

Title: Amplitudes

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URL: <https://pirsa.org/23060075>

$$\mathcal{L} \supset g_4 (\partial\phi)^4 + (\partial\phi)^6 + \dots$$

$A_6 =$  pole terms + local terms



$$\square \phi = 0$$

$$\Downarrow$$

$$P_i^2 = 0$$

integration-by-parts

$$\Downarrow$$

$$\sum_{i=1}^m P_i^m = 0$$

mom. cons.

$$\left( \partial^2 \phi^6 \right)_b \sum_{i,j} s_{ij} = 0$$

$$\phi^5 \partial^2 \phi = 0$$

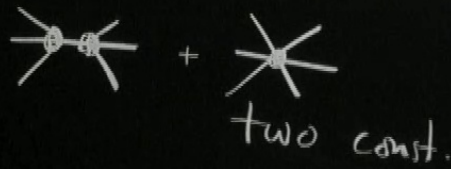
$$\phi^4 \partial_\mu \phi \partial^\mu \phi = \frac{1}{5} \partial_\mu (\phi^5) \partial^\mu \phi = \frac{1}{5} \partial_\mu (\phi^5 \partial^\mu \phi) - \frac{1}{5} \phi^5 \partial^2 \phi$$

total deriv.

→ 0 EOM

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$\rightarrow$  total deriv.  $\rightarrow$  0 EOM

$$\mathcal{L} \supset g_4 (\partial\phi)^4 + g_6 (\partial\phi)^6 + \dots$$

$$A_6 = \text{pole terms} + \text{local terms}$$

if  + two const.

$\phi \rightarrow \phi + \text{const} + v_\mu x^\mu \rightarrow$  "soft theorem" Adler zero

$$(\partial^2 \phi^6) \rightarrow \sum_{i < j} s_{ij} = 0$$

$$\rightarrow \phi^5 \partial^2 \phi = 0$$

$$\rightarrow \phi^4 \partial_\mu \phi \partial^\mu \phi = \frac{1}{5} \partial_\mu (\phi^5) \partial^\mu \phi = \frac{1}{5} \partial_\mu (\phi^5 \partial^\mu \phi) - \frac{1}{5} \phi^5 \partial^2 \phi$$

$\rightarrow$  total deriv.

$\rightarrow$  0 EOM

$\square \phi = 0$   
 $\Downarrow$   
 $P_i^2 = 0$

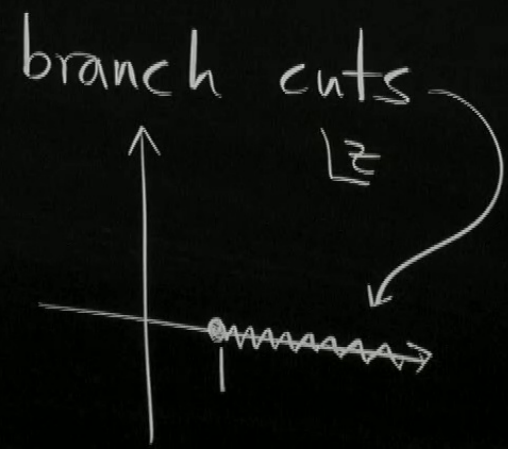
integration-by-parts  
 $\Downarrow$   
 $\sum_{i=1}^n P_i^m = 0$   
 mom. cons.



