Title: Talk 44 - Large N von Neumann Algebras and the renormalization of Newton's constant

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Abstract: In holography, the quantum extremal surface formula relates the entropy of a boundary state to the sum of two terms: the area term and the entropy of bulk fields inside the entanglement wedge. As the bulk effective field theory suffers from UV divergences, the second term must be regularized. It has been conjectured since the work of Susskind and Uglum that the renormalization of Newton's constant in the area term exactly cancels the difference between different choices of regularization for bulk entropy. In this talk, I will explain how the recent developments on von Neumann algebras appearing in the large N limit of holography allow to prove this claim within the framework of holographic quantum error correction, and to reinterpret it as an instance of the ER=EPR paradigm. This talk is based on the paper arXiv:2302.01938.

Large N von Neumann algebras and the renormalization of Newton's constant

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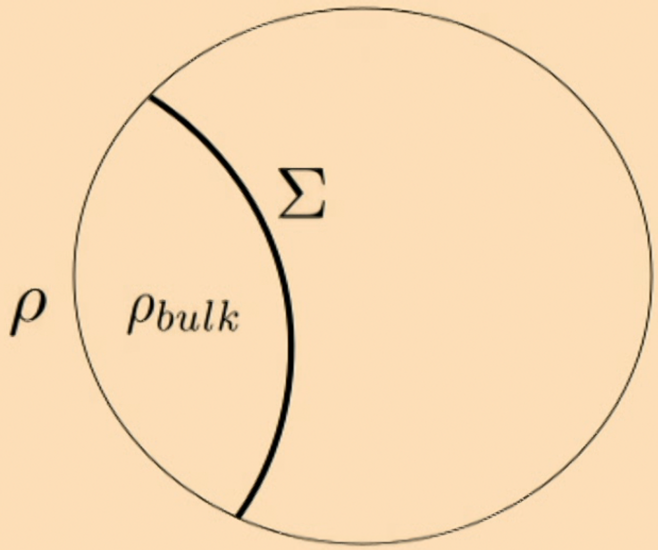
arXiv:2302.01938

The Quantum Extremal Surface Formula

The **Quantum Extremal Surface** (QES) Formula is one of the cornerstones of holography.

$$S(\rho) = \frac{A(\Sigma)}{4G_N} + S(\rho_{bulk}).$$

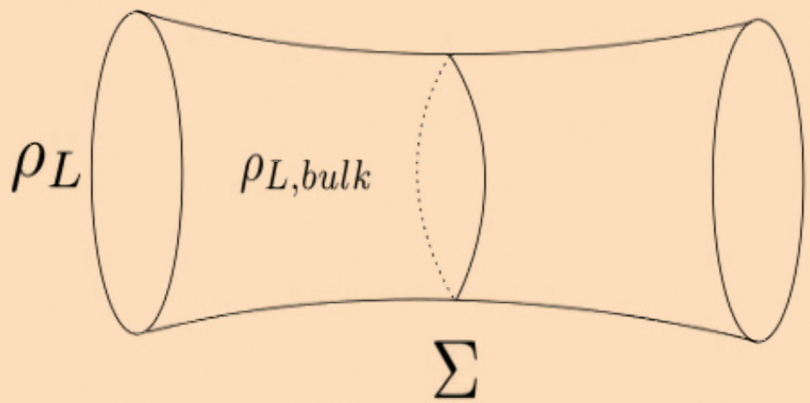
Σ is the quantum extremal surface associated to the subregion. It is defined by **extremizing** the RHS.



The Quantum Extremal Surface Formula

In the case of one side of a two-sided black hole, QES reduces to the calculation of **black hole entropy**.

$$S(\rho_L) = \frac{A(\Sigma)}{4G_N} + S(\rho_{L,bulk}).$$



Ambiguities in entropy formulas

- Even though entropy formulas are fundamental, it is not so straightforward to properly **define** each of their terms!

$$S(\rho_L) = \frac{A(\Sigma)}{4G_N} + S(\rho_{L,bulk}).$$

- In the bulk effective field theory description, $G_N = 1/N^2 = 0$ or is perturbatively small, and the entropy S_{bulk} of quantum fields across Σ is **infinite**: needs to be **regulated**.
- On the boundary, we are computing a UV-complete quantity: G_N (or $1/N^2$) needs to be taken **small but nonzero**.
- We face an apparent paradox: the right hand side looks **cutoff dependent** while the left hand side is finite and **cannot depend** on any cutoff!

