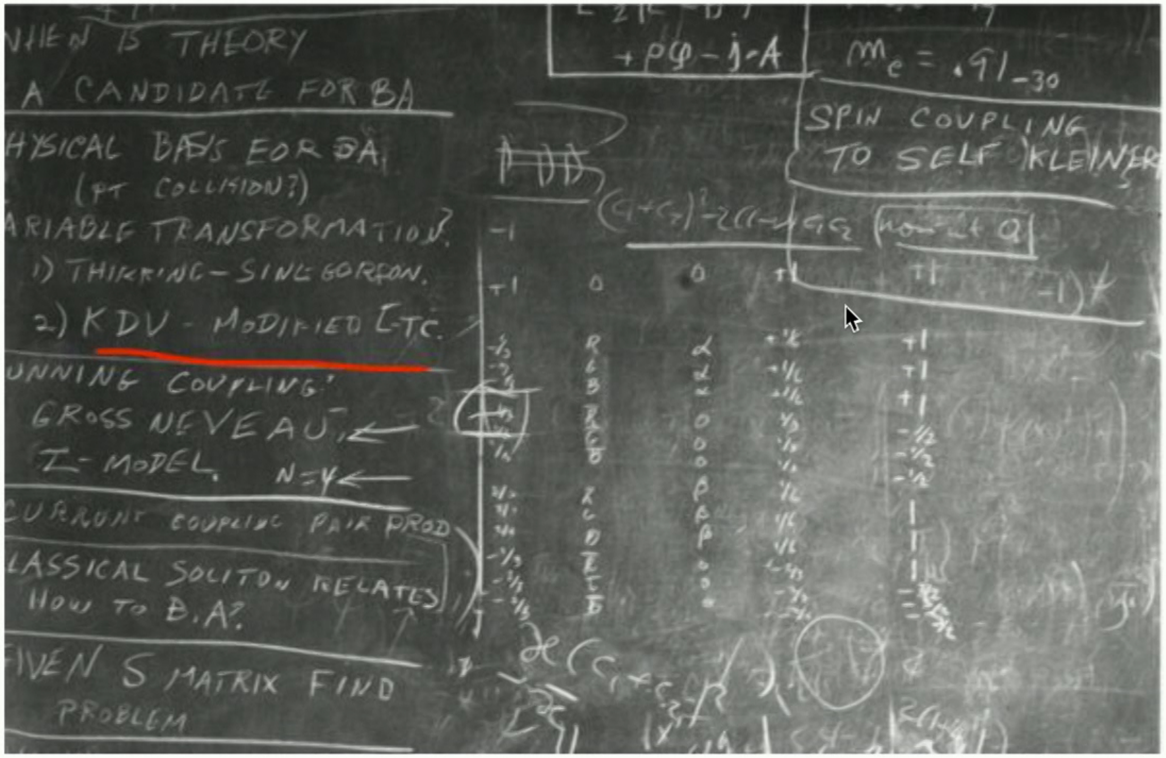

A CANDIDATE FOR BA
HYSICAL BASIS FOR BA
(PT COLLISION?)
ARIABLE TRANSFORMATION?
1) THIRRING – SINE GORDON.
2) KDV - MODIFIED ETC.
UNNING COUPLING
GROSS NEVEAU
N=4
PAIR PROD
LASSICAL SOLITON RELATES
HOW TO B.A.?
S MATRIX FIND
PROBLEM
+ρφ − j·A
m_e = .91_-30
SPIN COUPLING
TO SELF

Bethe Ansatz

An elegant method for solving quantum-mechanical models, introduced by Hans Bethe in 1931 in the case of Heisenberg's XXX model (1-dim. spin chain with space of states $(\mathbb{C}^2)^{\otimes N}$).

Namely, Bethe proposed an explicit formula for the eigenvectors of the Hamiltonian of the XXX model depending on certain parameters. These are indeed eigenvectors iff a certain system of equations is satisfied – Bethe Ansatz equations (BAE).

