Title: Talk 67 - Irreversibility, QNEC, and defects

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Abstract: In this talk, we will first present an analysis of infinitesimal null deformations for the entanglement entropy, which leads to a major simplification of the proof of the C, F and A-theorems in quantum field theory. Next, we will discuss the quantum null energy condition (QNEC) on the light-cone. Finally, we combine these tools in order to establish the irreversibility of renormalization group flows on planar d-dimensional defects, embedded in D-dimensional conformal field theories. This proof completes and unifies all known defect irreversibility theorems for defect dimensions below d=5. The F-theorem on defects (d=3) is a new result using information-theoretic methods. The geometric construction connects the proof of irreversibility with and without defects through the QNEC inequality in the bulk, and makes contact with the proof of strong subadditivity of holographic entropy taking into account quantum corrections.

Irreversibility, QNEC, and defects (II)

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It From Qubit Conference,
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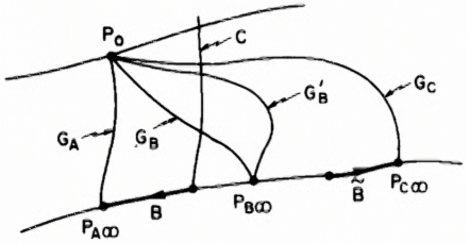
Laws of thermodynamics constraint the behavior of macroscopic systems. In recent years, Quantum Information methods have provided similar laws for quantum matter at zero temperature.

This has been important in QFT, which has fixed points, described by CFTs, and RG flows that connect them.



Intuitive notion of irreversibility: contributions from microscopic degrees of freedom decouple from the long distance physics.

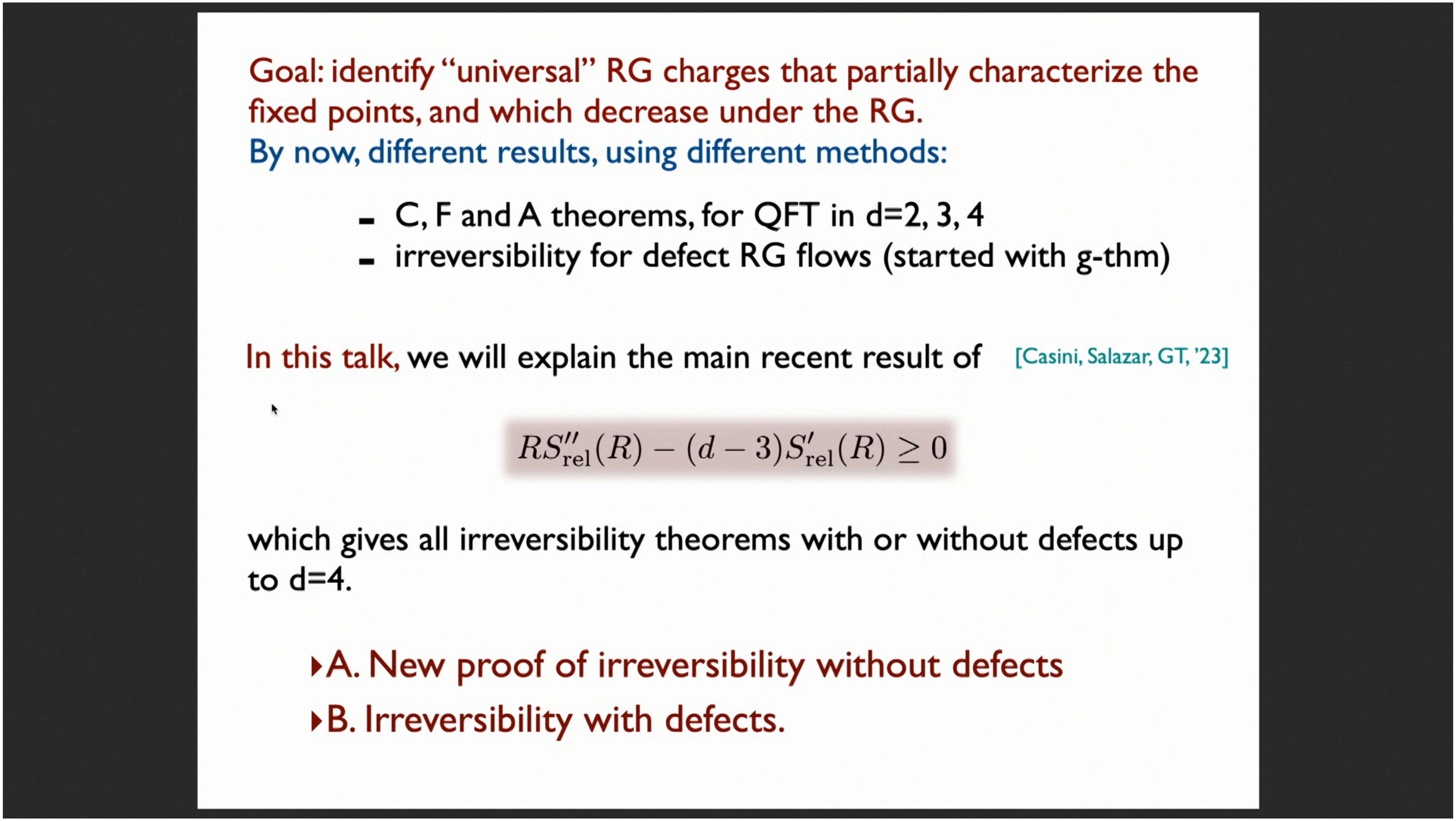
First irreversibility theorem:

"Irreversibility" of the flux of the renormalization group in a 2D field theory

A. B. Zomolodchikov

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR

1986



Goal: identify "universal" RG charges that partially characterize the fixed points, and which decrease under the RG.

By now, different results, using different methods:

