Title: Talk 88 - Type II_1 algebras for local subregions in quantum gravity

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Abstract: We argue that generic local subregions in semiclassical quantum gravity are associated with von Neumann algebras of type II_1, extending recent work by Chandrasekaran et.al. beyond subregions bounded by Killing horizons. The subregion algebra arises as a crossed product of the type III_1 algebra of quantum fields in the subregion by the flow generated by a gravitational constraint operator. We conjecture that this flow agrees with the vacuum modular flow sufficiently well to conclude that the resulting algebra is type II_\infty, which projects to a type II_1 algebra after imposing a positive energy condition. The entropy of semiclassical states on this algebra can be computed and shown to agree with the generalized entropy by appealing to a first law of local subregions. The existence of a maximal entropy state for the type II_1 algebra is further shown to imply a version of Jacobson's entanglement equilibrium hypothesis. We discuss other applications of this construction to quantum gravity and holography, including the quantum extremal surface prescription and the quantum focusing conjecture.

Generalized entropy for general subregions in quantum gravity

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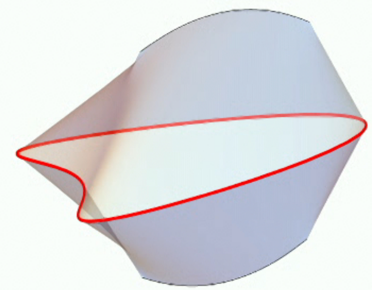
2306.01837 K. Jensen, J. Sorce, AJS

Generalized entropy

Fundamental quantity in semiclassical quantum gravity

$$S_{\text{gen}} = \frac{A}{4G_N\hbar} + S_{\text{EE}}$$



- UV-finite
- Relates geometry to entanglement

Applications:

- Generalized second law in black hole thermodynamics
- Quantum focusing conjecture
- Holographic entanglement entropy, Ryu-Takayanagi formula and quantum extremal surfaces
- Einstein equation from entanglement

Goals for the talk

Recent work has used von Neumann algebras to interpret S_{gen} as an algebraic entropy in configurations with boost symmetry (AdS-Schwarzschild, dS static patch) [Leutheusseur, Liu 2021; Witten 2021; Chandrasekaran, Longo, Penington, Witten 2022; Chandrasekaran, Penington, Witten 2022]

Goal: generalize to arbitrary subregions

1. Argue that subregions in quantum gravity are associated with type II von Neumann algebras
   a. Type II_1 for bounded subregions
   b. Type II_∞ for unbounded subregions
2. Entropy of semiclassical states in these algebras agrees with the subregion generalized entropy up to a constant
3. Maximal entropy state in the bounded case implies a version of Jacobson's entanglement equilibrium conjecture

Overview of von Neumann algebras

Used to describe subsystems in quantum mechanics and QFT:

$$\mathcal{A} \subset \mathcal{B}(\mathcal{H}), \qquad \mathcal{A}' = \{\text{operators that commute with } \mathcal{A}\}$$

Von Neumann algebras satisfy $\mathcal{A}'' = \mathcal{A}$

Simplest example: tensor factors (type I)

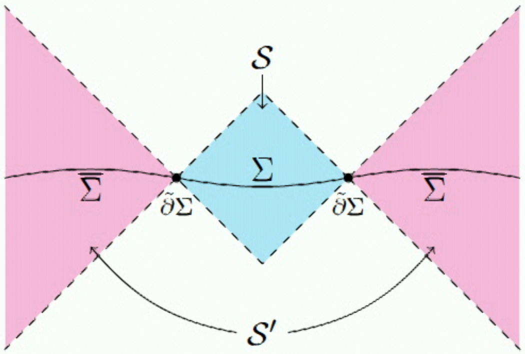
$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A'}, \quad \mathcal{A} = \{\mathsf{a} \otimes \mathbb{1}\}, \quad \mathcal{A}' = \{\mathbb{1} \otimes \mathsf{a}'\}$$

More general possibilities in infinite dimensions:

