Title: Amplitudes

Speakers: Henriette Elvang

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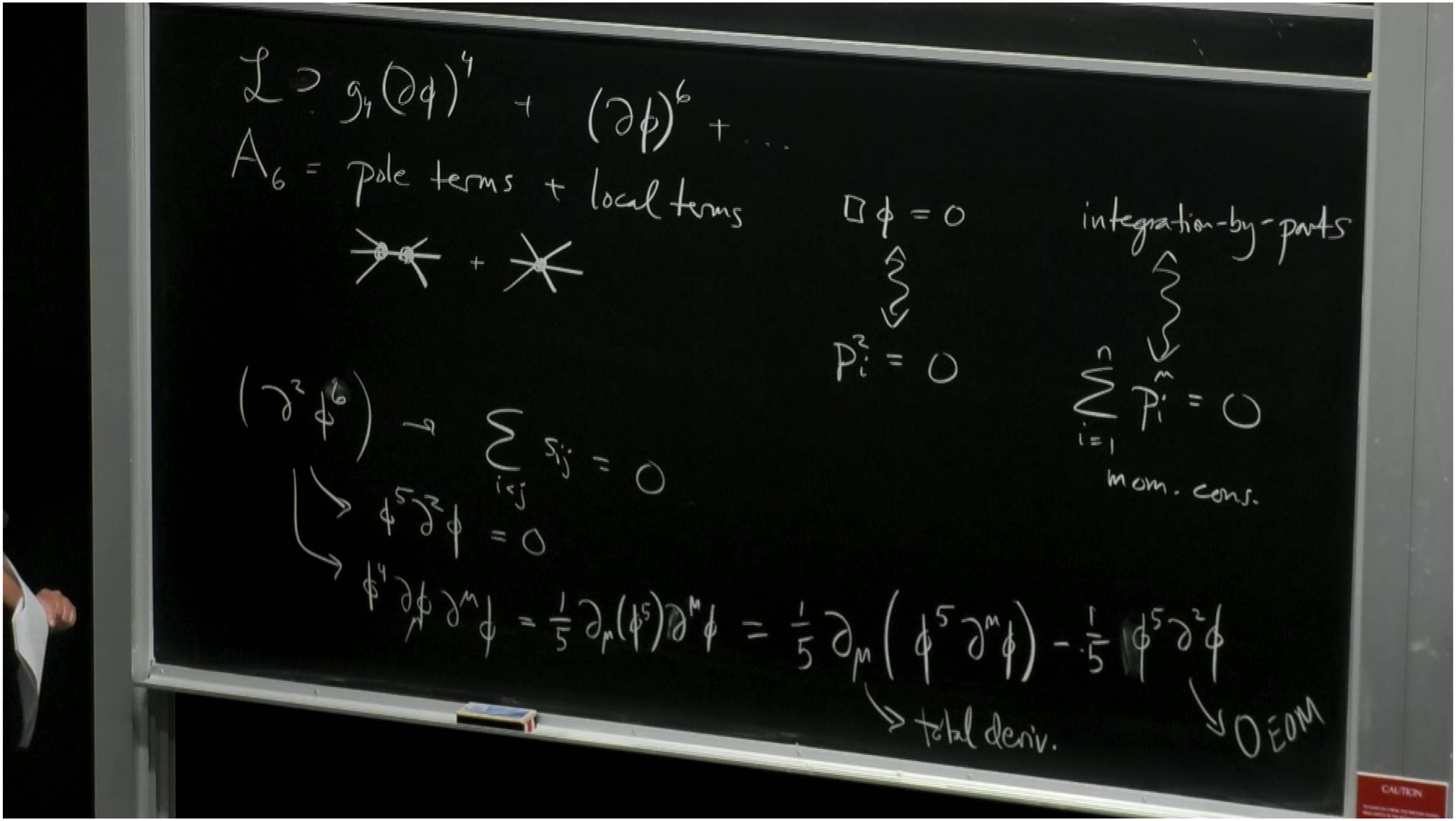
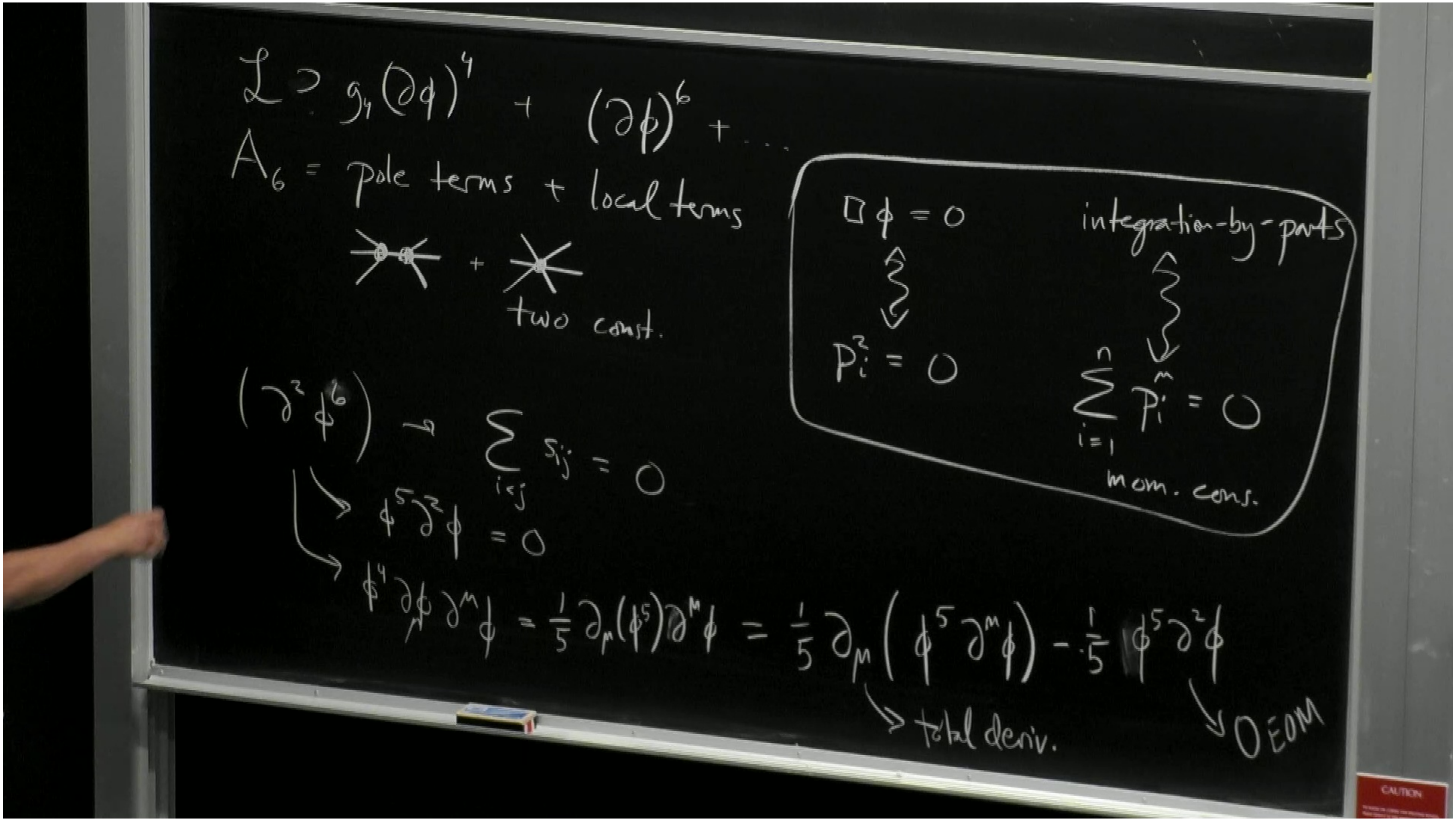

