Title: EFT of Gravity

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Effective Field Theory Methods for Gravity #2

Review: Constructing GR as QFT

John Donoghue
June 20, 2023
TRISEP

Global symmetries and currents
Using current as source

$\psi \to U(x)\psi$ local symmetry

$D_\mu \psi \to U(x) D_\mu \psi$ covariant deriv.

$[D_\mu, D_\nu] = i g \underline{\underline{F}}_{\mu\nu} = i g\left[\partial_\mu \underline{\underline{A}}_\nu - \partial_\nu \underline{\underline{A}}_\mu + g[\underline{\underline{A}}_\mu, \underline{\underline{A}}_\nu]\right]$

$T_{\mu\nu}$ as source $\Rightarrow$ local coord changes

$dx'^\mu = J^\mu{}_\nu dx^\nu$

$g_{\mu\nu}$ as source → local coord change

$$dx'^\mu = J^\mu{}_\nu(x)\, dx^\nu$$

$$g'_{\mu\nu} = J^{-1}{}_\mu{}^\alpha(x)\, J^{-1}{}_\nu{}^\beta(x)\, g_{\alpha\beta}$$

If $V'^\mu = J^\mu{}_\nu(x) V^\nu$ want $D_\mu V'^\lambda = J^{-1}{}_\mu{}^\nu J^\lambda{}_\sigma (D_\nu V^\sigma)$

or $D_\mu V^\lambda = \partial_\mu V^\lambda + \Gamma_{\mu\nu}{}^\lambda V^\nu$

with $$\Gamma'_{\mu\nu}{}^\lambda = (J^{-1})^{\mu'}_\mu (J^{-1})^{\nu'}_\nu J^\lambda_{\lambda'} \left(\Gamma_{\mu'\nu'}{}^{\lambda'} + (J^{-1})^{\lambda'}_\sigma \partial_{\mu'} J^\sigma_{\nu'} \right)$$

Field strength $$[D_\mu, D_\nu] V^\alpha = R_{\mu\nu\beta}{}^\alpha V^\beta$$

$$R_{\mu\nu\alpha}{}^\beta = \partial_\mu \Gamma_{\nu\alpha}{}^\beta - \partial_\nu \Gamma_{\mu\alpha}{}^\beta + \Gamma_{\mu\rho}{}^\beta \Gamma_{\nu\alpha}{}^\rho - \Gamma_{\nu\rho}{}^\beta \Gamma_{\mu\alpha}{}^\rho$$

If metricity $D\, g = 0$

$$R_{\mu\nu\alpha}{}^{\beta} = \partial_\mu \Gamma_{\nu\alpha}{}^{\beta} - \partial_\mu \Gamma_{\nu\alpha}{}^{\beta} + \Gamma_{\mu\rho}{}^{\beta}\Gamma_{\nu\alpha}{}^{\rho} - \Gamma\,\Gamma_{\nu\rho}{}^{\beta}\Gamma_{\nu\alpha}{}^{\rho}$$

<u>If</u> metricity $D_\mu g_{\alpha\beta} = 0$

$+$ symmetry $\underbrace{\Gamma_{\mu\nu}{}^{\lambda} = \Gamma_{\nu\mu}{}^{\lambda}}$ $\Big\}$ <u>obtain GR</u>

with

$$\Gamma_{\mu\nu}{}^{\lambda} = \frac{1}{2} g^{\lambda\sigma}\left[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right]$$

<u>Cousins of GR</u>

Cousins of GR

- No symmetry $\Gamma_{\mu\nu}{}^{\lambda} \neq \Gamma_{\nu\mu}{}^{\lambda}$ - torsion "Einstein Cartan"
- Non metricity $D_\mu g_{\alpha\beta} \neq 0$ "metric affine"