Type	Tensor factors $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A'}$	Density matrices $\rho_A, \rho_{A'}$	States $\langle \mathsf{a} \rangle_\Psi$	Modular operators Δ_Ψ	Trace $\mathrm{Tr}(\mathsf{a})$	Entropy $S(\rho_A)$	Relative entropy $S_{\mathrm{rel}}(\Phi\|\|\Psi)$	Example
I	✔	✔	✔	✔*	✔	✔	✔	qubits, harmonic oscillator
II	✗	✔	✔	✔	✔	✔	✔	gravitational subregions
III	✗	✗	✔	✔	✗	✗	✔	QFT subregions

Quantum gravity in the $G_N \to 0$ limit

Leading behavior: quantum field theory in curved spacetime



$\mathcal{A}_{\text{QFT}} \quad \to \quad$ fields in $\mathcal{S}$

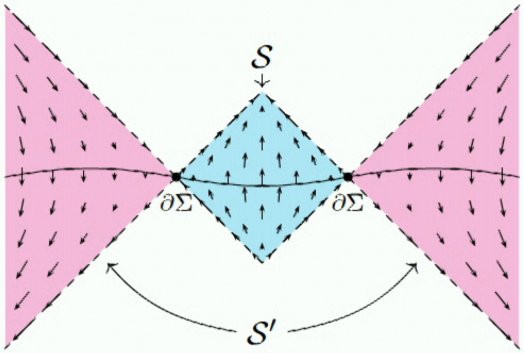
$\mathcal{A}'_{\text{QFT}} \quad \to \quad$ fields in $\mathcal{S}'$

- $\mathcal{A}_{\text{QFT}}, \mathcal{A}'_{\text{QFT}}$ act on a nonfactorized Hilbert space $\mathcal{H}_{\text{QFT}}$
- Include free gravitons (interactions suppressed by G_N)
- Algebras are type III_1

Gravitational constraints

Constraints $\mathcal{C}[\xi]$ arise in gravity from diffeomorphism invariance

$\mathcal{C}[\xi]$ generates the action of diffeomorphisms on kinematical algebras $\implies$ gauge invariant operators commute with $\mathcal{C}[\xi]$.



Boost generator on $\mathcal{A}_{\mathsf{QFT}}$, $\mathcal{A}'_{\mathsf{QFT}}$

$$H^g_\xi = \int_{\Sigma\cup\bar{\Sigma}} d\Sigma_b \, T^b{}_a \, \xi^a$$

$$\xi^a \overset{\partial\Sigma}{=} 0, \quad \nabla_a \xi_b \overset{\partial\Sigma}{=} \kappa n_{ab}$$

To avoid triviality, must introduce a local observer and/or asymptotic boundary $\to$ anchor for dressing observables

$$\mathcal{H}_{\mathsf{obs}} = L^2(\mathbb{R}), \qquad H_{\mathsf{obs}} = \hat{q}$$
$$\mathcal{H}_{\mathsf{ADM}} = L^2(\mathbb{R}), \qquad H_{\mathsf{ADM}} = \hat{q}'$$

Gravitational constraint: $\mathcal{C}[\xi] = H^g_\xi + H_{\mathsf{obs}} + H_{\mathsf{ADM}}$

Crossed product algebra

Gauge-invariant algebra can be represented on

$$\mathcal{H}_{\mathcal{S}} = \mathcal{H}_{\mathsf{QFT}} \otimes \mathcal{H}_{\mathsf{obs}}$$

consists of operators that commute with $\mathcal{C} = H^g_\xi + H_{\mathsf{obs}} = H^g_\xi + \hat{q}$

$$\mathcal{A}^{\mathcal{C}} = \{e^{i\hat{p}H^g_\xi}\mathsf{a}e^{-i\hat{p}H^g_\xi}, \hat{q}\}, \quad \mathsf{a} \in \mathcal{A}_{\mathsf{QFT}}$$
$$(\mathcal{A}^{\mathcal{C}})' = \{\mathsf{a}', \hat{q} + H^g_\xi\}, \quad \mathsf{a} \in \mathcal{A}'_{\mathsf{QFT}}$$