This method has subsequently been applied to many other integrable models, both discrete (QM) and continuous (2d QFTs), and proved to be surprisingly successful.

Feynman's notes, Caltech Lunch Seminar, 01/22/1987

(summary of the beginning)

Bethe Ansatz

Many different two-dimensional field theories have been proposed as models to learn from.

Sometimes, surprisingly, they can be solved; for example

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Non-linear Schrödinger
Thirring
sine-Gordon
Gross-Neveu (running coupling constant)
$O(N)$ σ-**model**
Two-dimensional statistical mechanics (Onsager, Baxter)

All solved by the same method: guessing the form of eigenvectors

Bethe Ansatz (1931)

Mystery: When will it work?

Connection to classical solitons [later in the notes: **Quantum KdV**]

Why study?

7/42

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(4) *Fun*

Some recent works linking BA & 4d gauge theory

Lipatov (1993), Faddeev-Korchemsky (1994) QCD $\rightsquigarrow$ XXX model

Minahan-Zarembo (2002), Beisert-Staudacher (2003) N=4 4d SYM (AdS_5/CFT_4) Gromov-Kazakov-Leurent-Volin (2013) QQ-system

Nekrasov-Shatashvili (2009) N=2 4d SYM with Ω-background $\rightsquigarrow$ Yang-Yang functions of integrable systems

Gaiotto-Witten (2011) S-duality in N=4 4d SYM $\rightsquigarrow$ Gaudin model

Costello (2013), Costello-Witten-Yamazaki (2017), Costello-Gaiotto-Yagi (2021) 4d Chern-Simons theory $\rightsquigarrow$ integrable models

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Gaiotto-Lee-Vicedo-Wu (2020) **Kondo problem** & Gaudin model

Gaudin model

Space of states: $(\mathbb{C}^2)^{\otimes N}$, or more generally, $\otimes_{i=1}^{N} V_{\lambda_i}$

$V_\lambda, \lambda \in \mathbb{Z}_{\geq 0}$ – finite-dim. rep. of $\mathfrak{sl}_2$ of dim. $\lambda + 1$ (spin $\lambda/2$)

Basis of $\mathfrak{sl}_2$: $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad e = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

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Gaudin Hamiltonians (for mutually distinct $z_i \in \mathbb{C}$):

$$H_i = \sum_{j \neq i} \frac{e^{(i)} \otimes f^{(j)} + f^{(i)} \otimes e^{(j)} + \frac{1}{2} h^{(i)} \otimes h^{(j)}}{z_i - z_j}, \quad i = 1, \ldots, N$$

(appear on the RHS of the KZ equations).
They commute with each other and the diagonal action of $\mathfrak{sl}_2$.

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Problem: diagonalize them on $\otimes_{i=1}^N V_{\lambda_i}$

More precisely, the decomposition of $\otimes_{i=1}^N V_{\lambda_i}$ under the diagonal $\mathfrak{sl}_2$ action is preserved by the H_i's.

Hence we consider the problem of finding eigenvectors and eigenvalues of the H_i's on the *subspace of highest weight vectors* in $\otimes_{i=1}^N V_{\lambda_i}$ w.r.t. diagonal $\mathfrak{sl}_2$ (i.e. annihilited by the diagonal e) *of weight*

$$\lambda_\infty := \sum_{i=1}^N \lambda_i - 2m$$

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For $m = 0$, this subspace is spanned by

$$|0\rangle = \otimes_{i=1}^{N} v_{\lambda_i}$$

where v_{λ_i} is the *highest weight vector* in V_{λ_i}.