With $S_g = \int d^4x \sqrt{-g}\, \frac{2}{\kappa^2} R + S_m$

$\kappa^2 = 32\pi G$

obtain $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$

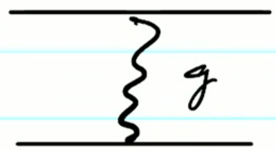
<u>With</u> $S_g = \int d^4x \sqrt{-g}\, \frac{2}{\kappa^2} R + S_m$

$\kappa^2 = 32\pi G$

obtain $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$

$$i\partial_t \psi = \left[-\frac{\nabla^2}{2m} + m\phi_g \right] \psi$$

$$\dot{\vec{p}} = -i[H, \vec{p}] = -m\vec{\nabla}\phi_g = m\vec{a}$$



$\rightarrow V(r) = -G\frac{m_1 m_2}{r}$

and Feynman rules

Still to be done — general background $\bar{g}_{\mu\nu}$
— ghosts

First calculation:

 $\sim \quad \Delta\mathcal{L} = \frac{1}{\varepsilon}\left[a R^2 + b R_{\mu\nu} R^{\mu\nu} \right]$

$(\partial g)^4$

$\neq \# R$

EFT via two examples

1) QED with heavy top quark

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi$$

$$Z = \int [dA_\mu][d\psi][d\bar{\psi}]\, e^{i\int d^4x\, \mathcal{L}(A,\psi)}$$

$$= \int [dA_\mu]_{\Lambda = m_t}\, e^{i\int d^4x\, \mathcal{L}_{eff}(A)}$$

with $\mathcal{L}_{eff} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{(\frac{2}{3}e)^2}{240\pi^2 m_t^2} F_{\mu\nu} \Box F^{\mu\nu}$

2) Linear sigma model

$$\mathcal{L}_\sigma = \frac{1}{2}\partial_\mu\sigma\,\partial^\mu\sigma + \frac{1}{2}\partial_\mu\vec{\phi}\,\partial^\mu\vec{\phi} + \frac{\mu^2}{2}(\sigma^2 + \vec{\phi}^2) - \frac{\lambda}{4}(\sigma^2 + \vec{\phi}^2)^2$$

2) Linear sigma model

$$\mathcal{L}_\sigma = \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2}\partial_\mu \vec{\phi}\, \partial^\mu \vec{\phi} + \frac{\mu^2}{2}(\sigma^2 + \vec{\phi}^2) - \frac{\lambda}{4}(\sigma^2 + \vec{\phi}^2)^2$$

with SSB $\langle \sigma \rangle = v = \sqrt{\frac{\mu^2}{\lambda}}$

$$Z = \int [d\sigma][d\vec{\phi}]\, e^{i\int d^4x\, \mathcal{L}_\sigma(\sigma, \vec{\phi})}$$

$$= \int [d\vec{\pi}]_{n=m_\sigma}\, e^{i\int d^4x\, \mathcal{L}_{eff}(\vec{\pi})}$$

With $\mathcal{L} = \frac{v^2}{4} Tr(\partial_\mu U^\dagger \partial^\mu U) + \ell_1 \left[Tr(\partial_\mu U^\dagger \partial^\mu U)\right]^2$

$$+ \ell_2 Tr[\partial_\mu U^\dagger \partial_\nu U] Tr(\partial^\mu U^\dagger \partial^\nu U)$$

with

$$= \int [d\vec{\pi}]_{\mu = m_\sigma} e^{i \int d^4x \, \mathcal{L}_{eff}(\vec{\pi})}$$

With $\mathcal{L} = \frac{v^2}{4} \mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U) + \ell_1 [\mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U)]^2$

$$+ \ell_2 \mathrm{Tr}[\partial_\mu U^\dagger \partial_\nu U] \mathrm{Tr}(\partial^\mu U^\dagger \partial^\nu U)$$

with

$U = \exp\left[i \frac{\vec{\tau} \cdot \vec{\pi}}{v} \right]$ and

$$\ell_1^r = \frac{v^2}{8m_\sigma^2} + \frac{1}{192\pi^2}\left[\ln \frac{m_\sigma^2}{\mu^2} - \frac{35}{6}\right]$$

$$\ell_2^r = \frac{1}{384\pi^2}\left[\ln \frac{m_\sigma^2}{\mu^2} - \frac{11}{6}\right].$$

$$= \int [d\vec{\pi}]_{h=m_\sigma} e^{i \int d^4x \, \mathcal{L}_{eff}(\vec{\pi})}$$

With $\mathcal{L} = \frac{v^2}{4} \mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U) + \ell_1 [\mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U)]^2$

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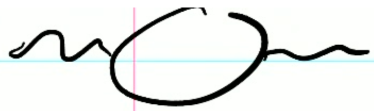
!

