

Title: Gravitational Wave Theory

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URL: <https://pirsa.org/23060083>

Errata  $t_{ab} = \frac{1}{32\pi} \langle \partial_a h_{cd}^{\text{TT}}, \partial_b h^{cd} \rangle$

$$1) \frac{dE}{dt} = \oint_{\Sigma} t_{0i} n^i ds = \oint t_{0r} ds = \oint h_{cd} (\partial_t h^{cd}) \Rightarrow \int h_{cd} \left( \frac{\partial_t h^{cd}}{c} \right)$$

2) Post Newtonian basics

$$g_{0i} = \frac{3}{c} \dot{\Phi}_i$$

$$g_{00} = - \left( 1 + \frac{2\Phi}{c^2} \right)$$

$$g_{ij} = \left( 1 - \frac{2\Phi}{c^2} \right) \delta_{ij} + \frac{7}{c^2} \dot{\Phi}_i \dot{\Phi}_j$$

E+

$g = M_1 + M_2$

CAUTION  
 Do not touch the board when it is in use.  
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Errata  $t_{ab} = \frac{1}{32\pi} \langle \partial_a h_{cd}^{\text{TT}}, \partial_b h^{cd} \rangle$

1)  $\frac{dE}{dt} = \oint_{\partial V} t_{0i} n_i d\sigma = \oint t_{0r} dS = \oint \dot{h}_{cd} (\partial_r h^{cd}) \Rightarrow \int \dot{h}_{cd} \left( \frac{\partial}{\partial t} h^{cd} \right)$

2) Post-Newtonian basics

$\rho_{0i} = \frac{W_i}{c^2}$

$\rho_{00} = -\left(1 + 2\frac{\Phi}{c^2}\right)$

$\rho_{ij} = \left(1 - 2\frac{\Phi}{c^2}\right) + \frac{X_{ij}}{c^2}$

frequency / wavelength

Brief Review & Ex

- Waves & propagation

$$\left\{ \begin{array}{l} \square \bar{h}_{ab} = 0 \\ \partial_a \bar{h}^{ab} = 0 \end{array} \right\} \bar{h}_{ab} = \bar{A}_{ab} e^{i k_c x^c}$$
$$k^c k_c = 0; \quad k^c A_{ab} = 0$$

z-prop  $\vec{k} = \left(\frac{\partial}{\partial t}\right)^a + \left(\frac{\partial}{\partial z}\right)^a$ , Also  $\vec{k}^a = \gamma^a k^a$  is null!

$$\hookrightarrow \vec{k}_a z^a = \gamma t + \gamma z$$

$$[e^{i \vec{k}_a z^a} + cc] \sim \sin(\gamma t + \gamma z)$$

??

$\Rightarrow$  Above eqns don't fix  $\sin(\omega t + k z)$   
frequency/wavelength



Brief Review & Ex

- Waves & propagation

$$\left\{ \begin{array}{l} \square \bar{h}_{ab} = 0 \\ \partial_a \bar{h}^{ab} = 0 \end{array} \right\} \bar{h}_{ab} = \bar{A}_{ab} e^{i k_c x^c}$$

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$$\Rightarrow \vec{k}_a x^a = \gamma t + \gamma z$$

$$\left[ e^{i \vec{k}_a x^a} + c.c \right] \sim \sin(\gamma t + \gamma z)$$

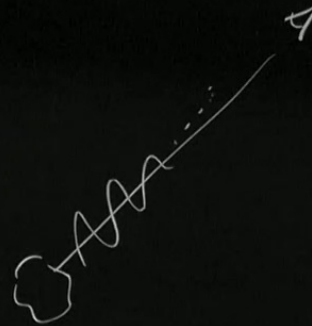
$$\sin(\omega t + k z)$$

$\Rightarrow$  Above eqns don't fix frequency/wavelength

- generation  $\bar{h}_{ab} = \frac{2G}{c^4 r} \ddot{Q}_{ab}(t-r)$

- at a given  $r$ !