$\mathcal{A}^{\mathcal{C}}$ is the crossed product of $\mathcal{A}_{\mathsf{QFT}}$ by the action of H^g_ξ

Still need to determine the type of $\mathcal{A}^{\mathcal{C}}$

Geometric modular flow

Claim: H_ξ^g generates modular flow of some state $|\Psi\rangle \in \mathcal{H}_{\mathsf{QFT}}$, implying that $\mathcal{A}^{\mathcal{C}}$ is type II_∞ (see e.g. [Takesaki 1973; Connes 1973; Witten 2021])

$$\text{Modular operator} \quad \Delta_\Psi \text{``}=\text{''} \rho_\Psi (\rho'_\Psi)^{-1}$$

$$\text{Modular Hamiltonian} \quad h_\Psi = -\log \Delta_\Psi \text{``}=\text{''} -\log \rho_\Psi + \log \rho'_\Psi$$

Formally split $H_\xi^g = H_\xi^\Sigma - H_\xi^{\bar{\Sigma}}$, and construct density matrix

$$\rho_\Psi = \frac{e^{-\beta H_\xi^\Sigma}}{Z_\xi}$$

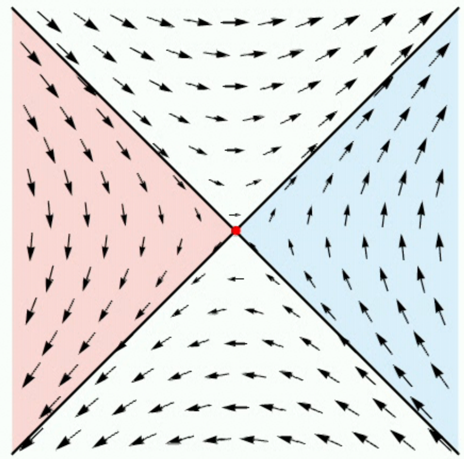
$\mathcal{A}_{\mathsf{QFT}}$ correlation functions are thermal (KMS) in this state

Geometric modular flow

Issue: Splitting H_ξ^g is UV-divergent. Need to be careful near the entangling surface $\partial\Sigma$:

Subregion locally looks like Rindler space near $\partial\Sigma$, whose vacuum modular Hamiltonian is the boost generator

[Bisognano, Wichmann 1975]



Conjecture: choosing H_ξ^g to generate the local boost ensures the existence of a state $|\Psi\rangle$ with

$$h_\Psi = \beta H_\xi^g, \qquad \beta = \frac{2\pi}{\kappa} \quad \text{(Unruh temperature)}$$

Modular operators

In type II algebras, modular operator factorizes $\Delta_{\widehat{\Phi}} = \rho_{\widehat{\Phi}}(\rho'_{\widehat{\Phi}})^{-1}$

Can determine $\Delta_{\widehat{\Phi}}$ by solving the relation

$$\langle\widehat{\Phi}|\,\widehat{\mathsf{a}}\,\widehat{\mathsf{b}}\,|\widehat{\Phi}\rangle = \langle\widehat{\Phi}|\,\widehat{\mathsf{b}}\,\Delta_{\widehat{\Phi}}\,\widehat{\mathsf{a}}\,|\widehat{\Phi}\rangle$$

For states $|\widehat{\Phi}\rangle = |\Phi\rangle_{\mathsf{QFT}} \otimes f(q)$, density matrices are

$$\rho_{\widehat{\Phi}} = \frac{1}{\beta} e^{i\hat{p}\frac{h_\Psi}{\beta}} f\big(\hat{q} - \frac{h_\Psi}{\beta}\big) e^{\beta\hat{q}} \Delta_{\Phi|\Psi} f^*\big(\hat{q} - \frac{h_\Psi}{\beta}\big) e^{-i\hat{p}\frac{h_\Psi}{\beta}}$$
$$\rho'_{\widehat{\Phi}} = \frac{1}{\beta} \Delta_{\Psi|\Phi}^{-\frac{1}{2}} J_{\Phi|\Psi} J_\Psi e^{\beta\hat{q}} \big| f\big(\hat{q} + \frac{h_\Psi}{\beta}\big)\big|^2 J_\Psi J_{\Psi|\Phi} \Delta_{\Psi|\Phi}^{-\frac{1}{2}}$$