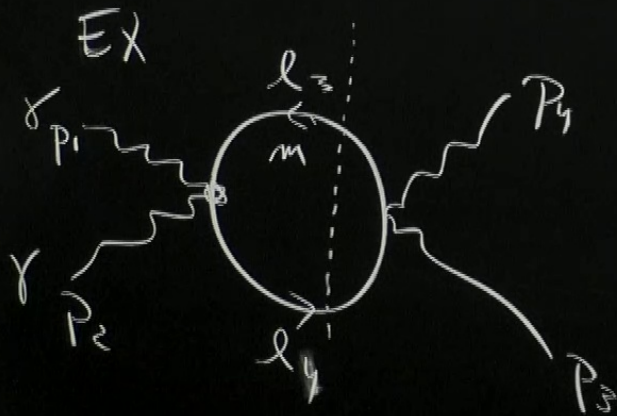
Loop amplitudes ← integrals over rational functions

- logarithms
- di-logarithms
- polylogarithms
- Wasser-logarithms
- Wasser-non-logarithms

$$\int \frac{dz}{1-z} = -\log(1-z)$$



Branch cuts in amplitudes appear starting at the threshold of physical particle production.



$$\text{CM frame } E_{\text{cm}}^2 = S = (p_1 + p_2)^2 = (l_3 + l_4)^2$$

Pick  $l_3^M = (m, \vec{0}) = l_4^M$  ↑ mom. cons.

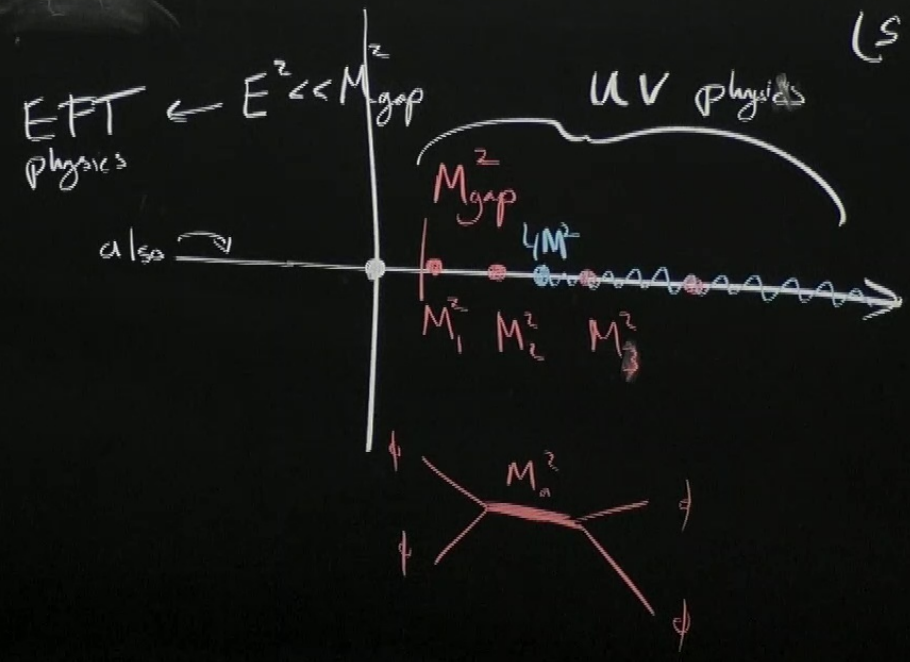
→ create the loop-particle at rest

when  $S = (m + m)^2 = 4m^2$

threshold.

# General amplitude

$$\sqrt{\alpha}$$



CAUTION  
Do not touch the screen when the screen is on. The screen is very sensitive to touch. Do not touch the screen when the screen is on. Do not touch the screen when the screen is on.



# Bootstrap of the S-matrix

$$Z \rightarrow \mathbb{R}$$

4pt amplit.

Low-energy

$\phi$  massless

$s, u$

$$s+t+u=0$$

$$A_4 = \underbrace{\text{diagrams}}_{\text{pole terms}} + \left[ b_0 \phi^4 + \underbrace{b_{1,0}s + b_{1,1}u}_{\partial^2 \phi^4} \right]$$

The diagrams show three tree-level exchange processes for a 4-point amplitude. The first is s-channel exchange, the second is t-channel exchange, and the third is u-channel exchange. Each diagram has external legs labeled with  $\phi$  and arrows indicating particle flow.

$$+ \underbrace{b_{2,0}s^2 + b_{2,1}su + b_{2,2}u^2}_{\partial^4 \phi^4} + \dots$$

↑ massive particles "integrated out"  $\rightarrow M_{\text{gap}}$





# Bootstrap of the S-matrix

$$Z \rightarrow \mathbb{R}$$

4pt amplit.