$\mathcal{L} \supset g_4 (\partial\phi)^4 + (\partial\phi)^6 + \ldots$
$A_6 =$ pole terms + local terms
$\Box\phi = 0$
integration-by-parts
$p_i^2 = 0$
$\sum_{i=1}^{n} p_i^\mu = 0$
mom. cons.
$(\partial^2\phi^6) \to \sum_{i<j} s_{ij} = 0$
$\phi^5\partial^2\phi = 0$
$\phi^4\partial_\mu\phi\,\partial^\mu\phi = \frac{1}{5}\partial_\mu(\phi^5)\partial^\mu\phi = \frac{1}{5}\partial_\mu(\phi^5\partial^\mu\phi) - \frac{1}{5}\phi^5\partial^2\phi$
total deriv.
0 EOM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mom. cons.

$$\mathcal{L} \supset g_4(\partial\phi)^4 + g_6(\partial\phi)^6 + \dots$$

A_6 = pole terms + local terms

two const.

if $\phi \to \phi + \text{const} + v_\mu x^\mu \to$ "soft theorem" Adler zero

$$(\partial^2\phi^6) \to \sum_{i<j} s_{ij} = 0$$

$$\to \phi^5\partial^2\phi = 0$$

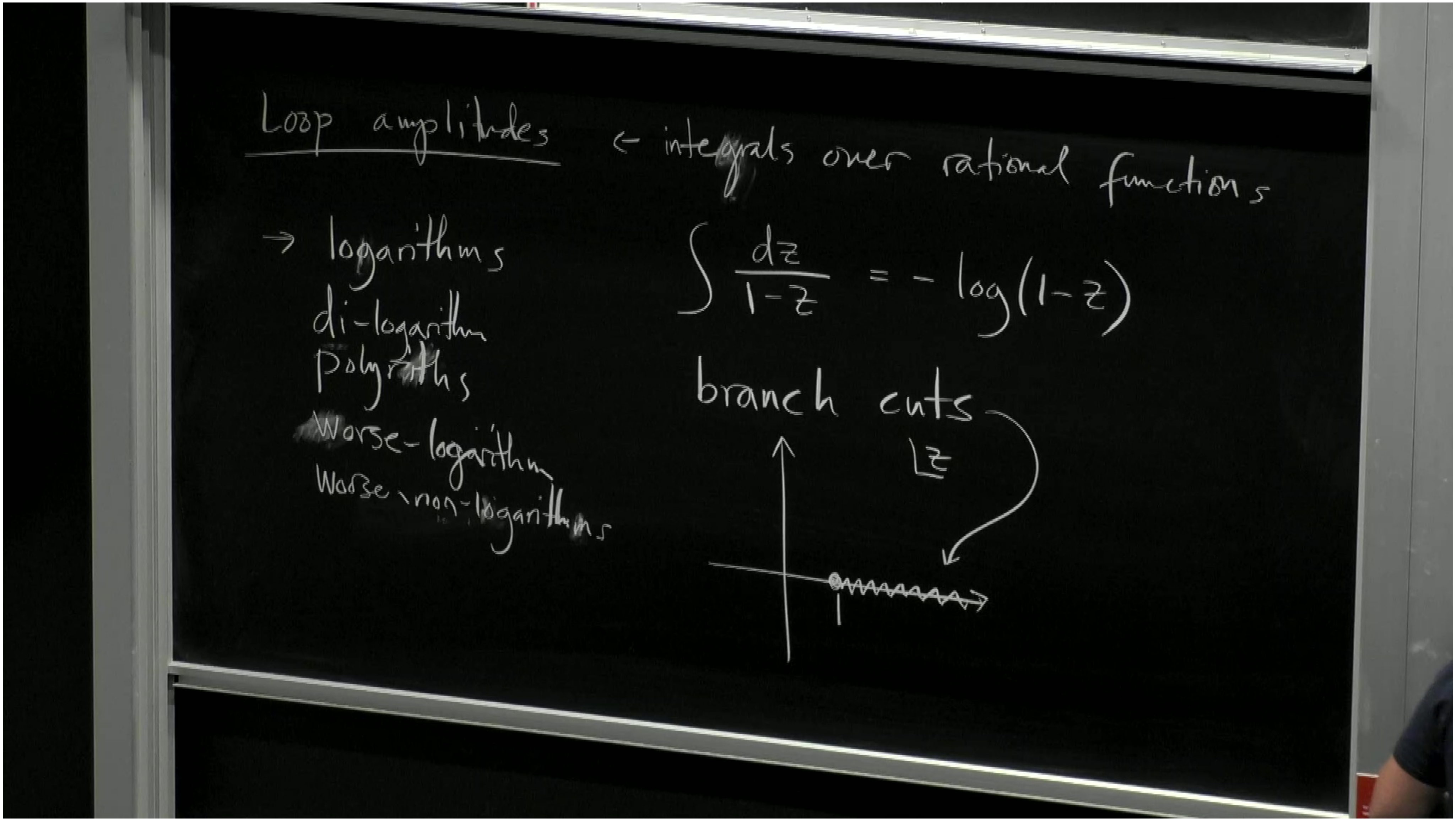
$$\to \phi^4\partial_\mu\phi\,\partial^\mu\phi = \frac{1}{5}\partial_\mu(\phi^5)\partial^\mu\phi = \frac{1}{5}\partial_\mu(\phi^5\partial^\mu\phi) - \frac{1}{5}\phi^5\partial^2\phi$$