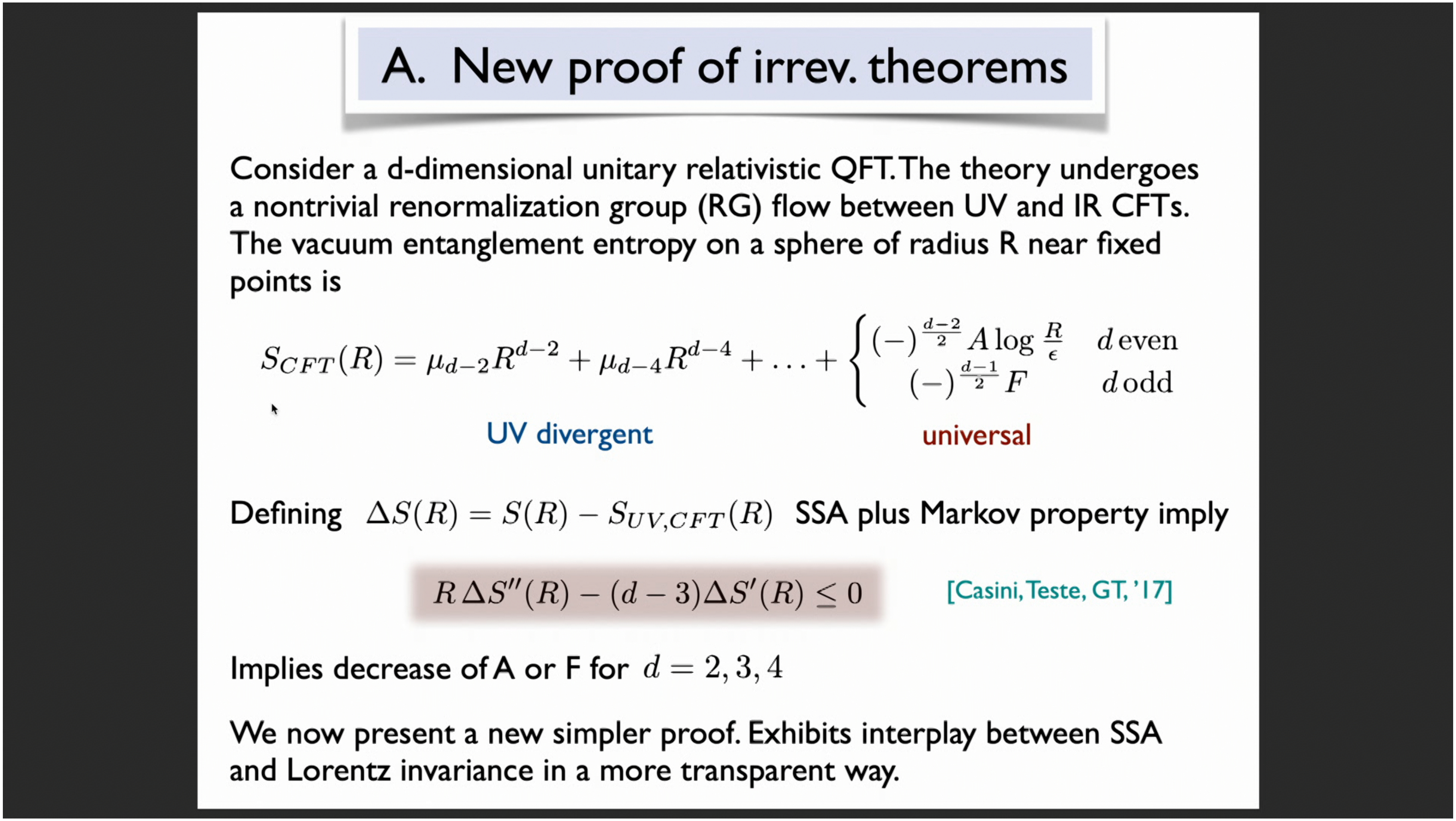
- C, F and A theorems, for QFT in d=2, 3, 4
- irreversibility for defect RG flows (started with g-thm)

In this talk, we will explain the main recent result of [Casini, Salazar, GT, '23]

$$RS''_{\text{rel}}(R) - (d-3)S'_{\text{rel}}(R) \geq 0$$

which gives all irreversibility theorems with or without defects up to d=4.

- ▸A. New proof of irreversibility without defects
- ▸B. Irreversibility with defects.



A. New proof of irrev. theorems

Consider a d-dimensional unitary relativistic QFT. The theory undergoes a nontrivial renormalization group (RG) flow between UV and IR CFTs. The vacuum entanglement entropy on a sphere of radius R near fixed points is

$$S_{CFT}(R) = \mu_{d-2} R^{d-2} + \mu_{d-4} R^{d-4} + \ldots + \begin{cases} (-)^{\frac{d-2}{2}} A \log \frac{R}{\epsilon} & d\,\text{even} \\ (-)^{\frac{d-1}{2}} F & d\,\text{odd} \end{cases}$$

UV divergent

universal

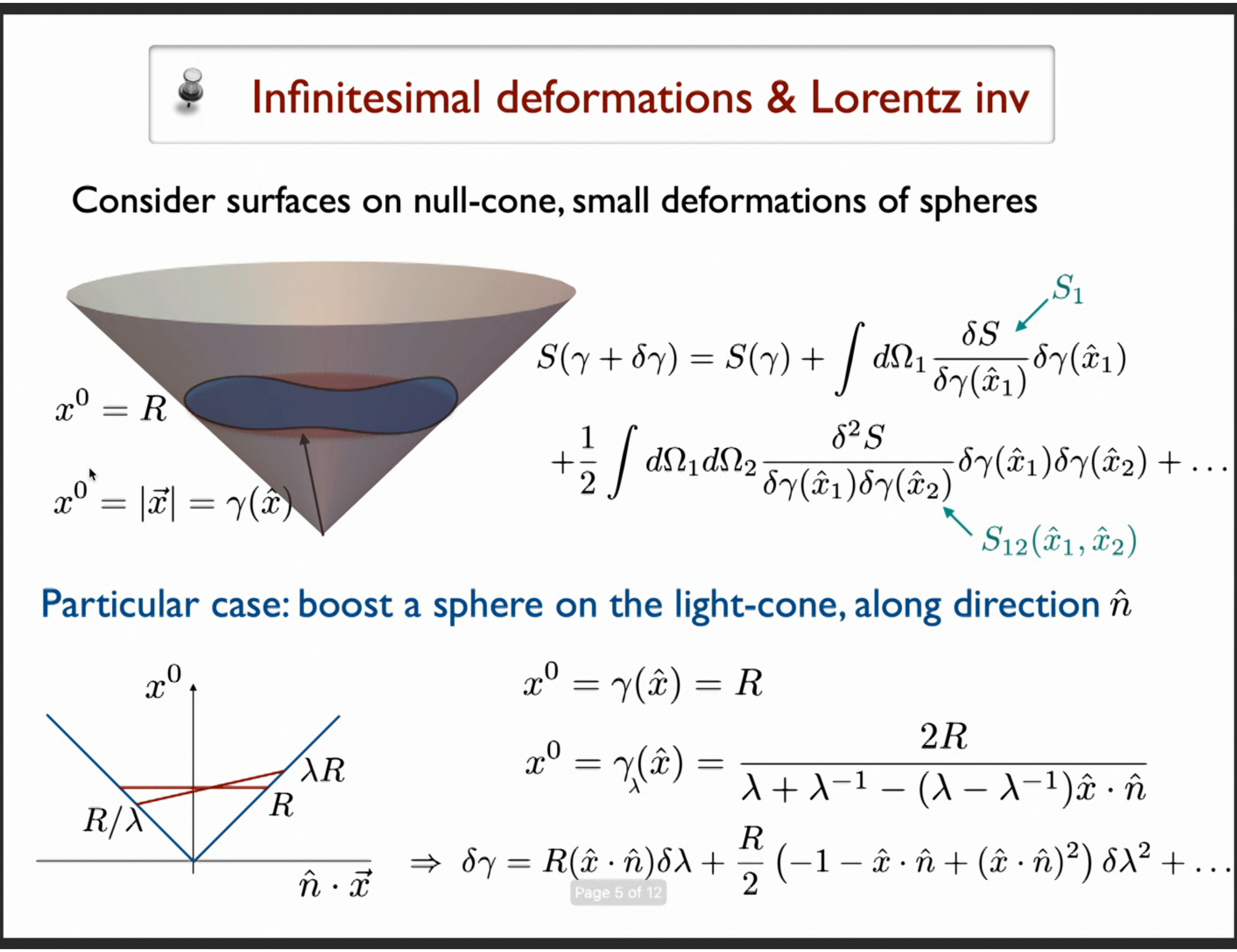
Defining $\Delta S(R) = S(R) - S_{UV,CFT}(R)$ SSA plus Markov property imply

$$R\,\Delta S''(R) - (d-3)\Delta S'(R) \leq 0$$

[Casini, Teste, GT, '17]

Implies decrease of A or F for $d = 2, 3, 4$

We now present a new simpler proof. Exhibits interplay between SSA and Lorentz invariance in a more transparent way.