Low-energy

$\phi$  massless

$s, u$

$$s+t+u=0$$

$$A_4 = \underbrace{\text{[Feynman diagrams: s-channel, t-channel, u-channel exchange]} + \text{[Feynman diagram: contact]} + \left[ b_0 \phi^4 + \underbrace{b_{1,0}s + b_{1,1}u}_{\partial^2 \phi^4} \right]}_{\text{pole terms}}$$

$$+ \underbrace{b_{2,0}s^2 + b_{2,1}su + b_{2,2}u^2 + \dots}_{\partial^4 \phi^4}$$

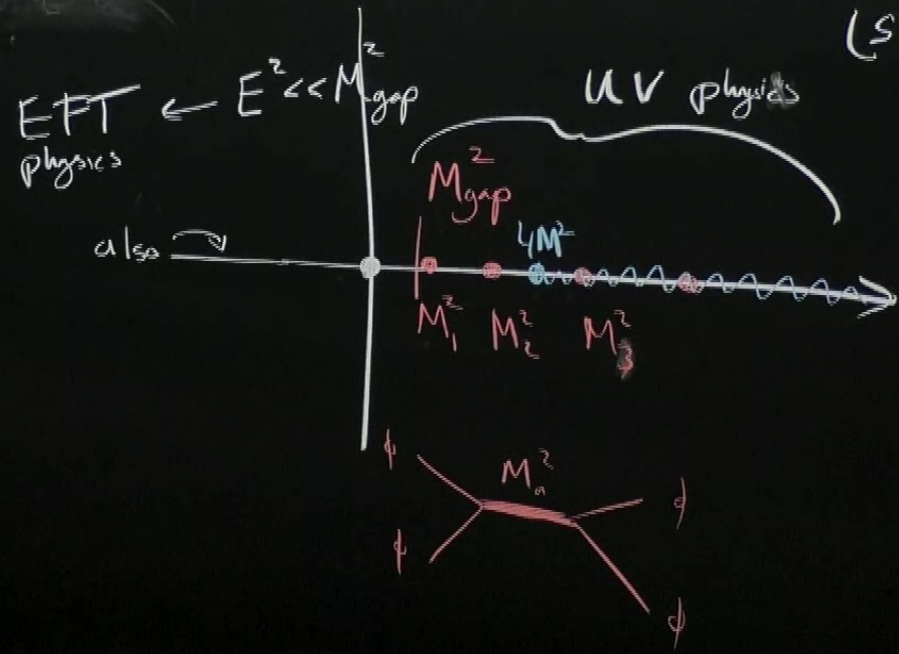
Unknown UV physics. massive particles "integrated out"  $\rightarrow M_{\text{gap}}$



# General amplitude

$$\sqrt{\alpha}$$

$$\frac{1}{s - M^2 + i\epsilon}$$



Unknown UV physics.

We apply this  $\rightarrow$  S-matrix bootstrap  
in the context of  $\mathcal{N}=4$  SYM at  
large  $N \leftarrow$  rank of gauge group.

Why?  $\rightarrow$  technical simplification of the analysis  
 $\rightarrow$  includes the open superstring amplitude  
(Veneziano)



We apply this <sup>→ S-matrix bootstrap</sup> in the context of  $U=4$  SYM at large  $N \leftarrow$  rank of gauge group.