Bethe Ansatz in Gaudin model

For $w \in \mathbb{C}, w \neq z_i$, let

$$f(w) = \sum_{i=1}^{N} \frac{f^{(i)}}{w - z_i}$$

Define the Bethe vector

$$|w_1, w_2, \ldots, w_m\rangle := f(w_1)f(w_2)\ldots f(w_m)|0\rangle$$

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Lemma. This vector is an **eigenvector of the Gaudin Hamiltonians** iff the following system of Bethe Ansatz equations is satisfied:

$$\sum_{i=1}^{N} \frac{\lambda_i/2}{w_j - z_i} - \sum_{s \neq j} \frac{1}{w_j - w_s} = 0, \qquad j = 1, \ldots, m$$

Eigenvalues of the Gaudin Hamiltonians

$$H_i \, |w_1, w_2, \ldots, w_m\rangle = \mu_i \, |w_1, w_2, \ldots, w_m\rangle$$

Let $\quad v(z) := \sum_{i=1}^{N} \frac{\lambda_i(\lambda_i+2)/4}{(z-z_i)^2} + \sum_{i=1}^{N} \frac{\mu_i}{z-z_i}.$

$$v(z) = u(z)^2 - \partial_z u(z), \qquad u(z) = \sum_{i=1}^{N} \frac{\lambda_i/2}{z-z_i} - \sum_{j=1}^{m} \frac{1}{z-w_j}$$

Miura transformation

$$\partial_z^2 - v(z) = (\partial_z - u(z))(\partial_z + u(z)).$$

PSL_2-opers describe the spectrum

PSL_2-oper (a.k.a. projective connection) is a differential operator

$$\partial_z^2 - v(z): \quad K^{-1/2} \longrightarrow K^{3/2}$$

(transforms as the stress tensor in CFT)

The joint **spectrum of the Gaudin Hamiltonians:**

$\mathfrak{sl}_2$-opers on $\mathbb{CP}^1$

- with regular singularities at $z_i, i = 1, \ldots, N$, and ∞;
- with leading terms $\lambda_i(\lambda_i + 2)/4, i = 1, \ldots, N$, and $\lambda_\infty(\lambda_\infty + 2)/4$;

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- with leading terms $\lambda_i(\lambda_i + 2)/4, i = 1, \ldots, N$, and $\lambda_\infty(\lambda_\infty + 2)/4$;
- with **trivial monodromy**

These conditions $\Leftrightarrow$ the $\mathfrak{sl}_2$-oper is the Miura transformation of first-order diff. operator with reg. sing. & residues λ_i at z_i, λ_∞ at z_∞.

Generalization to an arbitrary simple Lie algebra $\mathfrak{g}$

It is easy to construct analogues of the (quadratic) Gaudin Hamiltonians using an invariant bilinear form on $\mathfrak{g}$:

$$H_i = \sum_{j \neq i} \frac{\sum_a J_a^{(i)} J^{a(j)}}{z_i - z_j}, \quad i = 1, \ldots, N$$

Questions:

- Are there higher order commuting Hamiltonians forming a commutative subalgebra $\mathcal{A} \subset U(\mathfrak{g})^{\otimes N}$?
- Is there an explicit formula for the eigenvectors?
- What are the corresponding Bethe Ansatz equations?
- Can we describe the spectrum in terms of geometric objects on $\mathbb{P}^1$ like opers?

Master Algebra

Feigin-F.-Reshetikhin (1994); F.'s ICMP'94 talk

Let $\widehat{\mathfrak{g}}$ be the affine Kac–Moody algebra associated to $\mathfrak{g}((t))$.

For $k \in \mathbb{C}$, let $\widetilde{U}(\widehat{\mathfrak{g}})_k$ be the completion of $U(\widehat{\mathfrak{g}})$ with level k.

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Example. The coefficients S_n of Sugawara current:

$$S(z) = \frac{1}{2} \sum_a : J^a(z) J_a(z) := \sum_{n \in \mathbb{Z}} S_n z^{-n-2}$$

Commutation relations: $[S_n, J^a_m] = -(k + h^\vee) m J^a_{n+m}$

where $h^\vee$ is the dual Coxeter number ($h^\vee = n$ for $\mathfrak{sl}_n$).