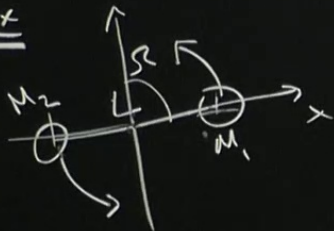
(in previous ex, at a given  $z$   
so "match")



Prop  $h = A e^{i\omega t + kz_{obs}} = e^{i\omega t} (A e^{iz_{obs}})$

gen  $h(t, z_{obs}) = \frac{2G}{c^4 r} \ddot{Q}(t-r)$

E+



$$Q \approx \int \delta x^2 \sim \int M R^2 (\sin^2 \Omega t)$$

$$(\sin \Omega t) \cos(\Omega t)$$

$$\approx \sin(2\Omega t) \text{ dependence}$$

$$P = M_1 \delta(r-R) \delta(\phi - \Omega t)$$

$$+ M_2 \delta(r-R) \delta(\phi - \Omega t - \pi)$$

① Compute amplitude [Bonus "hear it"]

② " Energy 2 BH,  $M = 10 M_\odot$   
 $D = 100 M_\odot$

③ " rate of change of distance

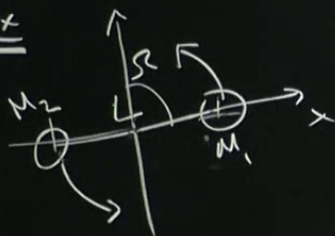
[Bonus: 2 BHs,  $M = 10^8 M_\odot$

$L = 0.05 \text{ pc}$

how long for  $\frac{\Delta L}{L} \sim 1\%$  ? ]



E+



$$Q \approx \int \delta x^2 \sim \int M R^2 (\sin^2 \Omega t)$$

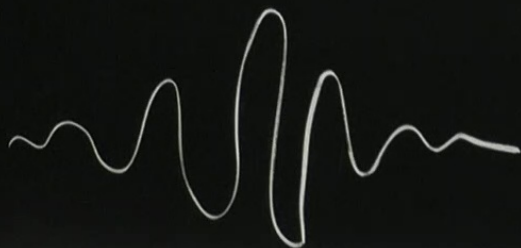
$$(\sin \Omega t) \cos(\Omega t)$$

$$\sim \sin(\underline{2\Omega t}) \text{ dependence}$$

$$S = M_1 \delta(r-R) \delta(\phi - \Omega t)$$

$$+ M_2 \delta(r-R) \delta(\phi - \Omega t - \pi)$$

$$L \approx L_0 \left(1 + \frac{1}{2} h_{xx}\right)$$



- ① Compute amplitude [Bonus "hear it"]
- ② " Energy 2 BH  $M = 10 M_\odot$
- ③ " rate of change of distance  $D = 100 M_\odot$

[Bonus: 2 BHs,  $M = 10^8 M_\odot$

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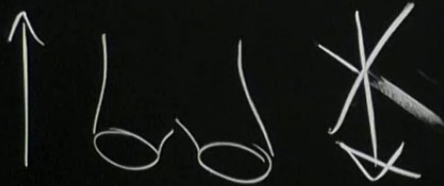
how long for  $\frac{\Delta L}{L} \sim 1\%$  ? ]



## Black holes

- Governed by 2 (+1?) parameters ( $M, a, q$ )
- Hide singularities, stable (ish!)
- Can join but not bifurcate (why?)
- Ultimate state conjecture.

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2m}{r}\right)} dr^2 + r^2 d\Omega^2$$



## QNM's

Kokkotas - Schmidt  
Liv. Rev. Relat

Berti - Cardoso - Starinets  
Review

Geodesics  $u^a u_a = -1$

$$k^a k_a = 0$$

$$\left(\frac{\partial}{\partial t}\right)^a, \left(\frac{\partial}{\partial \phi}\right)^a$$

$$E = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} \quad ; \quad L = r^2 \frac{d\phi}{d\tau}$$

Trajectories: massive particle

$$\frac{1}{2} E^2 = \frac{1}{2} \left(\frac{dr}{dt}\right)^2 + V(r)$$

Circular orbits

$$\text{with } V(r) = \frac{1}{2} - \frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$$

$\rightarrow \exists$  circ. orbits "stable"  $\forall r \geq 6M$

$L = 0.05 pc$   
 or  $\frac{M}{L} \sim 1\%$  ?  
 $(M, a, \varphi)$   
QNM's  
 Kokkotas-Schmidt  
 Liv. Rev. Relat  
 Bert-Cardoso-Starinets  
 Review

Geodesics  $u^a u_a = -1$   $k^a k_a = 0$   
 $\left(\frac{\partial}{\partial t}\right)^a$  ;  $\left(\frac{\partial}{\partial \phi}\right)^a$   $E = \left(1 - \frac{2M}{r}\right) \frac{dt}{dr}$  ;  $L = r^2 \frac{d\phi}{dr}$   
trajectories : massive particle  $\frac{1}{2} E^2 = \frac{1}{2} \left(\frac{dr}{dt}\right)^2 + V(r)$  ↓  
Circular orbits with  $V(r) = \frac{1}{2} - \frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$   
 → ∃ circ orbits "stable" if  $r \geq 6M$   
 → massless particles  $r = 3M$  Light ring / photon sphere