Why? → technical simplification of the analysis  
→ includes the open superstring amplitude (Veneziano)

$U=4$  SYM : (YM) gluons massless spin 1  
4 gluinos spin  $-\frac{1}{2}$   
6 scalars (real) spin 0  
↳ 3 complex scalars  $\Rightarrow z, \bar{z} \leftarrow$  hero

$\mathcal{N}=4$  SYM + h.d.

$$\mathcal{L} = \underbrace{-\frac{1}{4} \text{tr} F^2}_{\text{pure YM}} + \cancel{\text{tr} F^3} + \text{tr} F^4 + \text{tr} D^2 F^4 + \dots$$

*not susy*

*susy*

$\text{tr}(\partial^4 z^2 \bar{z}^2)$

$\text{tr}(\partial^6 z^2 \bar{z}^2)$

not susy

pure SYM

susy

susy

$A_4[z z \bar{z} \bar{z}] = -\frac{s}{u} + O(p^4) + O(p^6) + \dots$

Color-order  
 $\rightarrow A_4[z z \bar{z} \bar{z}] \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4})$

$\text{tr}(\partial^4 z^2 \bar{z}^2)$

$\text{tr}(\partial^6 z^2 \bar{z}^2)$



color-order

$$\rightarrow A_4[z\bar{z}\bar{z}\bar{z}] \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) + \dots$$

$$O(p^4) = b_{2,0} s^2 + \cancel{b_{2,1} s u} + \cancel{b_{2,2} u^2}$$

Susy  $\checkmark$

$$O(p^6) = b_{3,0} s^3 + b_{3,1} s^2 u + \cancel{b_{3,2} s u^2} + \cancel{b_{3,3} u^3}$$

Susy

$N=4$  SUSY requires  $s^2$ -factor

$$A(s, u) \equiv A[z\bar{z}\bar{z}\bar{z}] = -\frac{s}{u} + s^2 \sum_{k,q} a_{k,q} s^{k-q} u^q$$

$$A_4(z\bar{z}\bar{z}\bar{z}) \ln(T^{a_1} T^{a_2} T^{a_3} T^{a_4})$$

$$O(p^6) \quad b_{3,0} s^3 + b_{3,1} s^2 u + b_{3,2} s u^2 + b_{3,3} u^3$$

$\xrightarrow{\text{same}}$

$N=4$  SUSY requires  $s^2$ -factor

$$A(s, u) \equiv A[z\bar{z}\bar{z}\bar{z}] = -\frac{s}{u} + s^2 \sum_{k,q} a_{k,q} s^{k-q} u^q$$

$$a_{k,q} = a_{k,k-q} \quad N=4 \text{ SUSY}$$



$a_{0,0}$  coeff. of  $\text{tr} F^4$

$a_{1,0} = a_{1,1}$   $\mathcal{N}=4$  susy  $\text{tr}(D^2 F^4)$

$a_{2,0} = a_{2,2}$   
 $a_{2,1}$  }  $\text{tr}(D^4 F^4)$

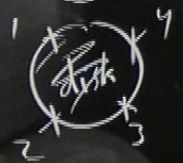
Veneziano open string

$a_{0,0} = \frac{\pi^2}{6} \alpha'^2$

$a_{1,0} = a_{1,1} = \frac{5}{3} \alpha'^3$

$a_{2,0} = a_{2,2} = \frac{\pi^4}{90} \alpha'^4$

$a_{2,1} = \frac{1}{4} \frac{\pi^4}{90} \alpha'^4$



$a_{k,q}^{\text{str}} = 0$

CAUTION  
 TO AVOID INJURY AND DAMAGE TO THE BOARD  
 DO NOT TOUCH THE BOARD WHEN IT IS HOT  
 ALWAYS WEAR GLOVES

CAUTION



large  $N \leftarrow$  rank of gauge group.