The Susskind—Uglum conjecture

$$S(\rho_L) = \frac{A(\Sigma)}{4G_N} + S(\rho_{L,bulk}).$$

- **Susskind—Uglum conjecture**: The renormalization of the area term (i.e. Newton's constant) exactly **cancels** that of the bulk entropy term!
- This talk: recent discussions on the **large N limit** of holography, as well as holographic **quantum error correction**, allow to formulate this conjecture precisely and prove it.

Outline

- I - Large N von Neumann algebras

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- II - Code subspace renormalization
- III - Proof of the Susskind—Uglum conjecture

Part I

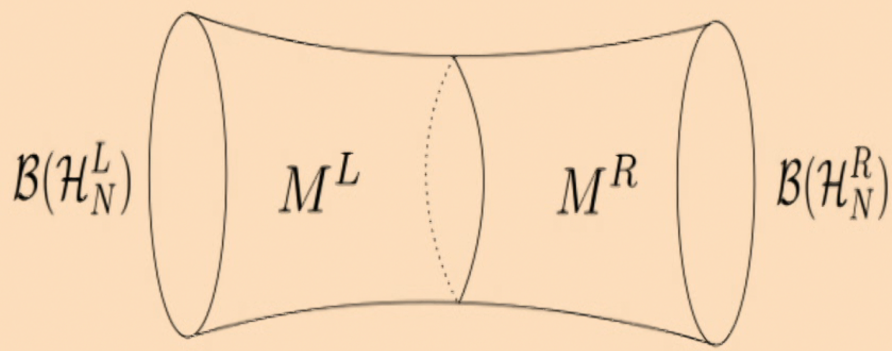
Large N von Neumann algebras

Large N and von Neumann types

- It has recently been realized that the discrepancy between the finite N and infinite N cases can be traced back to a **change of type** of von Neumann algebra.
- The **finite N** algebras, corresponding to the boundary UV-complete theories, have **type I**.
- The **infinite N** algebras, or perturbation theory in $1/N$, do **not** have type I. [Leutheusser, Liu, Witten]
- An algebra that does not have type I means that all its states are **infinitely entangled** with the rest of the system. Its underlying Hilbert space **does not factorize**.
- Then only **differences of entropy** (type II) or **no notion of entropy** at all (type III) can be defined.

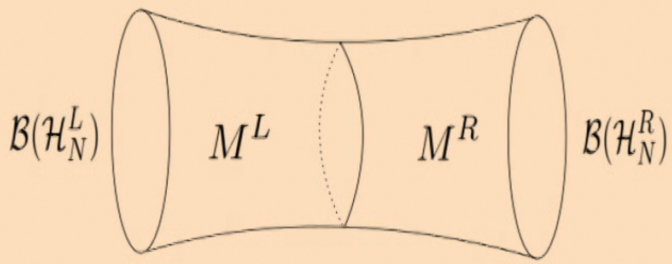
Bulk vs boundary

- In our context,



- The boundary is UV-complete: the Hilbert space factorizes, and the algebras $\mathcal{B}(\mathcal{H}_N^{L,R})$ are **type I** factors.
- The bulk is at large N: the Hilbert space doesn't factorize and the algebras $M^{L,R}$ **do not have type I**.