Geodesics  $u^a u_a = -1$   $k^a k_a = 0$

$\left(\frac{\partial}{\partial t}\right)^a$  ;  $\left(\frac{\partial}{\partial \phi}\right)^a$   $E = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau}$  ;  $L = r^2 \frac{d\phi}{d\tau}$

Trajectories: massive particle  $\frac{1}{2} E^2 = \frac{1}{2} \left(\frac{dr}{dt}\right)^2 + V(r)$

Circular orbits with  $V(r) = \frac{1}{2} - \frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$

→ ∃ circ. orbits "stable" for  $r \geq 6M$

→ massless particles  $r = 3M$  Light ring / photon sphere

Schmidt  
Rev. Relat  
—  
so - Stars  
new

CAUTION  
No climbing on the blackboard  
No leaning on the blackboard  
No sitting on the blackboard  
No standing on the blackboard

relation  $\bar{h}_{ab} = \frac{2G}{c^4 r} \ddot{Q}_{ab}(t-r)$   
 at a given  $r!$   
 it peaks at a given  $z$   
 so "match")

$h = A e^{i\omega t + k z_{05}} = e^{i\omega r} (A e^{i\omega t})$   
 $h(t, z_{05}) = \frac{2G}{c^4 r} \ddot{Q}(t-r)$



$L = 0.05 pc$   
 has log for  $\frac{M}{L} \sim 17?$

Black Holes

- Governed by 2 (+1?) parameters ( $M, a, q$ )
- Hide singularities, stable (ish!)
- Can't join but not bifurcate (uh?)
- Ultimate state conjecture.

$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 d\Omega^2$

$\uparrow$  bad  $\downarrow$

QNM's

Kokkotas-Schmidt

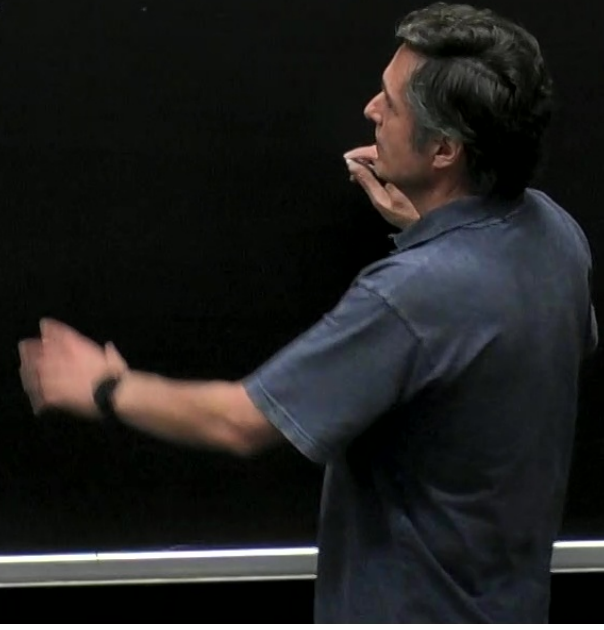
Lu-Pan-Beht

Bark-Cook-Sterns

Rever

$$\psi = \psi_{\text{BH}} + h \quad \psi = \psi_{\text{BH}} - \Phi \quad \boxed{\square \Phi = 0} \quad \mu^2 \Phi$$

$$\Phi = \sum_{lm} \frac{R_l(r)}{r} Y_{lm}(\theta, \phi) e^{-i\omega t}$$



CAUTION  
 Do not lean against the chalkboard.  
 This may cause the chalkboard to  
 move and fall. Do not touch the board  
 if it is moving or falling.