Thus, S_n are central elements of $\widetilde{U}(\widehat{\mathfrak{g}})_k$ when $k = -h^\vee$, **critical level**

The center of the completed enveloping algebra of $\widehat{\mathfrak{g}}$

Let $Z(\widehat{\mathfrak{g}})_k$ be the **center** of $\widetilde{U}(\widehat{\mathfrak{g}})_k$.

Theorem (Feigin-F.)

(1) $Z(\widehat{\mathfrak{g}})_{-h^\vee} \simeq \mathrm{Fun}\,\mathrm{Op}_{^LG}(D^\times)$

(2) If $k \neq -h^\vee$, then $Z(\widehat{\mathfrak{g}})_k = \mathbb{C}$.

$\mathrm{Op}_{^LG}(D^\times)$ – the space of ^LG-opers on $D^\times$, the punctured formal disc

Here ^LG – simple Lie group of adjoint type whose Lie algebra

$^L\mathfrak{g}$ is Langlands dual to $\mathfrak{g}$

Opers

If $\mathfrak{g} = \mathfrak{sl}_2$, then ${}^L G = PSL_2$ and PSL_2-oper on $D^\times$ is a second order differential operator $\partial_z^2 - v(z)$ where $v(z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-2}$.

Therefore, $\mathrm{Fun}\,\mathrm{Op}_{PSL_2}(D^\times)$ is a completion of $\mathbb{C}[v_n]^{\sim}_{n \in \mathbb{Z}}$.

The isomorphism $Z(V_{-2}(\mathfrak{sl}_2)) \simeq \mathrm{Fun}\,\mathrm{Op}_{PSL_2}(D_x)$

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${}^L G$-opers are, roughly speaking, gauge equivalence classes ${}^L\mathfrak{g}$-valued connections $\partial_z + A(z)$ on $D^\times$

(Drinfeld-Sokolov (1984), Beilinson-Drinfeld (2005))

$\mathrm{Fun}\,\mathrm{Op}_{{}^L G}(D^\times)$ is freely generated by $\ell = \mathrm{rk}(\mathfrak{g})$ series of elements $v_{i,n}, i = 1, \ldots, \ell; n \in \mathbb{Z}$, which under the F-F isomorphism correspond to *higher Sugawaras* in the center $Z(\widehat{\mathfrak{g}})_{-h^\vee}$.

Back to Gaudin model

Using $\widehat{\mathfrak{g}}$-conformal blocks, it is easy to construct a family of homomorphisms $Z(\widehat{\mathfrak{g}})_{-h^\vee} \to U(\mathfrak{g})^{\otimes N}$ depending on $\mathbf{z} = \{z_i, i = 1, \ldots, N\}$, such that

$$S(z) \mapsto \sum_{i=1}^{N} \frac{Cas^{(i)}}{(z - z_i)^2} + \sum_{i=1}^{N} \frac{H_i}{z - z_i}$$

Higher Sugawaras then give rise to *higher Gaudin Hamiltonians*.

The image $\mathcal{A}_{\mathbf{z}}$ is a commutative subalgebra of $U(\mathfrak{g})^{\otimes N}$ and the problem is to diagonalize its action on $\otimes_{i=1}^{N} V_{\lambda_i}$.

F-F-R constructed Bethe vectors and Bethe Ansatz equations using the free field (Wakimoto) realization of $\widehat{\mathfrak{g}}$. However, for $\mathfrak{g} \neq \mathfrak{sl}_2$ they do not always give rise to a basis of eigenvectors.

The spectrum and Langlands duality

Nonetheless, we can use the F-F isomorphism to describe the spectrum of $\mathcal{A}_{\mathbf{z}}$ on $\otimes_{i=1}^{N} V_{\lambda_i}$ directly without invoking Bethe vectors!