Quantum error correction



- It is a bit tricky to think about holographic **quantum error correction** in this context.
- The code should map the **large** N von Neumann algebra M^L to the **finite** N type I von Neumann algebra $\mathcal{B}(\mathcal{H}_N^L)$ on the boundary.
- Then one shouldn't trust the map for N-dependent operators that break the EFT: the code works **pointwise** at large N but not **uniformly**. [Faulkner, Li]

Entropy formula

- Recall the entropy formula:

$$S(\rho_L) = \frac{A(\Sigma)}{4G_N} + S(\rho_{L,bulk}).$$

- The bulk entropy term **cannot** be defined for the large N algebra! Entropy is **divergent**.
- The **full** large N algebra **cannot** be considered as a code subspace to prove this formula.
- What do we do instead? Single out small **subalgebras** of the large N algebra.
- How do we pick them?

Part II

Code subspace renormalization

Type I and bounded entropy

- What does a good regulated subalgebra look like?
- We want the regulated algebra to match the **bulk entropy term** in the large N limit.
- In order to hope for a finite entropy, the regulated algebra must have **type I**: either finite-dimensional or $\mathcal{B}(\mathcal{H})$ for $\mathcal{H}$ a separable Hilbert space.
- **Schmidt decompositions** can be defined for states on these algebras, and **von Neumann entropy** is defined in the usual way.

Constraints from complementary recovery

- All known QEC proofs of entropy formulas require an assumption of **complementary recovery**.
- We want to find a way to **regulate** the large N algebra that respects this structure.
- We want something like:

$$\begin{tikzcd} M^L \arrow[r, "\prime"] \arrow[d, "?"'] & M^R \arrow[d, "?"'] & \mathcal{H} \arrow[d] \\ M^L_\lambda \arrow[r, "\prime"] & M^R_\lambda & \mathcal{H}_\lambda \end{tikzcd}$$

- This is a **nontrivial constraint**.
- **Takesaki's theorem** (1972): "?" should be a **conditional expectation**, and $\mathcal{H}_\lambda$ should be a Hilbert space of **invariant states** under the conditional expectation.

Conditional expectations and Takesaki's theorem

$$
\begin{array}{ccccc}
M^L & \xrightarrow{\ \prime\ } & M^R & \qquad & \mathcal{H} \\
\mathcal{E}_\lambda \Big\downarrow & & \mathcal{E}'_\lambda \Big\downarrow & & \Big\downarrow \\
M^L_\lambda & \xrightarrow{\ \prime\ } & M^R_\lambda & & \mathcal{H}_\lambda
\end{array}
$$

- The map $\mathcal{E}_\lambda$ is called a **conditional expectation**. For $n_1, n_2 \in M^L_\lambda$ and $m \in M^L$,

$$\mathcal{E}_\lambda(n_1 m n_2) = n_1 \mathcal{E}_\lambda(m) n_2.$$

- States $|\Psi\rangle \in \mathcal{H}_\lambda$ satisfy

$$\langle\Psi|\, \mathcal{E}_\lambda(m)\, |\Psi\rangle = \langle\Psi|\, m\, |\Psi\rangle\,.$$

- The interpretation of $\mathcal{E}_\lambda$ is that it **integrates out** some entanglement in a way that is compatible with complementary recovery.

Conditional expectations onto a type I factor

- More precisely, if the factor M^L acts on a Hilbert space $\mathcal{H}$ and M^L_λ is a type I subfactor of M^L, then

$$M^L = M^L_\lambda \otimes M^{L,c}_\lambda .$$

- A conditional expectation $\mathcal{E}_\lambda : M^L \longrightarrow M^L_\lambda$ then has the form

$$\mathcal{E}(X \otimes X^c) = \langle \chi_\lambda | X^c | \chi_\lambda \rangle (X \otimes Id).$$

- The Hilbert space of **invariant states** under $\mathcal{E}_\lambda$ is of the form

$$\mathcal{H}^{inv} = \mathcal{H}_\lambda \otimes | \chi_\lambda \rangle .$$

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- So the conditional expectation integrates out the degrees of freedom of $M^{L,c}_\lambda$ by applying χ_λ!