total deriv. $\qquad$ 0 EOM

$\Box\phi = 0 \Leftrightarrow p_i^2 = 0$

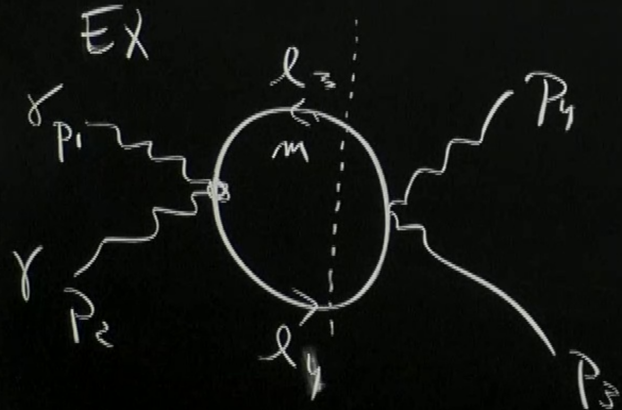
integration-by-parts $\Leftrightarrow \sum_{i=1}^{n} p_i^\mu = 0$

mom. cons.


Loop amplitudes ← integrals over rational functions
→ logarithms
di-logarithm
polylogarithms
worse-logarithm
worse non-logarithms
$\int \frac{dz}{1-z} = -\log(1-z)$
branch cuts
z
1

Branch cuts in amplitudes appear starting at the threshold of physical particle production.

Ex



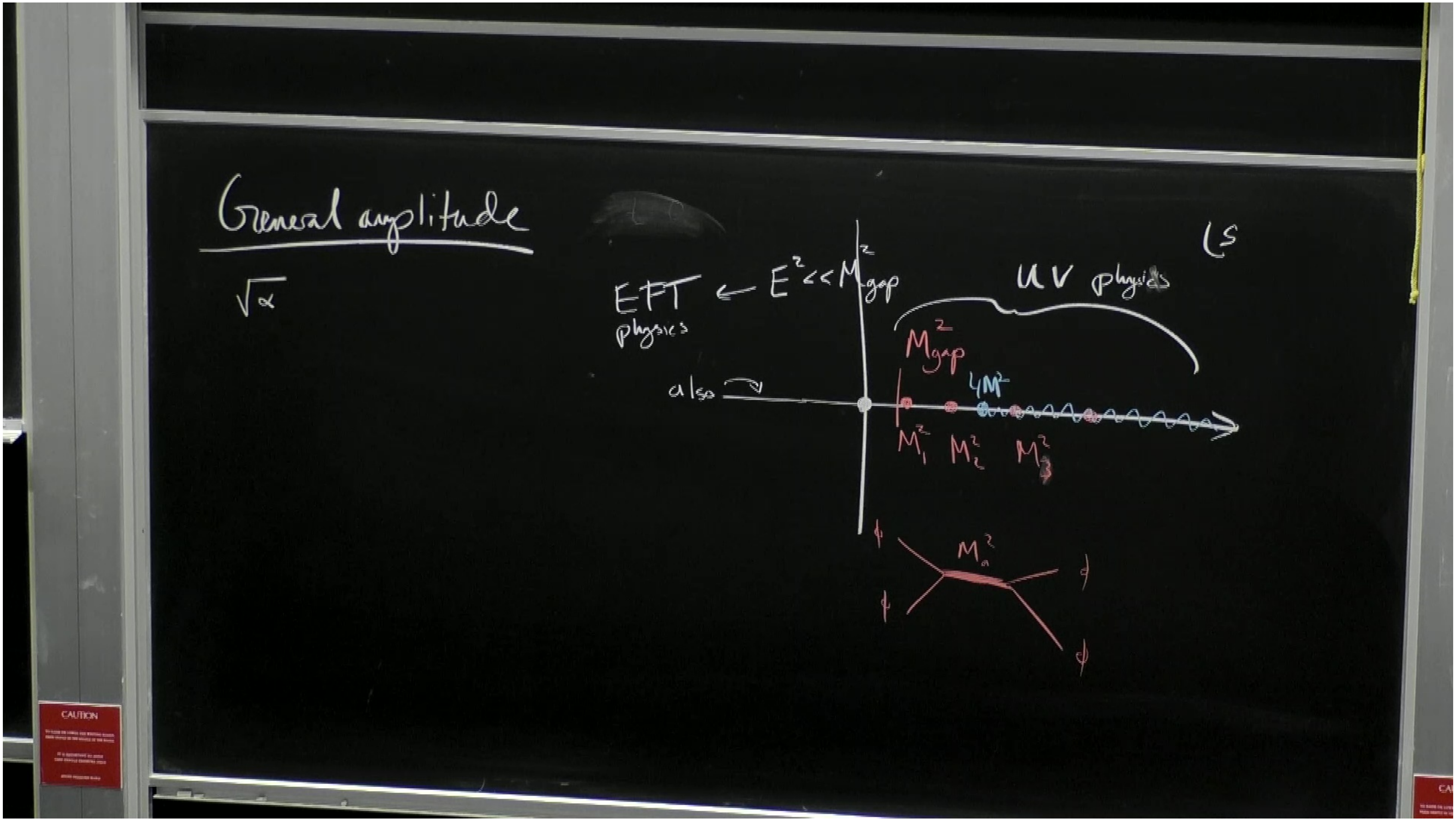
CM frame $E_{CM}^2 = s = (p_1+p_2)^2 = (l_3+l_4)^2$