Theorem (Feigin-F.-Rybnikov)
The joint spectrum of the algebra $\mathcal{A}_{\mathbf{z}}$ of generalized Gaudin Hamiltonians on $\otimes_{i=1}^{N} V_{\lambda_i}$ is in bijection with the set of ${}^L G$-opers on $\mathbb{P}^1$ with regular singularities at $z_i, i = 1, \ldots, N$, and ∞ with the "leading terms" determined by $\lambda_i, i = 1, \ldots, N$, and λ_∞ and **trivial monodromy**.

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Gaiotto-Witten (2011) interpreted this result as a consequence of S-duality of 4d SYM, which we will discuss in a moment.

There is also a generalization with irregular singularities, Feigin-F.-Toledano Laredo & Rybnikov (2007).

Langlands Correspondence

In mathematics, Langlands correspondence can be formulated in 3 different domains (in the framework of André Weil's *Rosetta Stone*):

Number Fields *Curves over* $\mathbb{F}_q$ *Curves over* $\mathbb{C}$

Langlands initially formulated his correspondence (in the late 1960s) in the first two domains, aiming to solve difficult questions in Number Theory using tools of Harmonic Analysis.

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Starting in the 1980s, in the works of Deligne, Drinfeld, Laumon and others, similar structures were found in the third domain, giving rise to the *geometric Langlands correspondence*.

However, there was a significant **difference** between the formulations in the first two domains and the third.

In the first two domains, we have the Hilbert space of **functions** on a certain natural discrete set with a measure, attached to a reductive algebraic group G, and a family of commuting **Hecke operators** acting on it. Langlands correspondence describes their joint spectra in terms of homomorpisms of the relevant Galois group to ${}^LG$.

On the other hand, in the geometric Langlands correspondence for a Rieamann surface X, we have a category of **sheaves** on the moduli stack of G-bundles on X and **Hecke functors** acting on this category. The geometric Langlands correspondence can be viewed as an equivalence between this category and another category, associated to ${}^LG$.

The prevailing wisdom in the subject was that a function-theoretic formulation was not appropriate, or even possible, for complex curves. **This turned out to be incorrect!**

S-duality

Kapustin-Witten (2006) linked the geometric/categorical Langlands correspondence for a Riemann surface X to the S-duality of (twisted topological) $N = 4$ 4d SYM theories with gauge groups G_c and ${}^L G_c$ on the 4-manifold $\Sigma \times X$.

Specifically, to the equivalence of the corresponding categories of A- and B-branes on the *Hitchin moduli spaces*, which naturally appear after the 2d compactification along X (e.g. Hecke functors become 't Hooft line operators acting on A-branes, etc.). This has inspred a great deal of research in this area.

S-duality has an explanation in terms of string theory (Vafa (1998)): namely, we realize $N = 4$ 4d SYM theories as (orbifolds of) compactifications on dual tori of Type IIA (or IIB) string theories on ALE spaces & applying T-duality twice, for both circles on the torus.

Analytic Langlands Correspondence

Etingof-F.-Kazhdan (2019-2021) proposed a novel **analytic version** of the Langlands correspondence for complex curves (i.e. *function-theoretic* instead of *sheaf-theoretic*), following earlier works by Teschner (2017) and Langlands (2018).

Moreover, the two versions (categorical & analytic) complement each other. We can use each of them to gain new insights about the other.

Analogy: correlation functions in 2D conformal field theory are single-valued bilinear combinations of (multi-valued) conformal and anti-conformal blocks.

Gaiotto-Witten (2021) have given an elegant interpretation of the analytic Langlands correspondence in terms of the S-duality and the *brane quantization* (Gukov-Witten (2008))

A brief summary of E-F-K

For each pointed Riemann surface X and a Lie group G there is a Hilbert space $\mathcal{H}_{X,G}$ of half-densities on Bun_G and a family of commuting operators on it:

- Hecke operators (integral);
- differential operators, holomorphic (Beilinson-Drinfeld) and anti-holomorphic.

$X = \mathbb{CP}^1$ – these differential operators are the generalized Gaudin Hamiltonians (and their complex conjugates)!