Code subspace renormalization schemes

This framework based on conditional expectations allows to regulate the bulk theory with several nested code spaces:

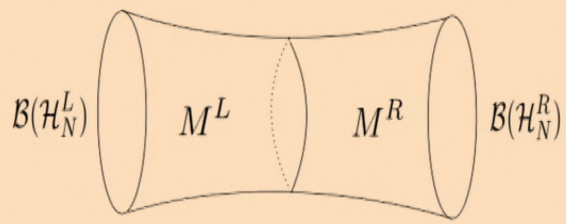
$$
\begin{tikzcd}
M^L \arrow[r, "'"] \arrow[d, "\mathcal{E}_\lambda"'] \arrow[dd, bend right=60, "\mathcal{E}_\mu"'] & M^R \arrow[d, "\mathcal{E}'_\lambda"] \arrow[dd, bend left=60, "\mathcal{E}'_\mu"] & \mathcal{H} \arrow[d] \\
M^L_\lambda \arrow[r, "'"] \arrow[d, "E_{\lambda\mu}"'] & M^R_\lambda \arrow[d, "E'_{\lambda\mu}"] & \mathcal{H}_\lambda \arrow[d] \\
M^L_\mu \arrow[r, "'"] & M^R_\mu & \mathcal{H}_\mu
\end{tikzcd}
$$

- The regulated algebras $M_{\lambda,\mu}$ are type I factors.

Part III

Proof of the Susskind—Uglum conjecture

Back to the code

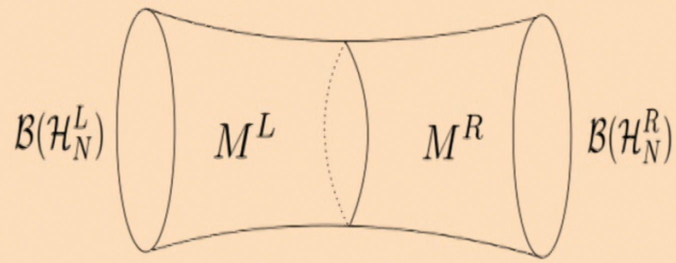


- We now have a consistent way of **renormalizing** the code subspace.
- Recall that there is an **encoding map** from the effective theory at large N to the finite N theory

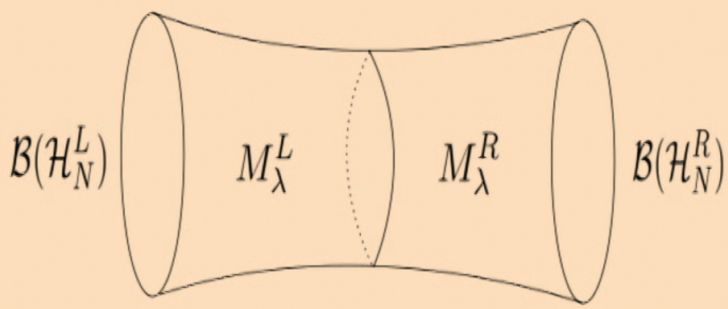
$$V_N : \mathcal{H} \longrightarrow \mathcal{H}_N^L \otimes \mathcal{H}_N^R.$$

- However closeness to isometry and reconstruction properties can only be formulated **pointwise**.
- Idea here: ask for **stronger** reconstruction properties, but only for **renormalized subalgebras**.

A smaller code

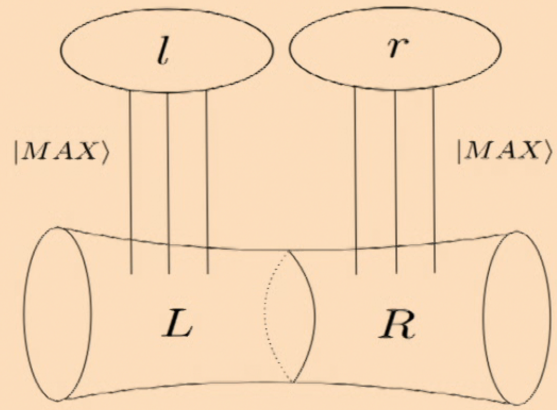


- Apply the RG flow in the bulk through $\mathcal{E}_\lambda$.



- Now the map is $V_N : \mathcal{H}_\lambda^L \otimes \mathcal{H}_\lambda^R \longrightarrow \mathcal{H}_N^L \otimes \mathcal{H}_N^R$.