↑ mom. cons.

pick $l_3^\mu = (m, \vec{0}) = l_4^\mu$

→ create the loop-particle at rest

when $\underbrace{s = (m+m)^2 = 4m^2}_{\text{threshold}}$


General amplitude
√α
EFT ← E² << M²gap
physics
UV physics
also
M²gap
4M²
M₁² M₂² M₃²
φ
M²a
φ
φ
φ
CAUTION

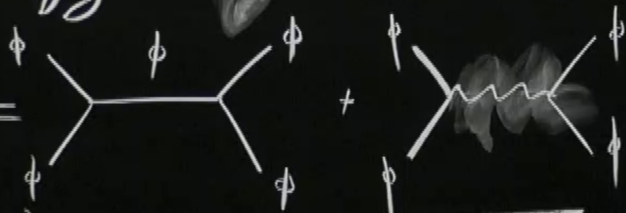
Bootstrap of the S-matrix

$2 \to 2$ 4pt ampl.

Low-energy ϕ massless

s, u , $s+t+u=0$

$$A_4 = $$



$$\underbrace{\quad\quad}_{\text{pole terms}} + \Big[\underset{\phi^4}{b_0} + \underbrace{b_{1,0}\, s + b_{1,1}\, u}_{\partial^2\phi^4}$$

$$+ \underbrace{b_{2,0}\, s^2 + b_{2,1}\, s u + b_{2,2}\, u^2}_{\partial^4\phi^4} + \ldots \Big]$$

massive particles $> M_{gap}^2$ "integrated out"

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$2 \to 2$ 4pt ampl.

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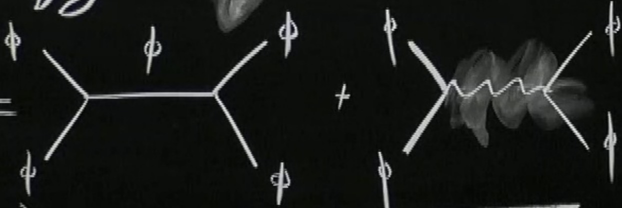
$s, u \quad , \quad s+t+u=0$