$$\Phi = \sum_{lm} \frac{R_{lm}(r)}{r} Y_{lm}(\theta, \phi) e^{-i\omega t}$$

$$\frac{d^2 R_{lm}}{dr^2} + [\omega^2 - V(r)] R_{lm} = 0$$

$$\omega_{l\neq 0} = \omega_R + i\omega_I$$

$$V = \left(1 - \frac{2M}{r}\right) \left[ \frac{l(l+1)}{r^2} + \frac{2M}{r^3} \right]$$

$$\Phi = \sum_{lm} \frac{R_{lm}(r)}{r} Y_{lm}(\theta, \phi) e^{-i\omega t}$$

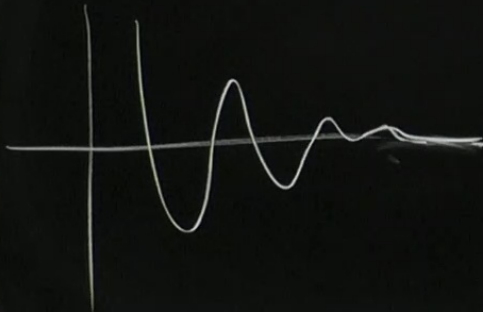
$$\frac{d^2 R_{lm}}{dr^2} + [\omega^2 - V(r)] R_{lm} = 0$$

$$V = \left(1 - \frac{2M}{r}\right) \left[ \frac{l(l+1)}{r^2} + \frac{2M}{r^3} \right]$$

$$\omega_{l\eta} = \omega_R + i\omega_I$$

→ decaying in time since

oscillatory behaviour



$$g = g_{\text{BH}} + h \quad // \quad g = g_{\text{BH}} \quad \Phi \quad \boxed{\square \Phi = 0} \quad \mu^2 \Phi$$

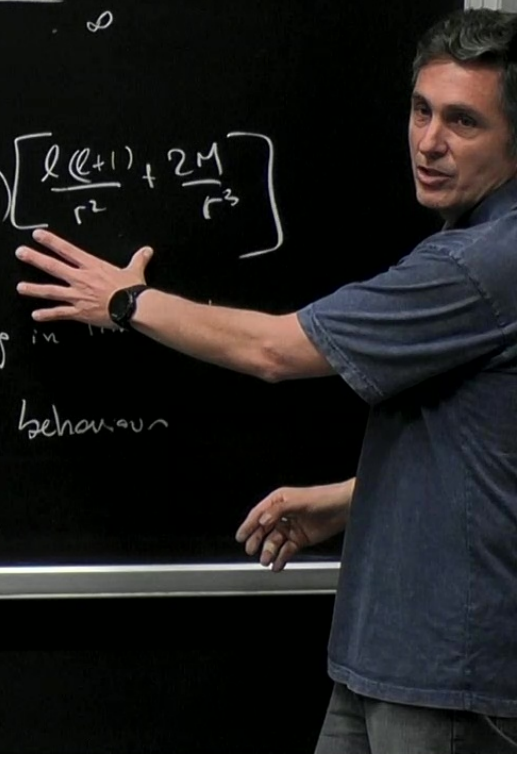
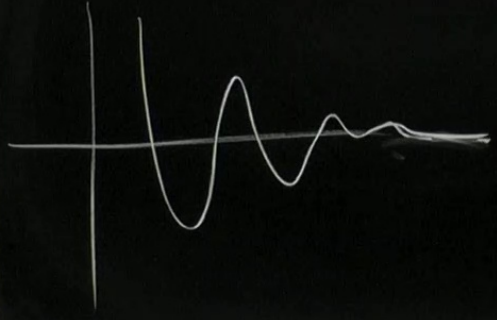
$$\Phi = \sum_{\ell m} \frac{R_{\ell m}(r)}{r} Y_{\ell m}(\theta, \phi) e^{-i\omega t}$$