Infinitesimal deformations & Lorentz inv

Consider surfaces on null-cone, small deformations of spheres

S_1

$$S(\gamma+\delta\gamma) = S(\gamma) + \int d\Omega_1 \frac{\delta S}{\delta\gamma(\hat{x}_1)}\delta\gamma(\hat{x}_1)$$

$$+\frac{1}{2}\int d\Omega_1 d\Omega_2 \frac{\delta^2 S}{\delta\gamma(\hat{x}_1)\delta\gamma(\hat{x}_2)}\delta\gamma(\hat{x}_1)\delta\gamma(\hat{x}_2) + \dots$$

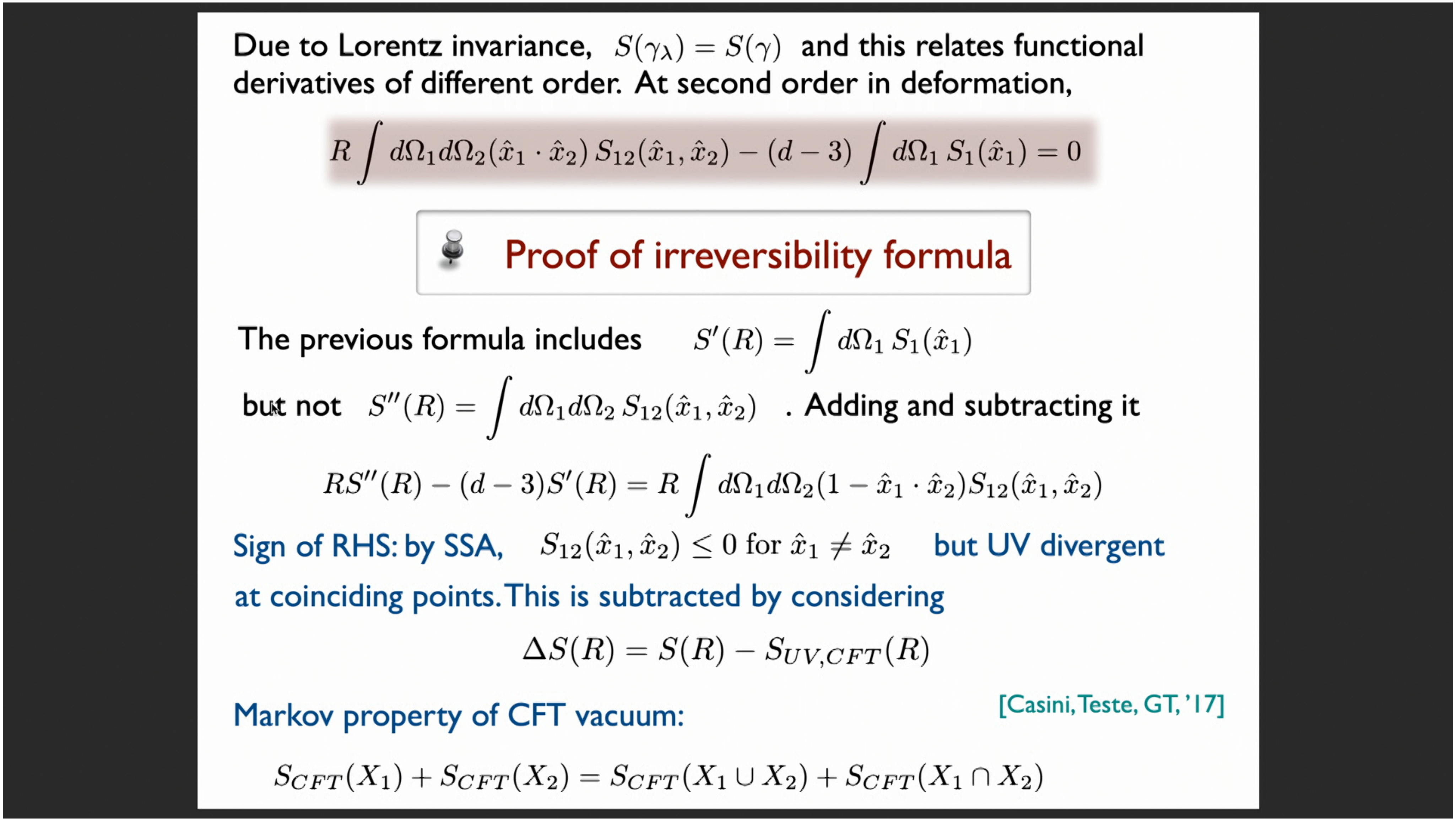
$S_{12}(\hat{x}_1, \hat{x}_2)$

Particular case: boost a sphere on the light-cone, along direction $\hat{n}$

$$x^0 = \gamma(\hat{x}) = R$$

$$x^0 = \gamma_{\lambda}(\hat{x}) = \frac{2R}{\lambda + \lambda^{-1} - (\lambda - \lambda^{-1})\hat{x}\cdot\hat{n}}$$

$$\Rightarrow \delta\gamma = R(\hat{x}\cdot\hat{n})\delta\lambda + \frac{R}{2}\left(-1 - \hat{x}\cdot\hat{n} + (\hat{x}\cdot\hat{n})^2\right)\delta\lambda^2 + \dots$$



Due to Lorentz invariance, $S(\gamma_\lambda) = S(\gamma)$ and this relates functional derivatives of different order. At second order in deformation,

$$R \int d\Omega_1 d\Omega_2 (\hat{x}_1 \cdot \hat{x}_2)\, S_{12}(\hat{x}_1, \hat{x}_2) - (d-3) \int d\Omega_1\, S_1(\hat{x}_1) = 0$$

Proof of irreversibility formula

The previous formula includes $S'(R) = \int d\Omega_1\, S_1(\hat{x}_1)$

but not $S''(R) = \int d\Omega_1 d\Omega_2\, S_{12}(\hat{x}_1, \hat{x}_2)$. Adding and subtracting it

$$RS''(R) - (d-3)S'(R) = R \int d\Omega_1 d\Omega_2 (1 - \hat{x}_1 \cdot \hat{x}_2) S_{12}(\hat{x}_1, \hat{x}_2)$$