$$A_4 =$$



pole terms

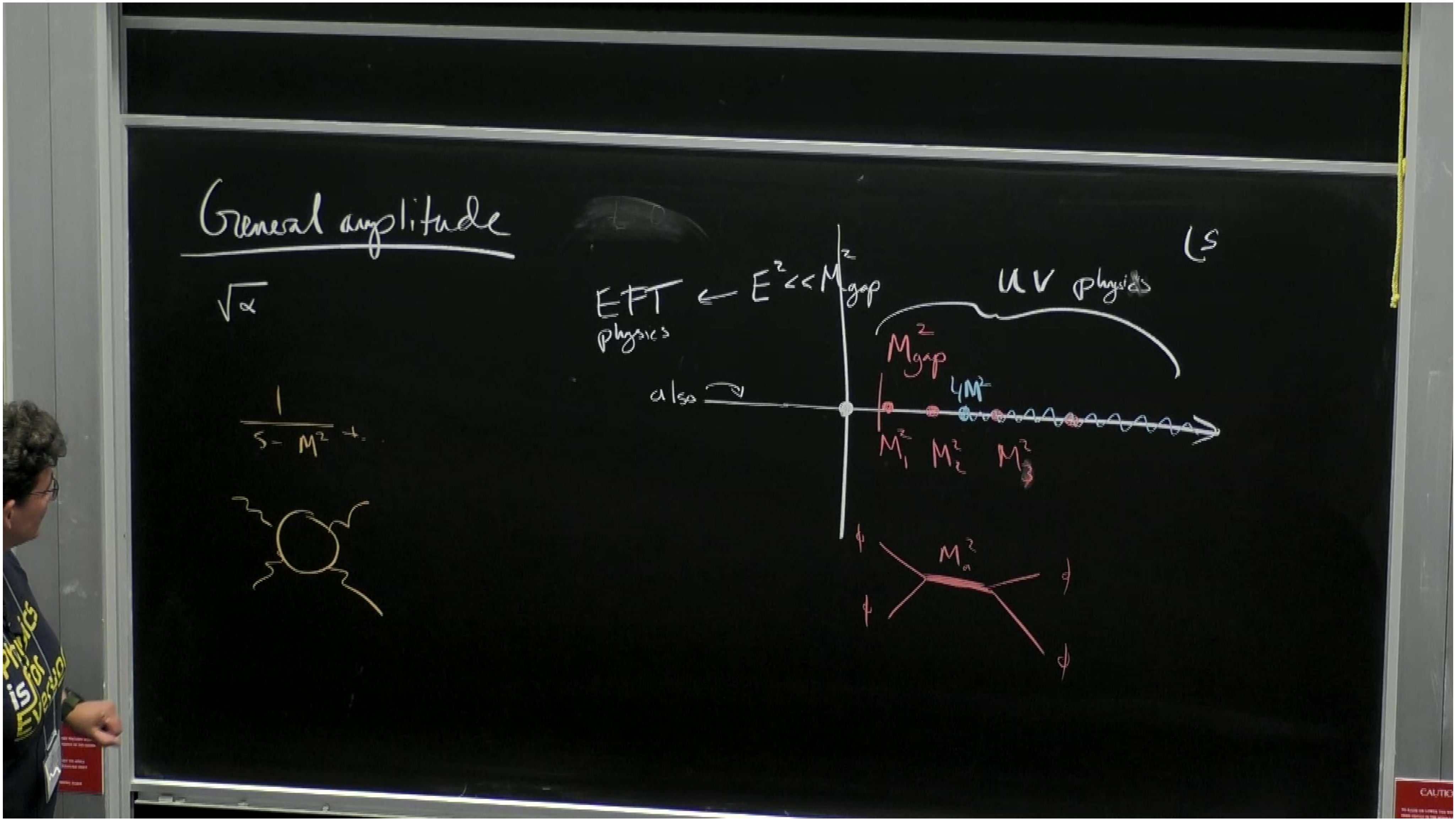
$$+\Big[\underbrace{b_0}_{\phi^4} + \underbrace{b_{1,0}\,s + b_{1,1}\,u}_{\partial^2\phi^4}$$

$$+ \underbrace{b_{2,0}\,s^2 + b_{2,1}\,su + b_{2,2}}_{\partial^4\phi^4}\,u^2 + \ldots\Big]$$

massive particles, $\to M_{gap}^2$

"integrated out"

Unknown UV physics.



General amplitude
$\sqrt{\alpha}$
$\frac{1}{s-M^2}+\ldots$
EFT ← $E^2 \ll M^2_{gap}$
physics
UV physics
M^2_{gap}
$4M^2$
also
M_1^2 M_2^2 M_3^2
M_a^2

unknown UV physics.

S-matrix bootstrap

We apply this in the context of $\mathcal{N}=4$ SYM at large $N \leftarrow$ rank of gauge grp.

Why? $\nearrow$ technical simplification of the analysis

$\searrow$ includes the open superstring amplitude (Veneziano)

S-matrix bootstrap

We apply this in the context of $\mathcal{N}=4$ SYM at large N $\leftarrow$ rank of gauge grp.

<u>Why</u>? $\nearrow$ technical simplification of the analysis

$\searrow$ includes the open superstring amplitude (Veneziano)

$\mathcal{N}=4$ SYM : YM gluons massless spin 1

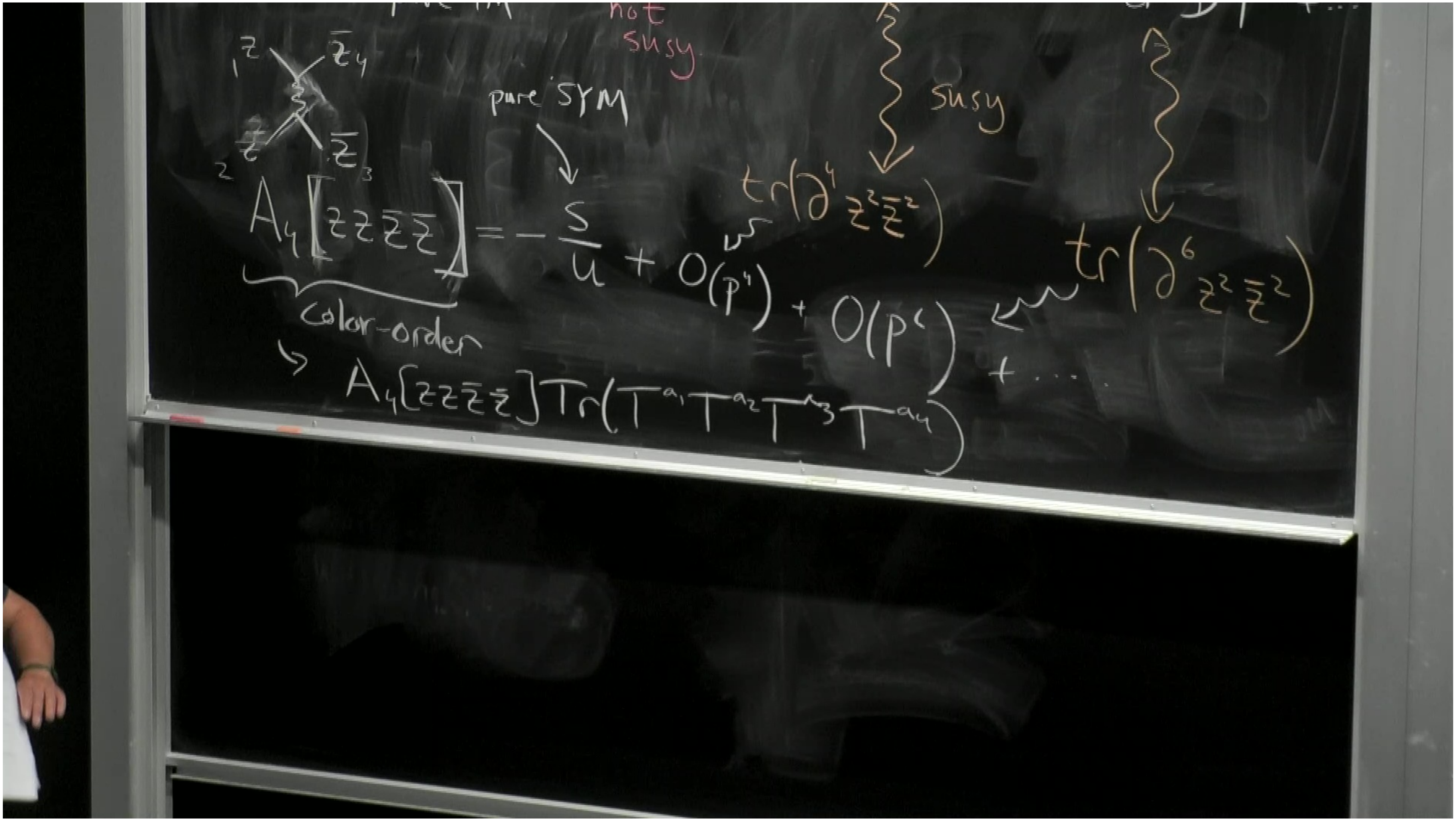
- 4 gluinos spin-$\frac{1}{2}$
- 6 scalars (real) spin 0
  - $\hookrightarrow$ 3 complex scalars $\supset Z, \bar{Z}$ $\leftarrow$ hero