Conjecture: The joint spectrum of these commuting operators can be identified with the set of ${}^L G$-opers on X whose monodromy is in the *split real form* ${}^L G(\mathbb{R})$ of ${}^L G(\mathbb{C})$.

This is the analytic Langlands correspondence for **curves over** $\mathbb{C}$.

This, and the existence of the commuting Hecke operators, implies that the spectrum can be expressed in terms of ${}^L G$-opers on $\mathbb{CP}^1$ (with singularities at our points) whose **monodromy** satisfies certain conditions (depending on the types of these representations).

The simplest case: finite-dimensional representations of $\mathfrak{g}$. Then the spectrum consists of ${}^L G$-opers with **trivial monodromy**.

Other types of representations $\Longrightarrow$ other monodromy conditions.

[A closely related description of the spectra of the Gaudin Hamiltonians has been obtained in some cases by Nekrasov-Rosly-Shatashvili (2011) by other methods.]

This kind of description suggests the following **modern version of Bethe Ansatz:**

It is no longer about finding explicit formulas for eigenvectors of the commuting quantum Hamiltonians but about describing their joint spectrum in terms of *dual classical geometric objects* (e.g. ${}^LG$-opers).

Such a description of the spectrum can be seen as a particular **duality**, which may well be related to a fundamental duality of QFT and/or String Theory. Our **challenge** is then to determine what it is.

(Finding a master algebra of commuting quantum Hamiltonians and finding its spectrum can also be helpful.)

For example, in the case of the Gaudin model, mathematically this duality is a special case of the Langlands duality, and it is a manifestation of S-duality of $N = 4$ 4d SYM theories, which can be derived from Type IIA/B String Theory.

I will now describe other examples.

Deformations

- We can consider $\widehat{\mathfrak{g}}$ *away from the critical level*. This corresponds to changing the coupling constant of the $N = 4$ 4d (twisted) SYM theory. We still have S-duality but there is no longer a *classical* side; both sides are quantum (**quantum Langlands**).
- We can stay at the critical level but deform $U(\widehat{\mathfrak{g}})$ to the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ (or the Yangian $Y(\mathfrak{g})$). Then Gaudin model gets deformed to a **quantum spin chain** of XXZ (or XXX) type, and on the other side opers become q-opers.
- We can do both deformations (**quantum q-Langlands**).
- We can go from $\mathfrak{g}$ to $\widehat{\mathfrak{g}}$, and hence from $\widehat{\mathfrak{g}}$ to a double loop algebra. As the result, we obtain **affine Gaudin models** and opers become affine opers. (We can also turn on q.)

Quantum Langlands

Recall the **F-F isomorphism:** $Z(\widehat{\mathfrak{g}})_{-h^\vee} \simeq \mathrm{Fun}\,\mathrm{Op}_{^LG}(D^\times)$

The RHS is actually the *classical $\mathcal{W}$-algebra $\mathcal{W}(^L\mathfrak{g})$* which is the Poisson algebra of functions on the phase space of $^L\mathfrak{g}$-**KdV system**.

The center $Z(\widehat{\mathfrak{g}})_{-h^\vee}$ also has a natural Poisson structure, and the F-F isomorphism is in fact an *isomorphism of Poisson algebras*.

Both algebras can be deformed: $\mathcal{W}(^L\mathfrak{g}) \rightsquigarrow \mathcal{W}_{^L\beta}(^L\mathfrak{g})$, where $^L\beta$ is a small parameter.

$Z(\widehat{\mathfrak{g}})_{-h^\vee} \rightsquigarrow \mathcal{W}_\beta(\mathfrak{g})$, where β is large.

Duality (F-F (1991)): $\mathcal{W}_\beta(\mathfrak{g}) \simeq \mathcal{W}_{^L\beta}(^L\mathfrak{g})$ if $\displaystyle {}^L\beta = \frac{1}{n_{\mathfrak{g}}\beta}$

This is connected to both T-duality and S-duality.