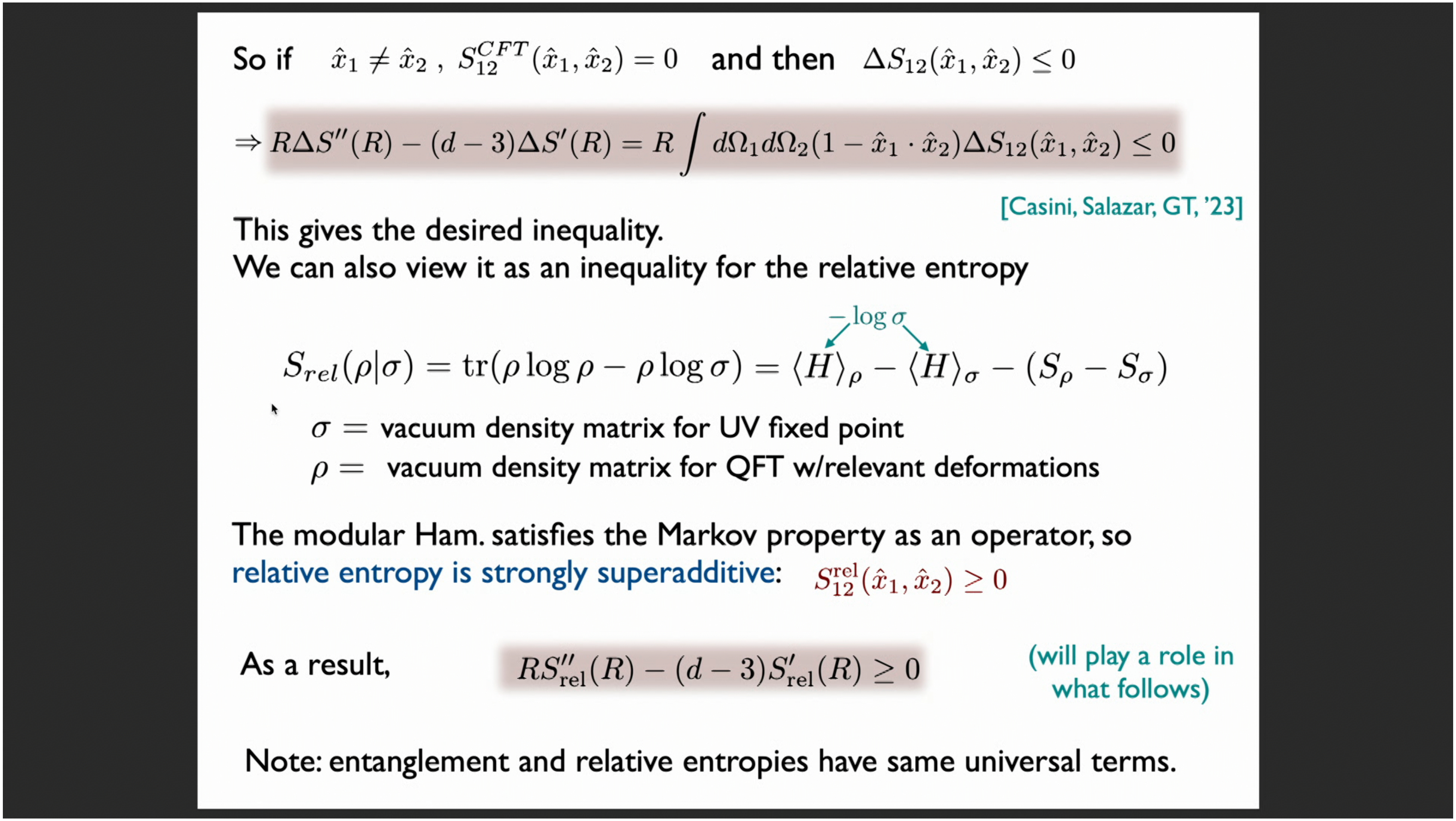
Sign of RHS: by SSA, $S_{12}(\hat{x}_1, \hat{x}_2) \leq 0 \text{ for } \hat{x}_1 \neq \hat{x}_2$ but UV divergent at coinciding points. This is subtracted by considering

$$\Delta S(R) = S(R) - S_{UV,CFT}(R)$$

Markov property of CFT vacuum:

[Casini, Teste, GT, '17]

$$S_{CFT}(X_1) + S_{CFT}(X_2) = S_{CFT}(X_1 \cup X_2) + S_{CFT}(X_1 \cap X_2)$$



So if $\hat{x}_1 \neq \hat{x}_2$, $S_{12}^{CFT}(\hat{x}_1,\hat{x}_2) = 0$ and then $\Delta S_{12}(\hat{x}_1,\hat{x}_2) \leq 0$

$$\Rightarrow R\Delta S''(R) - (d-3)\Delta S'(R) = R\int d\Omega_1 d\Omega_2 (1 - \hat{x}_1\cdot\hat{x}_2)\Delta S_{12}(\hat{x}_1,\hat{x}_2) \leq 0$$

[Casini, Salazar, GT, '23]

This gives the desired inequality.
We can also view it as an inequality for the relative entropy

$$-\log\sigma$$

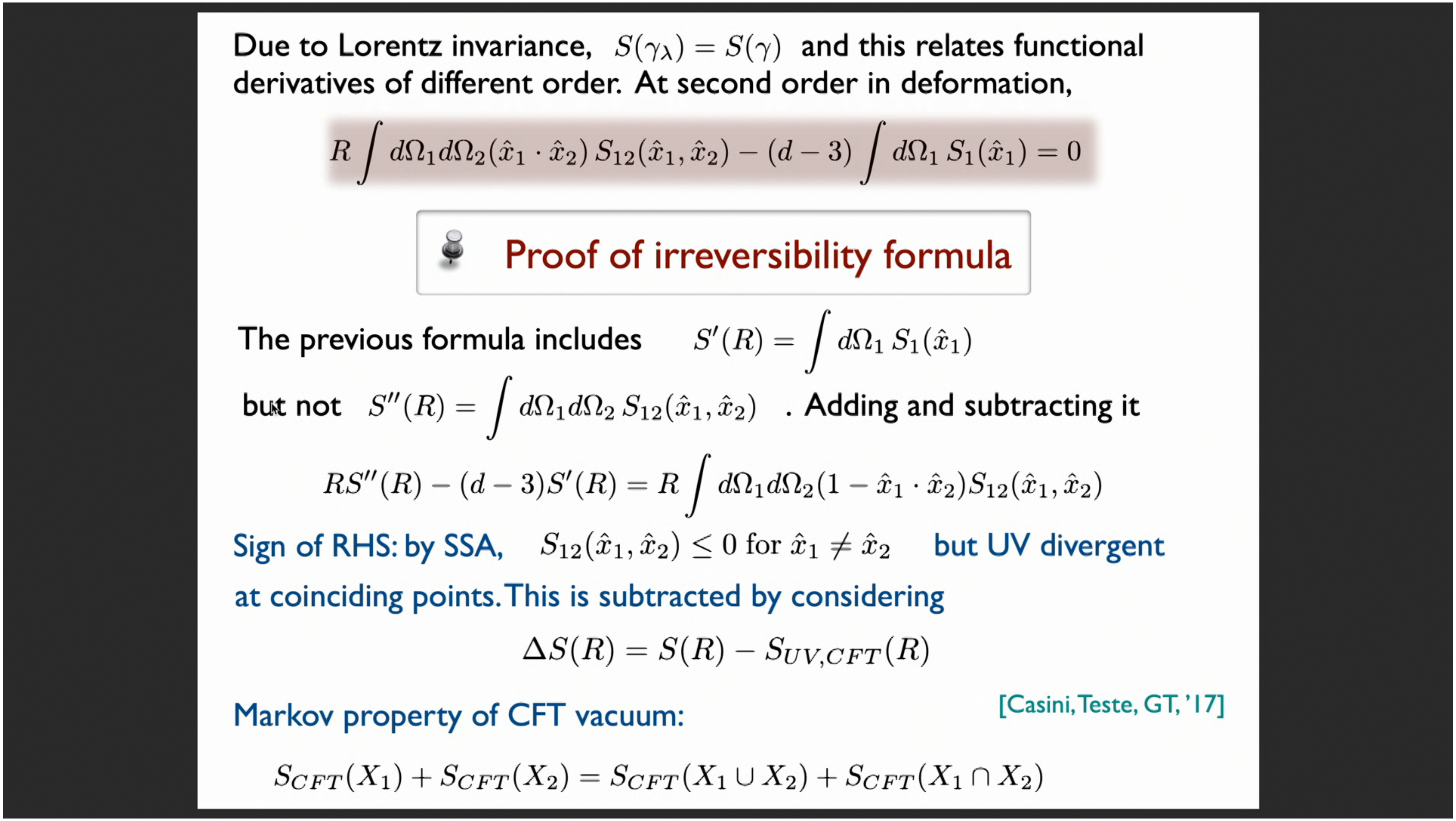
$$S_{rel}(\rho|\sigma) = \mathrm{tr}(\rho\log\rho - \rho\log\sigma) = \langle H\rangle_\rho - \langle H\rangle_\sigma - (S_\rho - S_\sigma)$$