QED ċ heavy top

$$\Pi(q) = \frac{e_0^2}{12\pi^2}\left[\frac{1}{\epsilon} + \ln(4\pi) - \gamma \right.$$
$$\left. - 6\int_0^1 dx\, x(1-x)\ln\left(\frac{m^2 - q^2 x(1-x)}{\mu^2}\right) + \mathcal{O}(\epsilon)\right]$$
$$= \frac{e_0^2}{12\pi^2}\begin{cases} \frac{1}{\epsilon} + \ln(4\pi) - \gamma + \frac{5}{3} - \ln\frac{-q^2}{\mu^2} + \cdots & (|q^2| \gg m^2), \\ \frac{1}{\epsilon} + \ln(4\pi) - \gamma - \ln\frac{m^2}{\mu^2} + \frac{q^2}{5m^2} + \cdots & (m^2 \gg |q^2|). \end{cases}$$

$$\frac{e_0^2}{q^2[1+\Pi(q^2)]}$$

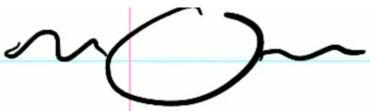


$$-6\int_0^1 dx\, x(1-x)\ln\left(\frac{m^2-q^2x(1-x)}{\mu^2}\right)+\mathcal{O}(\epsilon)\Bigg]$$

$$=\frac{e_0^2}{12\pi^2}\begin{cases}\dfrac{1}{\epsilon}+\ln(4\pi)-\gamma+\dfrac{5}{3}-\ln\dfrac{-q^2}{\mu^2}+\cdots & (|q^2|\gg m^2),\\[2mm] \dfrac{1}{\epsilon}+\ln(4\pi)-\gamma-\ln\dfrac{m^2}{\mu^2}+\dfrac{q^2}{5m^2}+\cdots & (m^2\gg |q^2|).\end{cases}$$

$$\frac{e_0^2}{q^2\left[1+\Pi(q^2)\right]}$$

Renormalize : $\frac{1}{\epsilon}$ disappear



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$$\frac{e_0^2}{q^2[1+\Pi(q^2)]}$$

$\ln M_t^2$

Renormalize: $\frac{1}{\epsilon}$ disappear

Look like Σ depends on M_t ?

$$\Pi(q) = \frac{e_0^2}{12\pi^2}\Big[\frac{1}{\epsilon} + \ln(4\pi) - \gamma$$

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$$\frac{e_0^2}{q^2[1+\Pi(q^2)]}$$

$\ln m_t^2$

Renormalize : $\frac{1}{\epsilon}$ disappear

Look like Σ depends on m_t ?

Renormalize $\Longrightarrow$ measure it

$e_0^2 = \quad e^2 \quad \frac{1}{\ldots}$

residual $\frac{g^2}{M_E^2}$

Appelquist Carazzone thm

Heavy effect renormaliz parameter

or suppressed

$\to$ no dn m_t

residual $\dfrac{g^2}{m_t^2}$

Appelquist Carazzone thm

Heavy effect renormaliz paramtr

or suppressed $1/m^2$

EFT Logic - Uncertainty princ.

Heavy $\longrightarrow$ Local

$\Longrightarrow$

EFT Logic - Uncertainty princ.