Write

$$a_{k,q} = \frac{1}{q!} \left( \frac{\partial}{\partial u} \right)^q \int_{\gamma_0} \frac{ds'}{2\pi i} \frac{A(s', u)}{s'^{k-q+3}} \Big|_{u=0}$$

$$a_{k,q} = \frac{1}{q!} \left( \frac{\partial}{\partial u} \right)^q \frac{\int_{\gamma_0} \sum s^{k-q} u^q}{s^{k-q+3}} = \frac{1}{s} u^q a_{k,q} \xrightarrow{\frac{1}{q!} \left( \frac{\partial}{\partial u} \right)^q} a_{k,q} \frac{1}{s}$$

CAUTION

DO NOT TOUCH THE BOARD OR THE BOARDER

IT IS PROHIBITED TO USE ANY OTHER OBJECTS ON THE BOARD

PLEASE REPORT ANY DAMAGE TO THE BOARD

Write

$$a_{k,q} = \frac{1}{q!} \left( \frac{\partial}{\partial u} \right)^q \int_{\mathcal{C}_0} \frac{ds'}{2\pi i} \frac{A(s', u)}{s'^{k-q+3}} \Big|_{u=0}$$

$$\frac{1}{q!} \left( \frac{\partial}{\partial u} \right)^q \frac{1}{s} u^q a_{k,q} \rightarrow a_{k,q} \frac{1}{s}$$

all  $k, q$

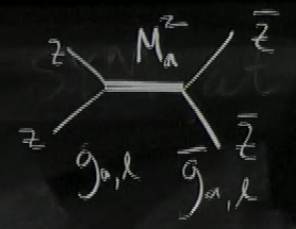
$a_{k,q}$  = pole-terms  
 + branch-cut  
 + big arcs  $\rightarrow 0$  assumption  $\frac{A(s,u)}{s^2} \rightarrow 0$   $|s| \rightarrow \infty$   
 Froissart-Martin bound.

↓ 6 scalars (red) spin 0  
 ↳ 3 complex scalars  $\supset \mathbb{Z} \bar{\mathbb{Z}} \leftarrow$  hero



+ (big arcs)  $\rightarrow 0$  as  $s \rightarrow \infty$  Froissart-Martin bound.

$$a_{k,q} = \frac{1}{q!} \left( \frac{\partial}{\partial u} \right)^q \left[ \sum_{\text{poles}} \frac{\text{Res}_{s=M_a^z} A(s', u)}{(M_a^z)^{k-q+3}} \right]$$



$$+ \left[ \int \frac{ds'}{4m^2} \text{Disc } A(s', u) \right]$$

$$\bullet \text{Res}_{s=M_a^z} A(s', u) = - \sum_{l=0}^{\infty} |g_{a,l}|^2 P_l \left( 1 + \frac{2u}{M_a^2} \right)$$

Legendre.

CAUTION  
DO NOT TOUCH THE BOARD WHEN  
OTHER PEOPLE ARE USING IT  
IT IS PROHIBITED TO DRINK  
AND EAT AT THE BOARD  
DURING PRESENTATION



$$\bullet \text{ Disc} = -2\text{Im} A = -32\pi \sum_{l=0}^{\infty} (2l+1) \text{Im}(a_l(s))$$

$$\text{Im}(a_l(s)) \geq 0 \quad \text{try unitarity.} \quad \times P_l\left(1 + \frac{2u}{s'}\right)$$

$$P_l(1 + \delta) = \sum_{q=0}^l v_{l,q} \delta^q$$

$\cos \theta$

$$v_{l,q} = \frac{\pi^q}{q!} \frac{l(l+1) - a(a+1)}{(q!)^2} > 0$$

Crossing sym  $\rightarrow a_{k,q} = a_{k,k-q}$  SUSY

$$a_{k,g} = \sum_{l=0}^{\infty} \int dM^2 \rho_l(M^2) \left( \frac{1}{M^2} \right)^{k+3} v_{l,g}$$

$$\rho_l(M^2) = \sum_a |g_{l,a}|^2 \delta(M - M_a) + 32\pi (\ell+1) \text{Im}(a_\ell(M^2))$$