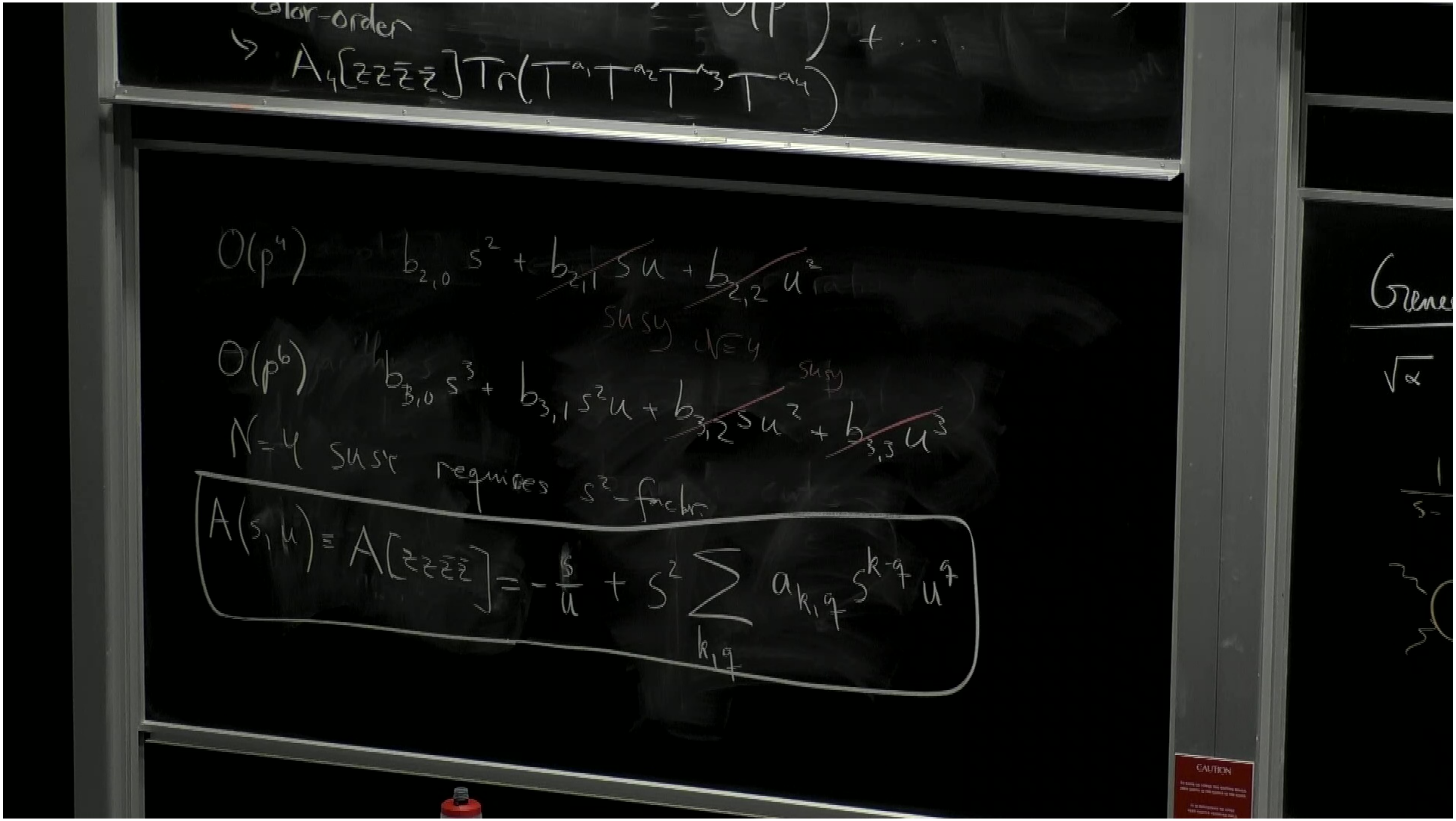
$\mathcal{N}=4$ SYM + h.d.