F-F isomorphism appears in the limit $\beta \to \infty$.

q-deformation

When we deform $\widehat{\mathfrak{g}}$ to $U_q(\widehat{\mathfrak{g}})$, the Gaudin model gets deformed to the XXZ spin chain for $\mathfrak{g} = \mathfrak{sl}_2$ and its generalizations.

In the simplest case, the space of states becomes $\otimes_{i=1}^N V_{\lambda_i}^q(z_i)$, where $V_{\lambda_i}^q$ is a finite-dimensional (level 0) representation of $U_q(\widehat{\mathfrak{g}})$. The parameters z_i are now the spectral paramers of these representations.

Commuting quantum Hamiltonians are the **transfer-matrices** $T_V(z)$, where $V \in \operatorname{Rep} U_g(\widehat{\mathfrak{g}})$, or more generally, $\operatorname{Rep} U_q(\widehat{\mathfrak{b}}_+)$.

The problem is to diagonalize them on $\otimes_{i=1}^N V_{\lambda_i}^q(z_i)$ (or more general representations of $U_g(\widehat{\mathfrak{g}})$).

For $\mathfrak{sl}_2$

Baxter (1972) Elegant reformulation of Bethe Ansatz:

Let $T(z)$ be the transfer-matrix of the 2-dim. rep. of $U_q(\widehat{\mathfrak{sl}}_2)$, and let $t(z)$ be one of its eigenvalues in $\otimes_{i=1}^N V_{\lambda_i}^q(z_i)$.

Consider the q-difference equation (**Baxter's TQ-relation**):

$$(D^2 - t(z)D + 1)Q(z) = 0, \qquad (D \cdot f)(z) = f(zq^2).$$

Then there is a unique solution $Q(z)$ which is a polynomial (*Baxter polynomial*), up to a universal factor that is the same for all eigenvalues in $\otimes_{i=1}^N V_{\lambda_i}^q(z_i)$. Its roots satisfy the Bethe Ansatz eqs.

Moreover, all solutions of this second order q-difference equation (q-oper!) are then polynomials, up to the same universal factor – this is the q-analogue of the **no monodromy** condition we have encountered in Gaudin model.

The TQ-relation is a q-analogue of the Miura transformation appearing in the formula for the eigenvalues of the Gaudin Hamiltonians.

There exist analogues of the Baxter TQ-relation for a general Lie algebra $\mathfrak{g}$ in terms of the q-characters. (F.-Reshetikhin (1998), F.-Hernandez (2014)):

There are now $\ell = \text{rank}(\mathfrak{g})$ Baxter polynomials $Q_i(z)$, and the eigenvalues of $t_V(z)$ can be written in terms of these $Q_i(z)$.

Beautiful fact: $Q_i(z)$'s are transfer-matrices of special ∞-dim. representations, which actually satisfy a system of relations among themselves, called the **QQ-system**. This system leads to a more concise description of the spectra of quantum Hamiltonians for $U_q(\widehat{\mathfrak{g}})$.

For $\mathfrak{sl}_2$: the q-Wronskian relation of B-L-Z (1996).

For $\mathfrak{sl}_n$: Bazhanov-Frassek-Łukowski-Meneghelli-Staudacher (2011).

For a general simple Lie algebra $\mathfrak{g}$: F.-Hernandez (2016 & to appear).

QQ-system for $\mathfrak{gl}(4|4)$ plays an important role in N=4 4d SYM (and AdS_5/CFT_4 correspondence) – *quantum spectral curve* of Gromov-Kazakov-Leurent-Volin (2013).