Areas in approximate codes



- The **area** associated to this subdivision is defined by the formula

$$A(\mathcal{H}^L_\lambda) = S(|CJ\rangle\,, L\ell).$$

- There is only one value of area per code subspace, but the area changes depending on the **choice of renormalized code subspace**!
- The **larger** the renormalized code subspace, the **smaller** the area term.

An approximate entropy formula

Following Akers and Penington, one can derive the following result:

- **Suppose** that for $|\Psi\rangle \in \mathcal{H}_\lambda$, and for all unitary operators U_λ^L, U_λ^R in M_λ^L and M_λ^R, there exist unitary operators $\tilde{U}_\lambda^L$ and $\tilde{U}_\lambda^R$ (chosen in a measurable way) in $\mathcal{B}(\mathcal{H}_N^L)$ and $\mathcal{B}(\mathcal{H}_N^R)$ such that

$$\| V_N U_\lambda^R U_\lambda^L |\Psi\rangle - \tilde{U}_\lambda^R \tilde{U}_\lambda^L V_N |\Psi\rangle \| \leq \delta_N,$$

  where δ_N decays faster than any polynomial in $1/N$.

- **Then**,

$$|S(|\Psi\rangle, M_\lambda^L) + A(\mathcal{H}_\lambda^L) - S(V_N |\Psi\rangle, \mathcal{B}(\mathcal{H}_N^L))| \xrightarrow[N\to\infty]{} 0.$$

Running the RG flow

- The crucial point is that this formula is valid for **any choice** of cutoff λ (as long as it doesn't depend on N).
- Then, we have **both** formulas for $|\Psi\rangle \in \mathcal{H}_\mu$:

$$\left|S(|\Psi\rangle\,, M^L_\mu) + A(\mathcal{H}^L_\mu) - S(V_N\,|\Psi\rangle\,, \mathcal{B}(\mathcal{H}^L_N))\right| \underset{N\to\infty}{\longrightarrow} 0,$$

$$\left|S(|\Psi\rangle\,, M^L_\lambda) + A(\mathcal{H}^L_\lambda) - S(V_N\,|\Psi\rangle\,, \mathcal{B}(\mathcal{H}^L_N))\right| \underset{N\to\infty}{\longrightarrow} 0.$$

- One can show that entropy **factors out** in the bulk:

$$S(|\Psi\rangle\,, M_\lambda) = S(|\Psi\rangle\,, M_\mu) + S(|\Psi\rangle\,, M_{\lambda\mu}).$$

- We get exactly **Susskind–Uglum**!

$$\left|A(\mathcal{H}^L_\mu) - (S(|\Psi\rangle\,, M_{\lambda\mu}) + A(\mathcal{H}^L_\lambda))\right| \underset{N\to\infty}{\longrightarrow} 0,$$

with $M_\lambda = M_\mu \otimes M_{\lambda\mu}$.

Susskind–Uglum as ER=EPR

- This proof based on quantum error correction provides a **reinterpretation** of Susskind–Uglum.
- What makes it work? The **bigger** the code subspace, the **smaller** the entropy of the CJ state (i.e. the area term) will be.
- This is because the missing entropy is now counted as part of the code subspace entropy!
- There is some entropy in the code that can be counted either as **bulk entropy** or as **geometry**. Whether it is one or the other amounts to making a choice of renormalization scale, which is completely **unphysical**.
- This is the paradigm of **ER=EPR**: no physical distinction between **entanglement** and emergent **geometry** in gravity.

Recap

- There is an algebraic way to **regulate** a von Neumann algebra with divergent entropy while preserving complementarity, through **conditional expectations**.
- By restricting the holographic code to the regulated subalgebras, one can prove an **entropy formula**.
- As the cutoff runs, the **bulk entropy** gets repackaged into the **area term**.
- There seems to be a close connection between **ER=EPR** and **renormalization**.
- Can we make sense of it in full quantum gravity?
  (ongoing with M. Marcolli and J. McNamara)

Thank you!