$$\mathcal{L} = \underbrace{-\frac{1}{4}\,\mathrm{tr}\,F^2}_{\text{pure YM}} + \underbrace{\cancel{\mathrm{tr}\,F^3}}_{\text{not susy}} + \mathrm{tr}\,F^4 + \mathrm{tr}\,D^2F^4 + \dots$$

$$\mathrm{tr}\,F^4 \;\overset{\text{susy}}{\longleftrightarrow}\; \mathrm{tr}\left(\partial^4 Z^2 \bar{Z}^2\right)$$

$$\mathrm{tr}\,D^2F^4 \;\longleftrightarrow\; \mathrm{tr}\left(\partial^6 Z^2 \bar{Z}^2\right)$$


not susy
susy
pure SYM
$A_4[zz\bar{z}\bar{z}] = -\frac{s}{u} + O(p^4) + O(p^6) + \ldots$
$\mathrm{tr}(\partial^4 z^2 \bar{z}^2)$
$\mathrm{tr}(\partial^6 z^2 \bar{z}^2)$
color-order
$A_4[zz\bar{z}\bar{z}]\,\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$


color-order
$\to A_4[z z \bar{z} \bar{z}]\,\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$
$O(p^4)$ $b_{2,0}\, s^2 + \cancel{b_{2,1}\, s u} + \cancel{b_{2,2}\, u^2}$
susy
$N=4$
$O(p^6)$ $b_{3,0}\, s^3 + b_{3,1}\, s^2 u + \cancel{b_{3,2}\, s u^2} + \cancel{b_{3,3}\, u^3}$
susy
$N=4$ susy requires s^2-factor
$A(s,u) \equiv A[z z \bar{z} \bar{z}] = -\frac{s}{u} + s^2 \sum_{k,q} a_{k,q}\, s^{k-q} u^q$

$$A_4[zz\bar{z}\bar{z}]\,\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$$

SUSY $\mathcal{N}=4$

$O(p^6)$ $\quad b_{3,0}\,s^3 + b_{3,1}\,s^2u + \cancel{b_{3,2}\,su^2} + \cancel{b_{3,3}\,u^3}$

same

susy

$\mathcal{N}=4$ SUSY requires s^2-factor:

$$\boxed{A(s,u) \equiv A[zz\bar{z}\bar{z}] = -\frac{s}{u} + s^2 \underbrace{\sum_{k,q} a_{k,q}\, s^{k-q} u^q}}$$

$$a_{k,q} = a_{k,k-q} \qquad \mathcal{N}=4 \text{ SUSY}$$

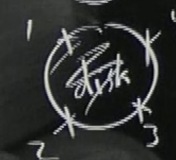
$a_{0,0}$ coeff. of $\mathrm{tr}\, F^4$

$a_{1,0} = a_{1,1}$ $\quad \mathcal{N}=4$ susy $\mathrm{tr}(D^2 F^4)$

$$\left.\begin{matrix} a_{2,0} = a_{2,2} \\ a_{2,1} \end{matrix}\right\} \text{ — '' — '' — } \mathrm{tr}(D^4 F^4)$$

Veneziano open string

$$a_{0,0} = \frac{\pi^2}{\underbrace{6}_{\zeta(2)}} \alpha'^2$$



$$a_{1,0} = a_{1,1} = \zeta(3)\, \alpha'^3$$

$$a_{2,0} = a_{2,2} = \frac{\pi^4}{90} \alpha'^4$$

$$a_{2,1} = \frac{1}{4} \frac{\pi^4}{90} \alpha'^4$$

$\vdots$

$$a^{str}_{k,q} \geq 0$$

large $N \leftarrow$ rank of gauge grp.

Write

$$a_{k,q} = \frac{1}{q!}\left(\frac{\partial}{\partial u}\right)^{q} \oint_{\mathcal{C}_0} \frac{ds'}{2\pi i}\, \underbrace{\frac{A(s',u)}{s'^{\,k-q+3}}}\Bigg|_{u=0}$$

$$a_{k,q}\,\frac{s^2\, s^{k-q}\, u^q}{s^{k-q+3}} = \frac{1}{s}\, u^q\, a_{k,q} \xrightarrow{\frac{1}{q!}\left(\frac{\partial}{\partial u}\right)^{q}} a_{k,q}\,\frac{1}{s}$$

Write

$$a_{k,q} = \frac{1}{q!}\left(\frac{\partial}{\partial u}\right)^q \oint_{\mathcal{C}_0} \frac{ds'}{2\pi i} \underbrace{\frac{A(s',u)}{s'^{\,k-q+3}}}\Bigg|_{u=0}$$

$$a_{k,q}\,\frac{s^2\, s^{k-q} u^q}{s^{k-q+3}} = \frac{1}{s} u^q a_{k,q} \xrightarrow{\frac{1}{q!}\left(\frac{\partial}{\partial u}\right)^q} a_{k,q}\frac{1}{s}$$

all k, q

$$a_{k,q} = \text{pole-terms} + \text{branch-cut} + \text{big arcs} \to 0 \quad \text{assumption} \quad \frac{A(s,u)}{s^2} \to 0 \quad |s| \to \infty$$

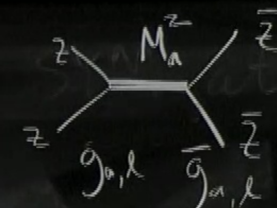
Froissart-Martin bound.