$$\frac{d^2 R_{\ell m}}{dr_*^2} + [\omega^2 - V(r)] R_{\ell m} = 0$$

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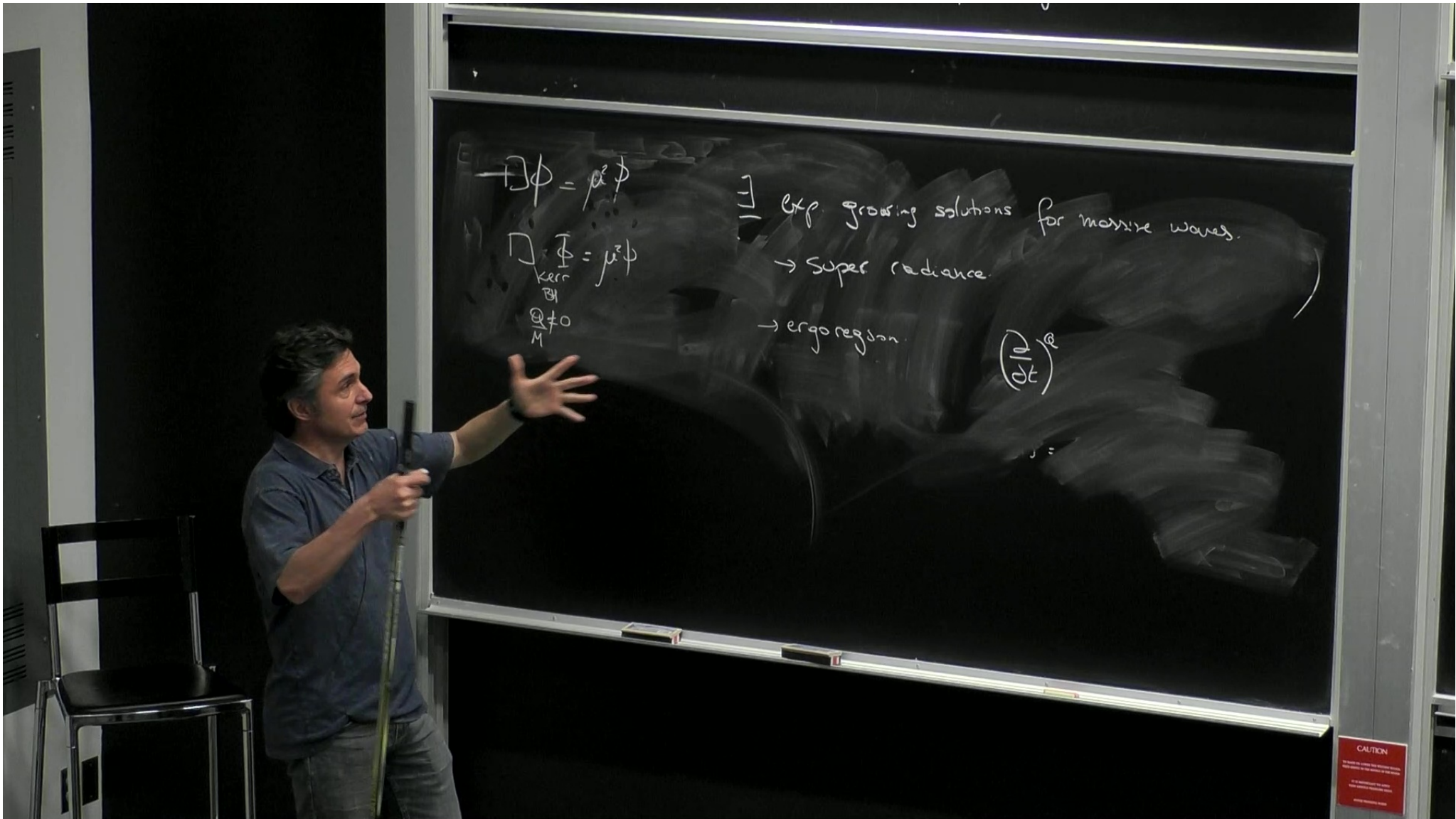
$$\omega_{\ell m \eta} = \omega_R + i\omega_I$$

→ decaying in time  
 oscillatory behaviour



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parameters  $(M, a, \varphi)$   
 ble (ish!)  
 furcate (why?)  
 ure.

QNM's  
 Kokkato...  
 Bert...  
 Rene



$$g = g_{\text{BH}} + h \quad // \quad g = g_{\text{BH}} \quad \Phi \quad \boxed{\square \Phi = 0} \quad \mu^2 \Phi$$

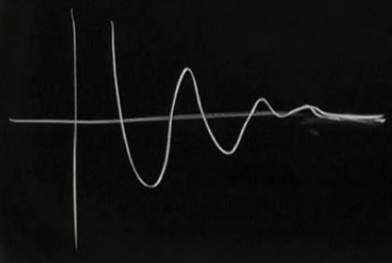
$$\Phi = \sum_{l,m} \frac{R_{lm}(r)}{r} Y_{lm}(\theta, \phi) e^{-i\omega t}$$

$$\frac{d^2 R_{lm}}{dr^2} + [\omega^2 - V(r)] R_{lm} = 0$$

$$V = \left(1 - \frac{2M}{r}\right) \left[ \frac{l(l+1)}{r^2} + \frac{2M}{r^3} \right]$$

$$\omega_{l \neq \eta} = \omega_R + i \omega_I \rightarrow \text{decaying in time since } \omega_I < 0$$

oscillatory behaviour



CAUTION

$$\nabla \phi = \mu$$

$$\nabla \cdot \vec{\Phi} = \mu$$

$\oplus$   
M

$$\omega_{lmn} = \omega_r^{(M, \alpha)} + i \omega_I^{(M, \alpha)}$$

→ Measuring more than 3 QNM.