$\sigma =$ vacuum density matrix for UV fixed point
$\rho =$ vacuum density matrix for QFT w/relevant deformations

The modular Ham. satisfies the Markov property as an operator, so relative entropy is strongly superadditive: $S_{12}^{\mathrm{rel}}(\hat{x}_1,\hat{x}_2) \geq 0$

As a result, $RS''_{\mathrm{rel}}(R) - (d-3)S'_{\mathrm{rel}}(R) \geq 0$ (will play a role in what follows)

Note: entanglement and relative entropies have same universal terms.



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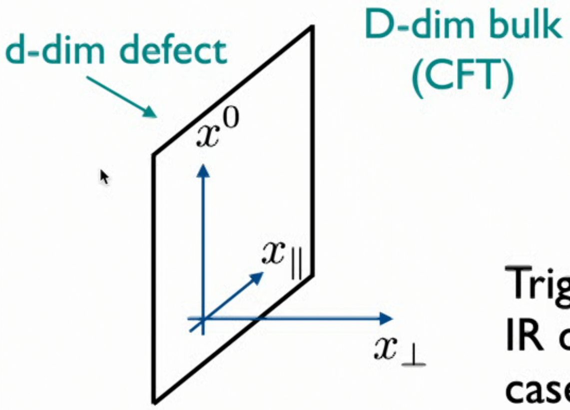
Markov property of CFT vacuum: [Casini, Teste, GT, '17]

$$S_{CFT}(X_1)+S_{CFT}(X_2)=S_{CFT}(X_1\cup X_2)+S_{CFT}(X_1\cap X_2)$$

B. Irreversibility with defects

Now consider D-dimensional CFT with a d-dimensional planar defect. This situation is of interest in condensed matter (Kondo model), gauge theories (Wilson lines), string theory (D-branes), theories with boundaries …



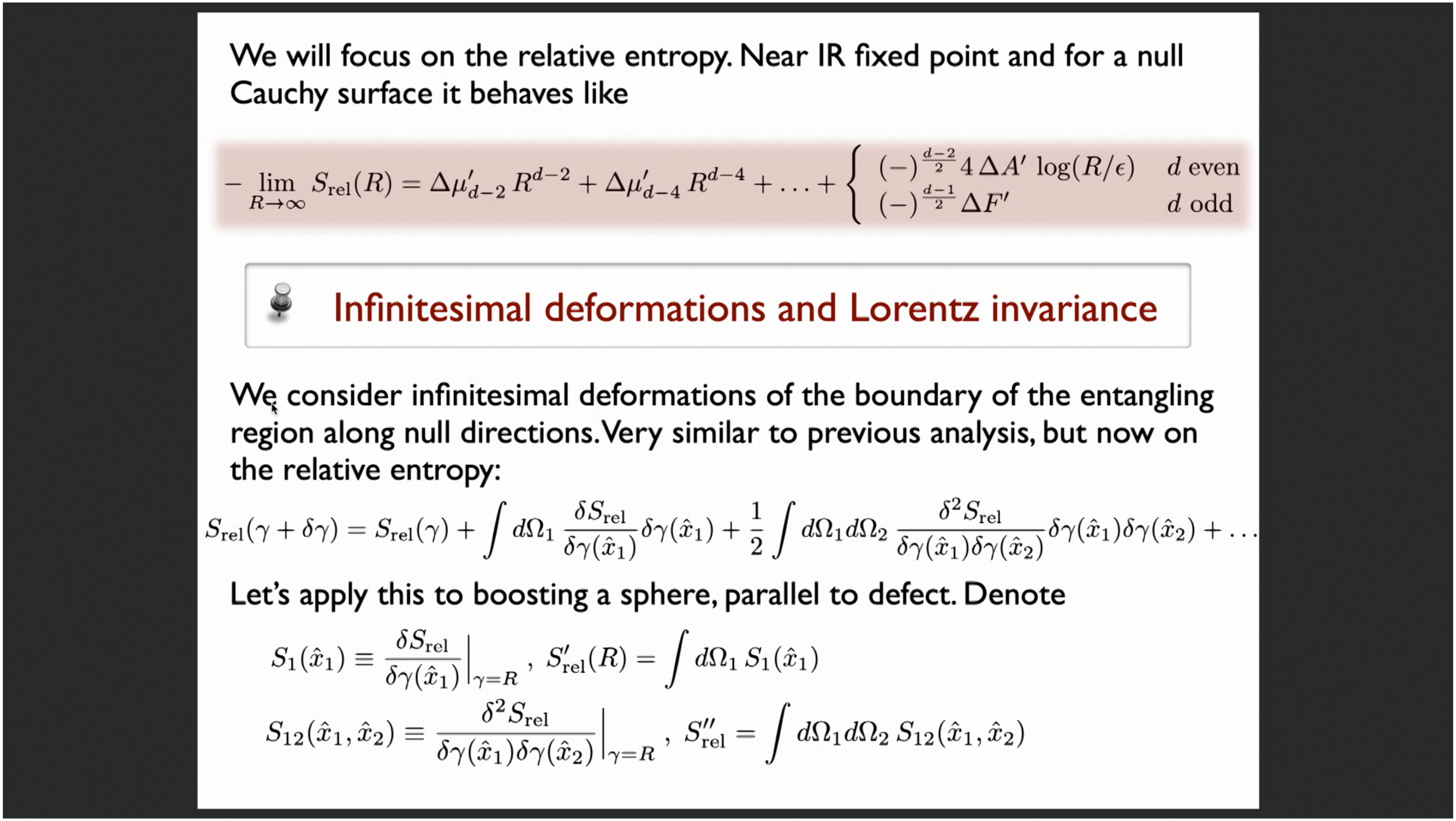
The defect is conformal in the UV; turn on relevant deformations on defect

$$S = S_{UV,CFT} + \int d^dx\, g\mathcal{O}$$

Triggers an RG flow, which we assume ends on an IR defect. The bulk is always conformal. Particular case D=d: equivalent to QFT with no defect.