6 scalars (red) spin 0

↳ 3 complex scalars ⊃ $Z, \bar{Z}$ ← here

+ big arcs $\to 0$ assump... s^2 Froissart-Martin bound.

$$a_{k,q} = \frac{1}{q!}\left(\frac{\partial}{\partial u}\right)^q \left[\sum_{\substack{a \\ \text{poles}}} \frac{\mathrm{Res}_{s=M_a^2} A(s',u)}{(M_a^2)^{k-q+3}} + \int_{4m^2} \frac{ds'}{s'^{\,k-q+3}}\, \mathrm{Disc}\, A(s',u)\right]$$



$$\mathrm{Res}_{s'=M_a^2} A(s',u) = -\sum_{\substack{\ell=0 \\ \text{spin}}}^{\infty} |g_{a,\ell}|^2\, P_\ell\left(1+\frac{2u}{M_a^2}\right)$$

Legendre.

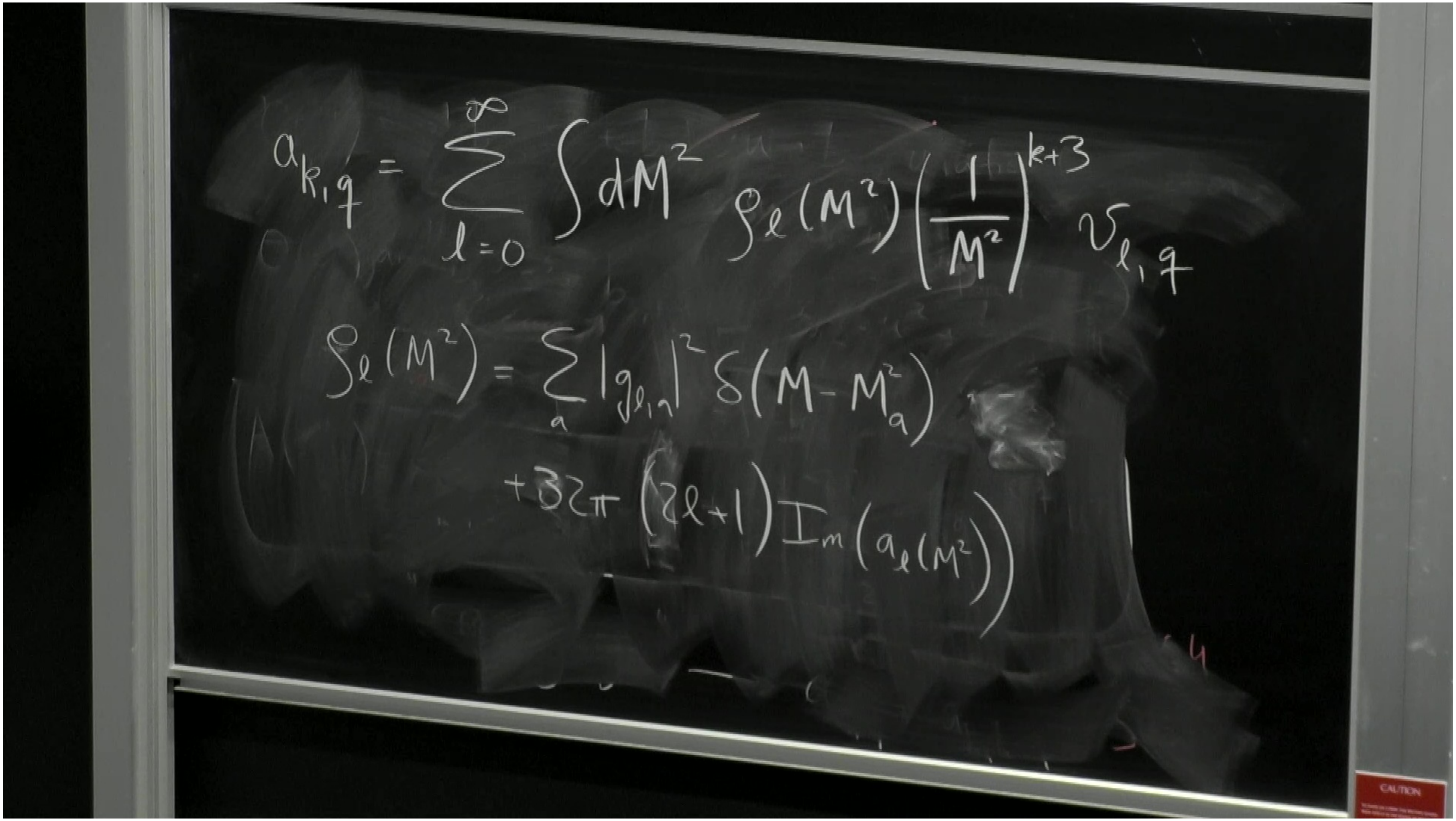
$$\text{Disc} = -2\,\text{Im}\,A = -32\pi \sum_{\ell=0}^{\infty} (2\ell+1)\,\text{Im}(a_\ell(s')) \times P_\ell\left(1+\frac{2u}{s'}\right)$$

$$\text{Im}(a_\ell(s)) \geq 0$$ by unitarity.

$$\underbrace{P_\ell(1+\delta)}_{\cos\theta} = \sum_{q=0}^{\ell} v_{\ell,q}\,\delta^q$$

$$v_{\ell,q} = \frac{\prod_{a=1}^{q}\left(\ell(\ell+1) - a(a+1)\right)}{(q!)^2} > 0$$

crossing sym $\longrightarrow$ $a_{k,q} = a_{k,k-q}$ SUSY



$a_{k,q} = \sum_{\ell=0}^{\infty} \int dM^2 \, \rho_\ell(M^2) \left(\frac{1}{M^2}\right)^{k+3} v_{\ell,q}$
$\rho_\ell(M^2) = \sum_a |g_{\ell,a}|^2 \delta(M - M_a^2)$
$+ 32\pi (2\ell+1) \, \mathrm{Im}\left(a_\ell(M^2)\right)$