Heavy $\Longrightarrow$ Local

$\Longrightarrow$ Local $\mathcal{L}$

$\Longrightarrow$ couplings

$$= \int [\;]_{\Lambda = m_t}$$

$\swarrow Q_t$

with $\mathcal{L}_{eff} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{(\frac{2}{3} e)^2}{240\pi^2 m_t^2} F_{\mu\nu} \Box F^{\mu\nu}$

2) Linear sigma model

$$\mathcal{L}_\sigma = \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2} \partial_\mu \vec{\phi} \partial^\mu \vec{\phi} + \frac{\mu^2}{2} (\sigma^2 + \vec{\phi}^2) - \frac{\lambda}{4} (\sigma^2 + \vec{\phi}^2)^2$$

with SSB $\langle \sigma \rangle = v = \sqrt{\frac{\mu^2}{\lambda}}$

$$Z = \int [d\sigma][d\vec{\phi}]\, e^{i \int d^4x\, \mathcal{L}_\sigma(\sigma, \vec{\phi})}$$

$$= \int [d\vec{\pi}]_{\Lambda = m_\sigma} e^{i \int d^4x\, \mathcal{L}_{eff}(\vec{\pi})}$$

$$m_t^2 \qquad\qquad\qquad\qquad m_t^2$$

Using

Power correction $\displaystyle\int \frac{d^4q}{(2\pi)^4}\, e^{iq\cdot x}\left[q^2\right]^n$

$= 1$

m_t^2 $\qquad\qquad$ m_t^2

Using

Power correction $\displaystyle\int \frac{d^4q}{(2\pi)^4}\, e^{iq\cdot x}\,[q^2]^n$

$= \Box^n\, \delta^4(x)$

$$m_t^2 \qquad\qquad\qquad\qquad m_t^2$$

Using

Power correction $\displaystyle\int \frac{d^4q}{(2\pi)^4}\, e^{iq\cdot x}\left[q^2\right]^n$

m^2_t $\qquad\qquad$ m^2_t

Using

Power correction $\quad \int \frac{d^4q}{(2\pi)^4} e^{iq\cdot x}[q^2]^n$

$= \Box^n \delta^4(x)$

Non analytic $\quad \int \frac{d^4q}{(2\pi)} e^{iq\cdot x} \ln(q^2)$

$= L(x) \sim \frac{1}{x^4}$

2) Nonlinear σ model

a) Usual notation $\quad \sigma = v + \tilde{\sigma}$

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$$\mathcal{L} = \frac{1}{2}\partial_\mu \vec{\Phi}\,\partial^\mu \vec{\Phi} + \frac{1}{2}\left[(\partial_\mu \tilde{\sigma})^2 - m_\sigma^2 \tilde{\sigma}^2\right] - \lambda v \tilde{\sigma}\,(\sigma$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \vec{\phi}\, \partial^\mu \vec{\phi} + \frac{1}{2}\left[(\partial_\mu \tilde{\sigma})^2 - m_0^2 \tilde{\sigma}^2\right] - \lambda v \tilde{\sigma}(\tilde{\sigma}^2 + \vec{\phi}^2) - \frac{\lambda}{4}(\tilde{\sigma}^2 + \vec{\phi}^2)^2$$

$\mathcal{L}_{GB}$

b) Better notation

$\vec{\tau} \leftarrow$ Pauli

$$\Sigma = \sigma + i\,\vec{\tau}\cdot\vec{\phi}$$

$$\frac{1}{2}\mathrm{Tr}(\Sigma^\dagger \Sigma) = \sigma^2 + \vec{\phi}^2$$

D) Better notation

$$\Sigma = \sigma + i\,\vec{\tau}\cdot\vec{\phi}$$

$$\frac{1}{2}\mathrm{Tr}(\Sigma^\dagger \Sigma) = \sigma^2 + \vec{\phi}^{\,2}$$

$$\mathcal{L} = \frac{1}{4}\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger\, \partial^\mu \Sigma\right) + \frac{\mu^2}{\cancel{2}\,4}\mathrm{Tr}(\Sigma^\dagger \Sigma) - \frac{\lambda}{16}\left(\mathrm{Tr}\,\Sigma^\dagger \Sigma\right)^2$$