Consider the EE on a sphere centered on the defect. One can try to prove irreversibility inequalities for its universal terms. But in general this does not work. The defect contributes energy; the universal terms in the EE and those of the free energy no longer agree.

[Kobayashi, Nishioka, Sato, Watanabe] [Jensen, O'Bannon, Robinson, Rodgers]



We will focus on the relative entropy. Near IR fixed point and for a null Cauchy surface it behaves like

$$-\lim_{R\to\infty} S_{\rm rel}(R) = \Delta\mu'_{d-2}\,R^{d-2} + \Delta\mu'_{d-4}\,R^{d-4} + \ldots + \begin{cases} (-)^{\frac{d-2}{2}}4\,\Delta A'\,\log(R/\epsilon) & d \text{ even} \\ (-)^{\frac{d-1}{2}}\Delta F' & d \text{ odd} \end{cases}$$

Infinitesimal deformations and Lorentz invariance

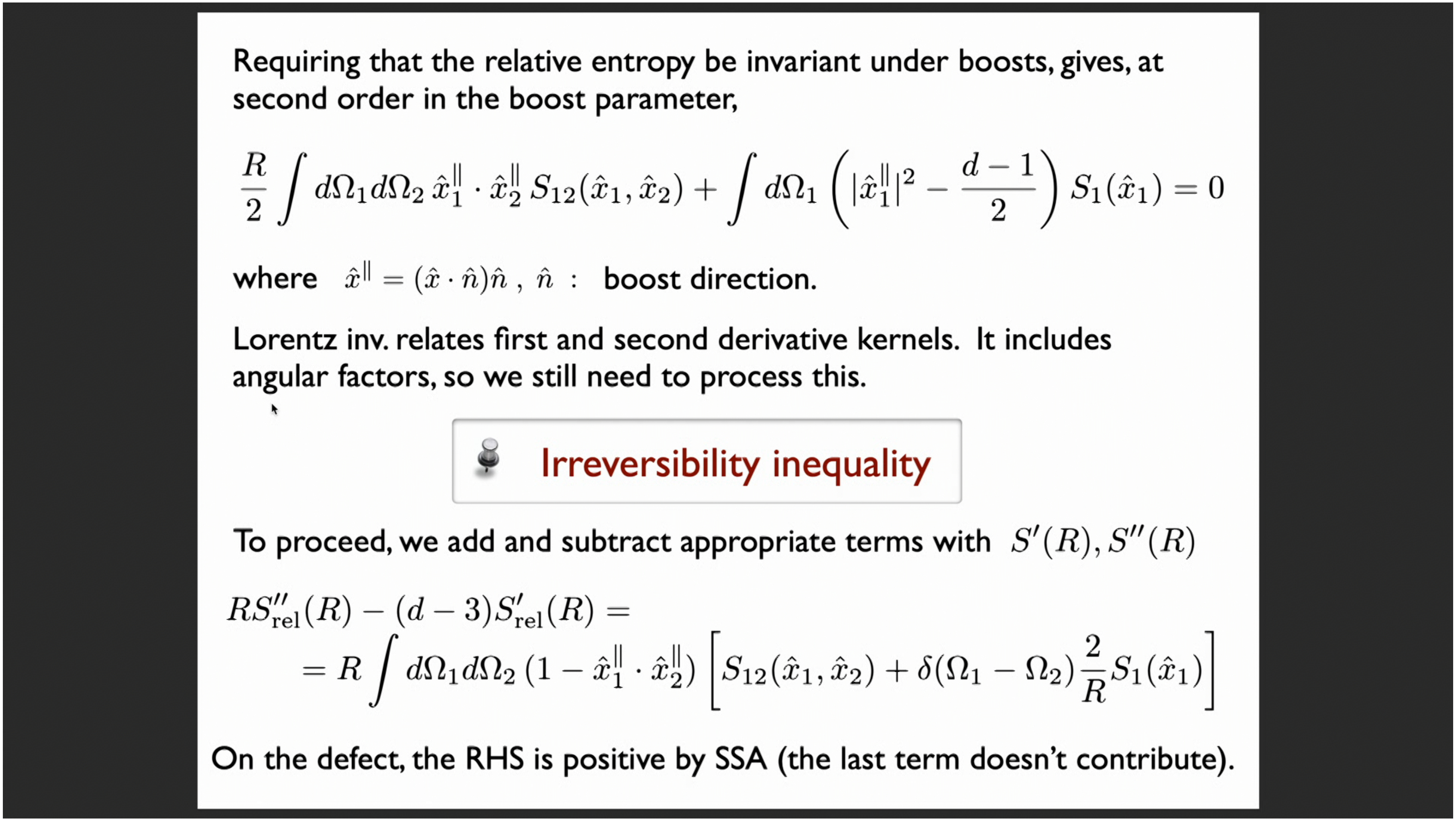
We consider infinitesimal deformations of the boundary of the entangling region along null directions. Very similar to previous analysis, but now on the relative entropy:

$$S_{\rm rel}(\gamma+\delta\gamma) = S_{\rm rel}(\gamma) + \int d\Omega_1\,\frac{\delta S_{\rm rel}}{\delta\gamma(\hat{x}_1)}\delta\gamma(\hat{x}_1) + \frac{1}{2}\int d\Omega_1 d\Omega_2\,\frac{\delta^2 S_{\rm rel}}{\delta\gamma(\hat{x}_1)\delta\gamma(\hat{x}_2)}\delta\gamma(\hat{x}_1)\delta\gamma(\hat{x}_2) + \ldots$$

Let's apply this to boosting a sphere, parallel to defect. Denote

$$S_1(\hat{x}_1) \equiv \frac{\delta S_{\rm rel}}{\delta\gamma(\hat{x}_1)}\Big|_{\gamma=R}\,,\quad S'_{\rm rel}(R) = \int d\Omega_1\,S_1(\hat{x}_1)$$

$$S_{12}(\hat{x}_1,\hat{x}_2) \equiv \frac{\delta^2 S_{\rm rel}}{\delta\gamma(\hat{x}_1)\delta\gamma(\hat{x}_2)}\Big|_{\gamma=R}\,,\quad S''_{\rm rel} = \int d\Omega_1 d\Omega_2\,S_{12}(\hat{x}_1,\hat{x}_2)$$



Requiring that the relative entropy be invariant under boosts, gives, at second order in the boost parameter,

$$\frac{R}{2}\int d\Omega_1 d\Omega_2\, \hat{x}_1^{\|}\cdot\hat{x}_2^{\|}\, S_{12}(\hat{x}_1,\hat{x}_2)+\int d\Omega_1\left(|\hat{x}_1^{\|}|^2-\frac{d-1}{2}\right)S_1(\hat{x}_1)=0$$

where $\hat{x}^{\|}=(\hat{x}\cdot\hat{n})\hat{n}\ ,\ \hat{n}\ :$ boost direction.