Moreover, it also appeared in the study of the spectra of **affine opers** that appear on the **dual side** of quantum KdV (Masoero-Raimondo-Valeri (2015)), which we'll discuss shortly.

q-opers

The QQ-system can be described (at least for simply-laced $\mathfrak{g}$) in terms of **Miura** (G, q)**-opers** (F.-Koroteev-Sage-Zeitlin (2020))

Closely related work on *fused flags* by Ekhammar-Shu-Volin (2021)

$$\boxed{\begin{array}{c}\text{spectrum of } U_q(\widehat{\mathfrak{g}}) \\ \text{Hamiltonians}\end{array}} \quad \leftrightarrow \quad \boxed{\begin{array}{c}({}^L G, q)\text{-opers} \\ \text{with Miura structure}\end{array}}$$

This is a q-deformation of the Langlands duality we discussed earlier:

$$\boxed{\begin{array}{c}\text{spectrum of } \mathfrak{g}\text{-Gaudin} \\ \text{Hamiltonians}\end{array}} \quad \leftrightarrow \quad \boxed{\begin{array}{c}{}^L G\text{-opers} \\ \text{with Miura structure} \\ \Leftrightarrow \ \text{no monodromy}\end{array}}$$

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Quantum q-Langlands and String Duality

When we turn on both parameters, q and $k + h^\vee$, we obtain **quantum q-Langlands** duality (Aganagic-F.-Okounkov (2017)):

Origin: Duality in little string theory on an ALE space times a torus, with non-zero string tension (which corresponds to $k + h^\vee \neq 0$).

Affine Gaudin models

We now keep $q = 1$ but replace $\mathfrak{g}$ by $\widehat{\mathfrak{g}}$, so $\widehat{\mathfrak{g}}$ should be replaced by $\widehat{\widehat{\mathfrak{g}}}$.

Then Gaudin model $\rightsquigarrow$ **affine Gaudin model** (Feigin-F. (2007)).

Classical L-operator of the Gaudin model (with irreg. sing. at ∞):

$$L = \sum_{i=1}^{N} \frac{A_i}{z - z_i} + \chi, \qquad A_i \in \mathfrak{g}^*, \qquad \text{with fixed}\, \chi \in \mathfrak{g}^*$$

In the affine Gaudin model:

$$L = \sum_{i=1}^{N} \frac{\partial_t + A_i(t)}{z - z_i} + \chi, \qquad \partial_t + A_i(t) \in \widehat{\mathfrak{g}}_1^*$$

Langlands duality for quantum KdV

Recall that for the finite Gaudin model:

spectrum of $\mathfrak{g}$-Gaudin Hamiltonians	$\leftrightarrow$	${}^L G$-opers with trivial monodromy

Now, for the affine Gaudin model:

spectrum of $\widehat{\mathfrak{g}}$-Gaudin Hamiltonians	$\leftrightarrow$	affine ${}^L\widehat{G}$-opers with trivial monodromy

Quantum KdV: Link via the QQ-system

$$\begin{tikzcd}[column sep=small]
\boxed{\begin{array}{c}\text{spectra of quantum}\\ \widehat{\mathfrak{g}}\text{-KdV Hamiltonians}\end{array}} \arrow[rr, leftrightarrow] \arrow[dr, "{[\mathrm{FH}]}"] & & \boxed{\begin{array}{c}\text{affine } {}^L\widehat{G}\text{-opers}\\ \text{no monodromy}\end{array}} \arrow[dl, "{[\mathrm{BLZ, MRV, MR}]}"'] \\
& \boxed{\begin{array}{c}\text{solutions of}\\ \text{the } QQ\text{-system}\end{array}} &
\end{tikzcd}$$

[BLZ] Bazhanov-Lukyanov-Zamolodchikov (2003)
[MRV] Masoero-Raimondo-Valeri (2015)
[MR] Masoero-Raimondo (2018)
[FH] F.-Hernandez (2016)

Open Problems

- Find the master algebra of affine Gaudin models (it can be viewed as an analogue the center of the enveloping algebra of a double loop algebra at its "critical level")
- Is there a String Theory explanation for the affine Gaudin models' Langlands duality?
- Is there a q-deformation of affine Gaudin models, and if so, what is the corresponding Bethe Ansatz?
- Are there applications of "Bethe Ansatz" to 4d gauge theories like QCD (Richard Feynman's dream)?