Generalized entropy

Entropy given by

$$S(\rho_{\widehat{\Phi}}) = \langle \widehat{\Phi} | - \log \rho_{\widehat{\Phi}} | \widehat{\Phi} \rangle = - \operatorname{Tr} \rho_{\widehat{\Phi}} \log \rho_{\widehat{\Phi}}$$

Assuming that observer is weakly entangled with the quantum fields by choosing $f(q)$ to be slowly varying, find

$$S(\rho_{\widehat{\Phi}}) \approx -S_{\text{rel}}(\Phi || \Psi) - \beta \langle H_{\text{obs}} \rangle + S_f^{\text{obs}} + \log \beta$$

Convert to generalized entropy by imposing the local constraints (Einstein equation) and applying local first law of subregions

$$S(\rho_{\widehat{\Phi}}) = \left\langle \frac{A}{4G_N} \right\rangle_{\widehat{\Phi}} + S_{\Phi}^{\text{QFT}} + S_f^{\text{obs}} + \text{const.}$$

Note: Reversing the logic, can demand $S(\rho_{\widehat{\Phi}}) = S_{\text{gen}}$, and derive the local Einstein equation, similar to holographic arguments [Lashkari, McDermott, Van Raamsdonk 2014; Faulkner, Guica, Hartman, Myers, Van Raamsdonk 2014; Swingle, Van Raamsdonk 2014; Faulkner, Haehl, Hijano, Parrikar, Rabideau, Van Raamsdonk 2017; Lewkowycz, Parrikar 2018]

Energy conditions and entanglement equilibrium

Impose positive observer energy by projecting $\Pi = \Theta(\hat{q})$

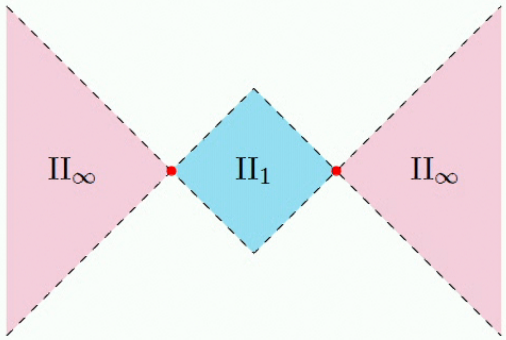
$$\mathcal{A}^{\Pi} = \Pi \mathcal{A}^{\mathcal{C}} \Pi \text{ is type } \mathrm{II}_1, \quad \mathrm{Tr}\, \mathbb{1} = 1$$

$\implies$ existence of maximal entropy state

$$|\Psi_{\text{max}}\rangle = |\Psi\rangle \otimes \sqrt{\beta} e^{-\frac{\beta q}{2}} \Theta(q)$$

Implies Jacobson's entanglement equilibrium conjecture [Jacobson 2015], which postulates that small causal diamonds in quantum gravity have a maximally entangled state.

For asymptotic subregions, H_{obs} is replaced by $-H_{\text{ADM}}$, and the projected algebra is type II_∞.



Summary and outlook

Summary of construction:

A1	QFT in CST, $\mathcal{A}_{\mathrm{QFT}}$ is type III_1	dressing, gravitational observables, bulk reconstruction
A2	Observer degree of freedom	Interpretation? code subspace, nonlinear gravitational dof, edge modes...
A3	Gravitational constraint	Diff-invariance/gauge-fixing, other constraints?
A4	Geometric modular flow	Construct proof? Derive corrections?
A5	Local first law	Needed to arrive at generalized entropy, relation to EE from entanglement
A6	Positive energy condition	Derivation from first principles? Always applicable?

Future applications: generalized second law, quantum focusing conjecture, quantum extremal surface prescription in holography