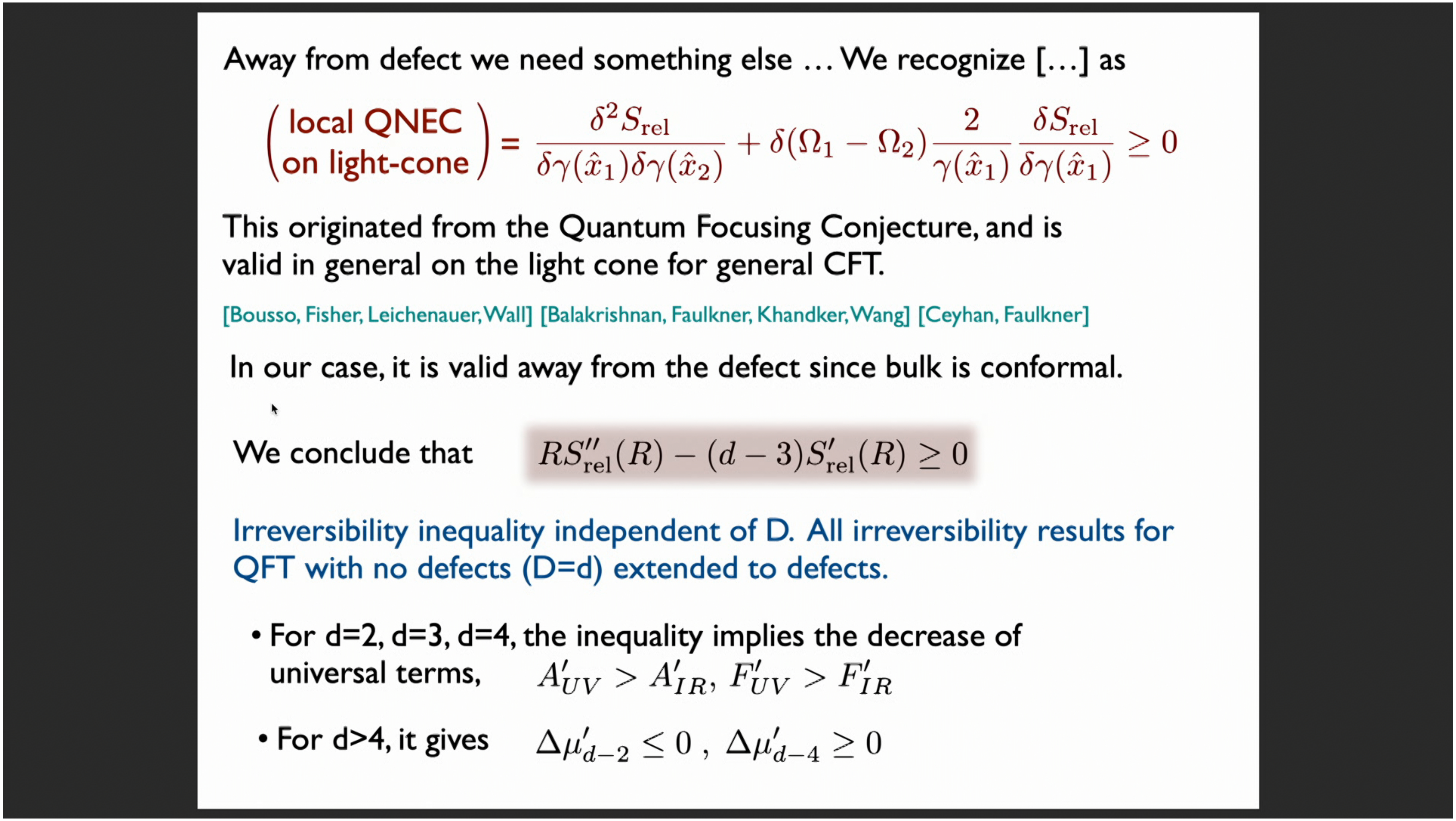
Lorentz inv. relates first and second derivative kernels. It includes angular factors, so we still need to process this.

Irreversibility inequality

To proceed, we add and subtract appropriate terms with $S'(R), S''(R)$

$$RS''_{\rm rel}(R)-(d-3)S'_{\rm rel}(R)=$$
$$=R\int d\Omega_1 d\Omega_2\,(1-\hat{x}_1^{\|}\cdot\hat{x}_2^{\|})\left[S_{12}(\hat{x}_1,\hat{x}_2)+\delta(\Omega_1-\Omega_2)\frac{2}{R}S_1(\hat{x}_1)\right]$$

On the defect, the RHS is positive by SSA (the last term doesn't contribute).



Away from defect we need something else … We recognize […] as

$$\left(\begin{matrix}\text{local QNEC}\\ \text{on light-cone}\end{matrix}\right) = \frac{\delta^2 S_{\rm rel}}{\delta\gamma(\hat{x}_1)\delta\gamma(\hat{x}_2)} + \delta(\Omega_1 - \Omega_2)\frac{2}{\gamma(\hat{x}_1)}\frac{\delta S_{\rm rel}}{\delta\gamma(\hat{x}_1)} \geq 0$$

This originated from the Quantum Focusing Conjecture, and is valid in general on the light cone for general CFT.

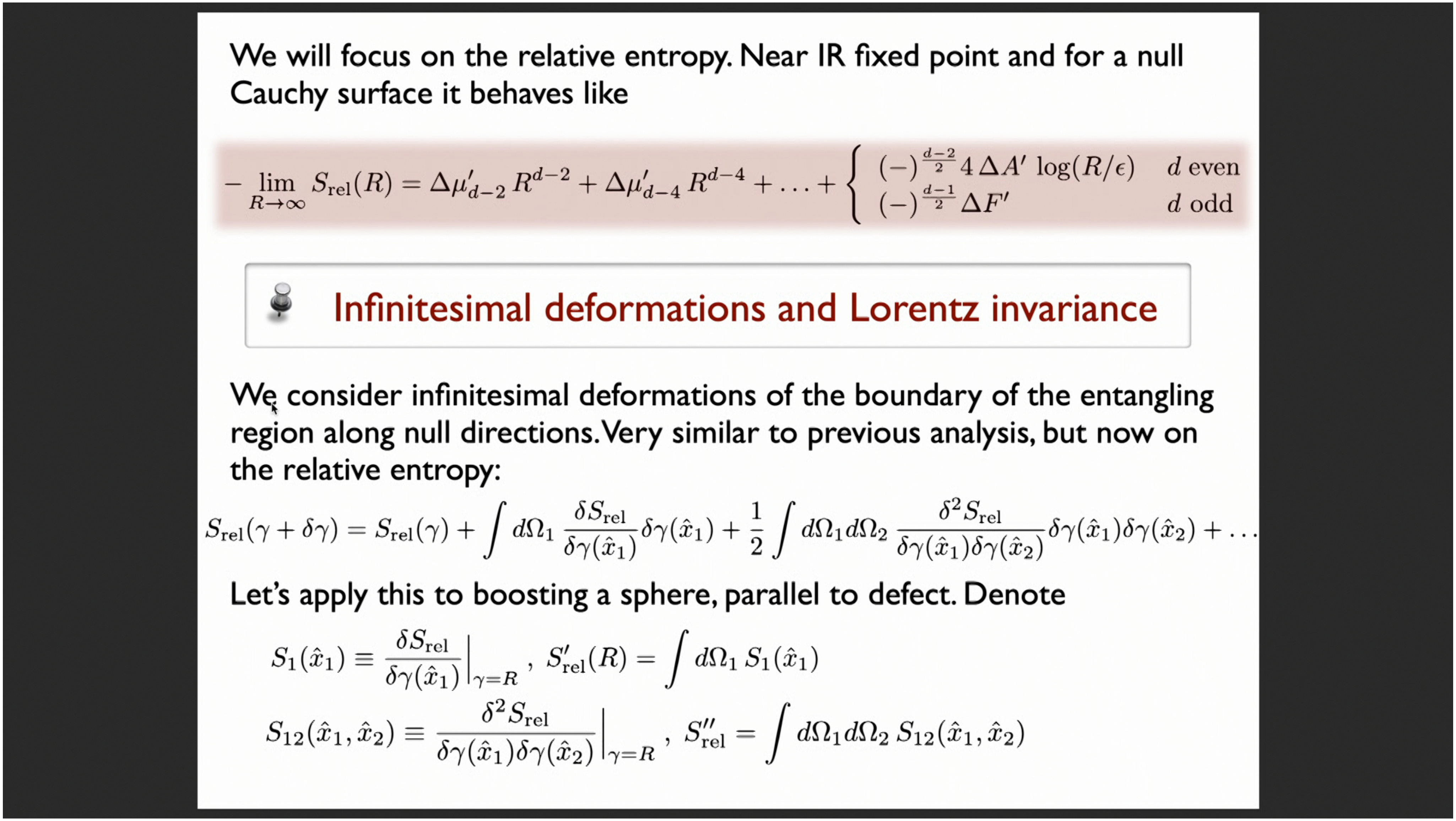
[Bousso, Fisher, Leichenauer, Wall] [Balakrishnan, Faulkner, Khandker, Wang] [Ceyhan, Faulkner]