Shows $\Sigma \to \underbrace{L^\dagger \Sigma R}_{SU(2)}$ $\qquad \mathrm{Tr}(\Sigma^\dagger \Sigma)_{min}$

Shows $\Sigma \to L^{\dagger} \Sigma R$ $\qquad \mathrm{Tr}(\Sigma^{\dagger}\Sigma)_{\min}$

$\underbrace{\qquad}_{SU(2)}$

c) Best notation

$$\Sigma = (v + S)\, U \qquad U = e^{\,i\frac{\vec{\tau}\cdot\vec{\pi}}{v}}$$

$$\begin{cases} S = \tilde{\sigma} + (\dots) \\ \pi = \phi + (\quad) \end{cases}$$

KE same

$$\begin{cases} S = \tilde{\sigma} + (\ldots) \\ \bar{\pi} = \bar{\phi} + (\quad) \end{cases}$$

KE same

$$\mathcal{L} = \frac{(v+S)^2}{4} \operatorname{Tr}(\partial_\mu U \partial^\mu U^\dagger) - \lambda v$$

$$\begin{cases} S = \tilde{\sigma} + (\cdots) \\ \bar{\pi} = \bar{\phi} + (\quad) \end{cases}$$

KE same

$$\mathcal{L} = \frac{(v+S)^2}{4} \mathrm{Tr}\left(\underline{\partial_\mu U} \, \partial^\mu U^\dagger\right) - \lambda v S^3 - \lambda S^4 + \frac{1}{2}\left[(\partial S)^2 - m_S^2 S^2\right]$$

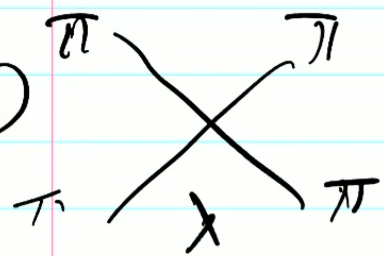
Haag's thm   Names don't matter

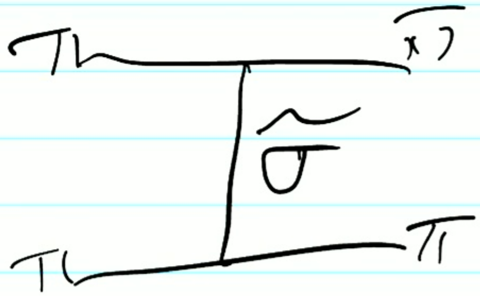
Scattering amps identical on shell

K.E. unchanged

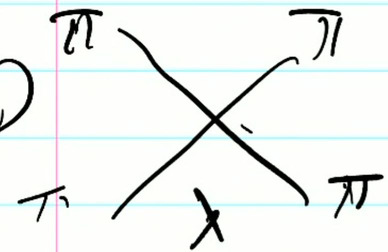
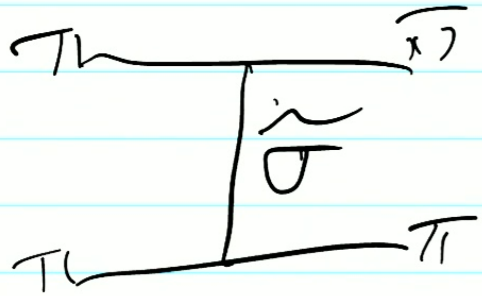
K.E. unchanged

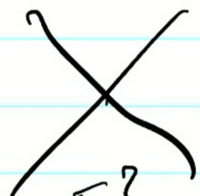
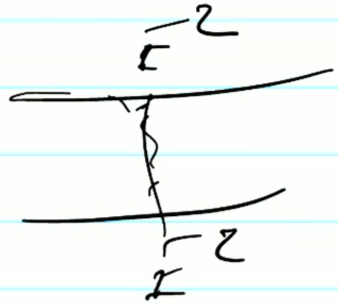
(a) + (c) give same amps PSM Ch 4

a) 