$$\text{Gas} = T_{ab}$$

$$\int \{ \dots \}$$

Black

$$ds^2 =$$





frequency / wavelength

$$\nabla^2 \phi = \rho$$

$$\omega_{lmn} = \omega_R^{(M, \alpha)} + i \omega_I^{(M, \alpha)}$$

$$\nabla \cdot \vec{\Phi} = \mu$$

→ Measuring more than 1 QNM

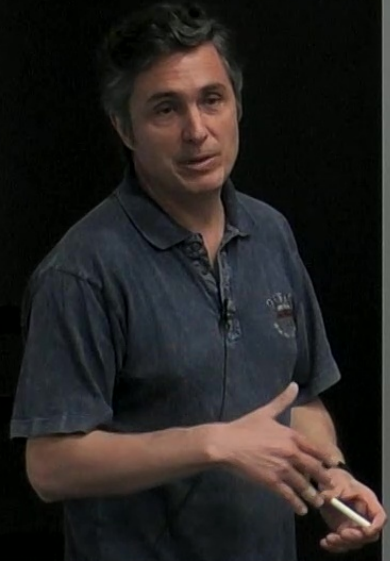
$\frac{\partial}{\partial t}$   
M

$$G_{ab} = T_{ab}$$

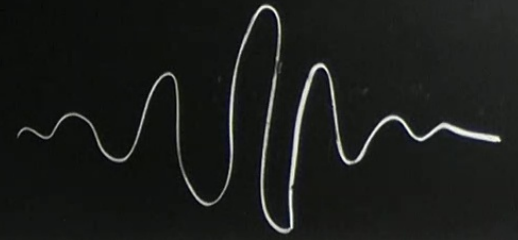
$$\Gamma_b = \Gamma_{ab}^a$$

$$\Gamma_b = 0 \Leftrightarrow \partial X^a = 0$$

$$g^d \partial_c \partial_d g_{ab} + \dots$$



$\Rightarrow$  Above eqns don't fix frequency/wavelength  
 $\sin(\omega t + kz)$



$\nabla \cdot \vec{\Phi} = \mu$

$\omega_{lmn} = \omega_R^{(M_1)} + i \omega_I^{(M_2)}$

$\rightarrow$  Measuring more than 3 QNM

$\times \left(\frac{v}{c}\right)$  small

$\times \zeta$  small

$\times \frac{M}{M_{\text{pl}}}$

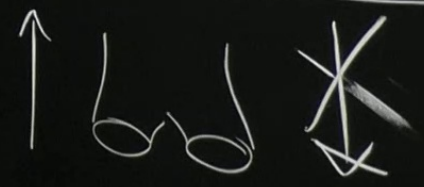
$\Gamma_b = \Gamma_{\text{pl}}^2$

- Precision
- Mass ratios?
- $\frac{M_1}{M_2} \approx 1 \rightarrow 20$

Black Holes

- Governed
- Hide sing
- Can't join
- Ultimate

$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2$





$$(e^{i\omega t + i k z}) \sim \sin(\omega t + k z)$$

⇒ Above eqns don't fix  $\omega$  and  $k$   
 $\sin(\omega t + k z)$

$$\square \phi = \mu$$

$$\square \Phi = \mu$$

Kerr BH  
M

$$\omega_{\text{EMH}} = \omega_z + i \omega_s$$

→ Measuring more than 3 QNM

×  $(\frac{v}{c})$  small

×  $G$  small

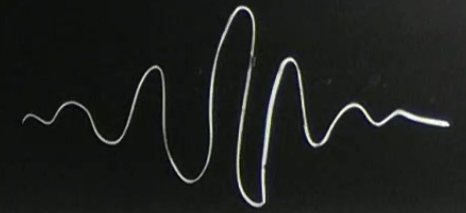
×  $\frac{M_c}{m_{\text{pl}}} \rightarrow$  small (EMRI)

× EFT, Amplitudes (Porto)

$$\Gamma_b = \Gamma_{\text{a}}^2$$

- Precision
- Mass ratios?
- $\frac{M_1}{M_2} \approx 1 \rightarrow 20$

$$L \approx L_0 (1 + \frac{1}{2} h_{xx})$$



### Black Holes

- Governed by
- Hide singular
- Can join
- Ultimate sta

$$ds^2 = -(1 - \frac{2M}{r}) dt^2 + \frac{1}{(1 - \frac{2M}{r})} dr^2 + r^2 d\Omega^2$$