In our case, it is valid away from the defect since bulk is conformal.

We conclude that $$R S''_{\rm rel}(R) - (d-3) S'_{\rm rel}(R) \geq 0$$

Irreversibility inequality independent of D. All irreversibility results for QFT with no defects (D=d) extended to defects.

- For d=2, d=3, d=4, the inequality implies the decrease of universal terms, $A'_{UV} > A'_{IR}$, $F'_{UV} > F'_{IR}$
- For d>4, it gives $\Delta\mu'_{d-2} \leq 0\ ,\ \Delta\mu'_{d-4} \geq 0$



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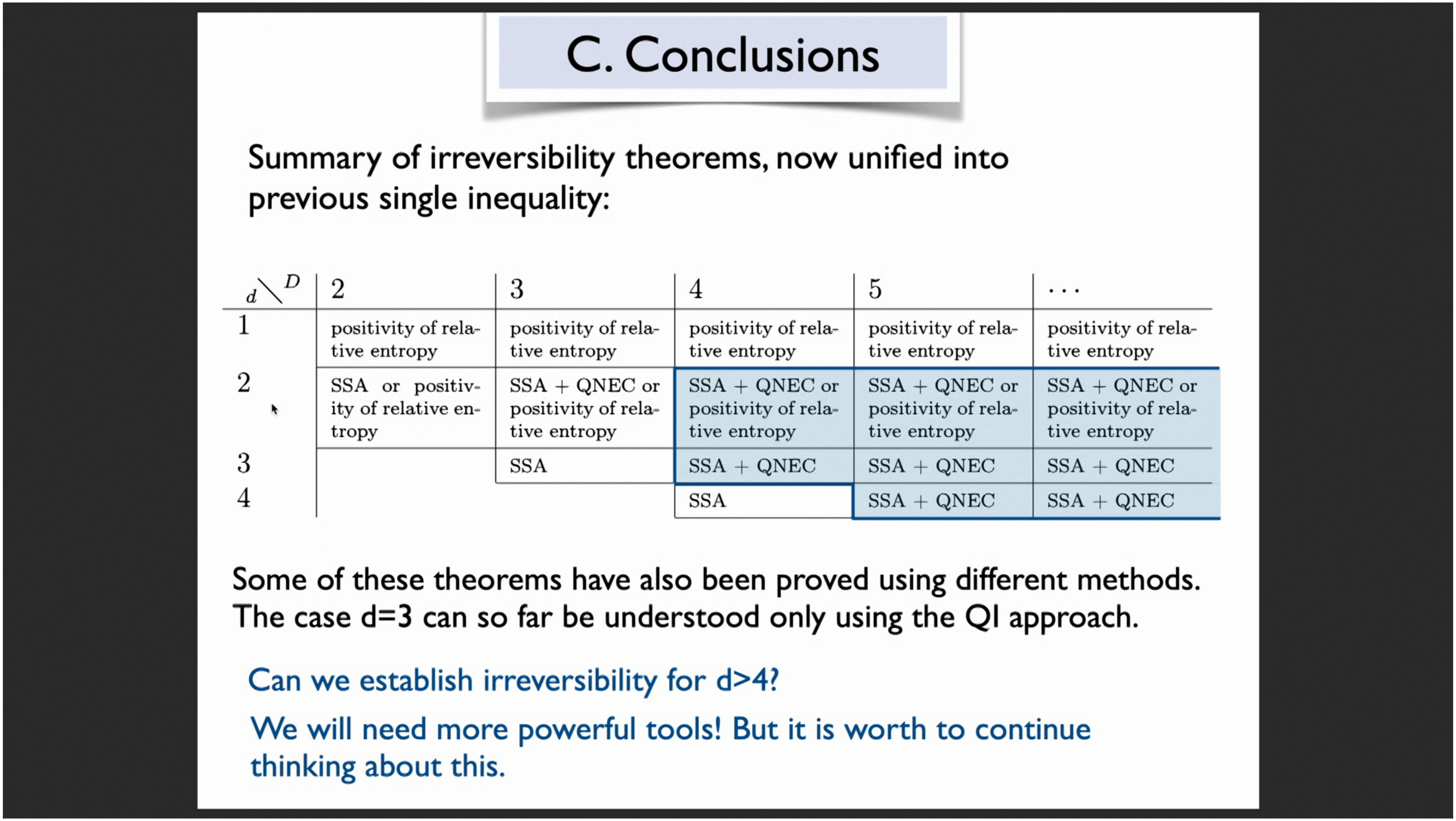

C. Conclusions

Summary of irreversibility theorems, now unified into previous single inequality:

$d \diagdown D$	2	3	4	5	$\cdots$
1	positivity of relative entropy	positivity of relative entropy	positivity of relative entropy	positivity of relative entropy	positivity of relative entropy
2	SSA or positivity of relative entropy	SSA + QNEC or positivity of relative entropy	SSA + QNEC or positivity of relative entropy	SSA + QNEC or positivity of relative entropy	SSA + QNEC or positivity of relative entropy
3		SSA	SSA + QNEC	SSA + QNEC	SSA + QNEC
4			SSA	SSA + QNEC	SSA + QNEC

Some of these theorems have also been proved using different methods. The case d=3 can so far be understood only using the QI approach.

Can we establish irreversibility for d>4?

We will need more powerful tools! But it is worth to continue thinking about this.