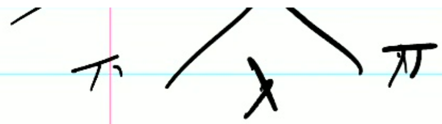


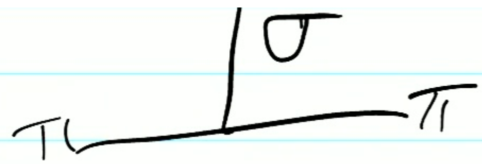
$= 1 - 1 + \frac{q^2}{m_\sigma^2}$

a)   $= 1 - 1 + \frac{g^2}{m_\sigma^2} + \frac{g^4}{m_\sigma^4}$

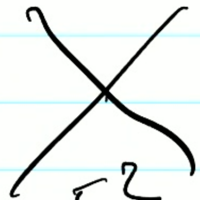
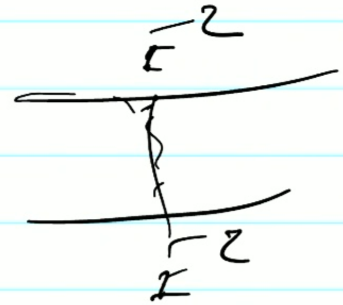
c)  $+$  $\frac{g^2}{m_\sigma^2}\left(1 + \frac{g^4}{m_\sigma^4}\right)$

$\frac{E^2}{v^2}$ $\qquad$ E^4/v^4





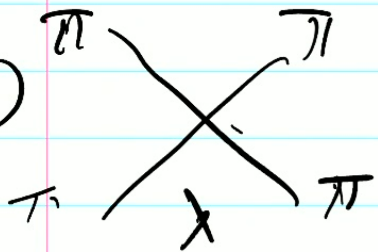
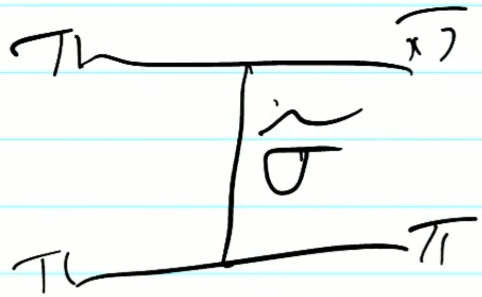
$\sim 1 \sim 1 + \frac{g^2}{m_\sigma^2} + \frac{g^4}{m_\sigma^4}$

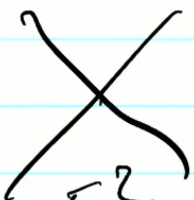
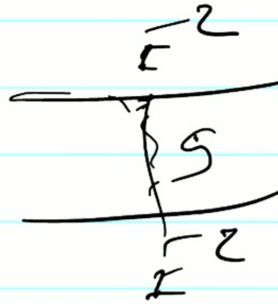
c)  $+$ 

$\frac{g^2}{m_\sigma^2}\left(+\frac{g^4}{m_\sigma^4}\right)$

$\frac{E^2}{v^2}$ $\frac{E^4}{v^4}$

$$\mathcal{L} = \frac{v^2}{4}\, \mathrm{Tr}\, \partial_\mu U\, \partial^\mu U^\dagger$$

a)   $= 1 - 1 + \frac{g^2}{m_\sigma^2} + \frac{g^4}{m_\sigma^4}$

c)  $+$  $\frac{1}{E^2 - m_\sigma^2}$ $\frac{g^2}{m_\sigma^2} \left(+ \frac{g^4}{m_\sigma^4} \right)$

$\frac{E^2}{v^2}$ $\qquad E^2$

$\qquad\qquad E^4/v^4$

$\mathcal{L} = v^2 \, \mathrm{Tr}\, \partial_\mu U \partial^\mu U^\dagger + \frac{v^2}{\ldots} [\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger)]^2$

But EFT is full QFT

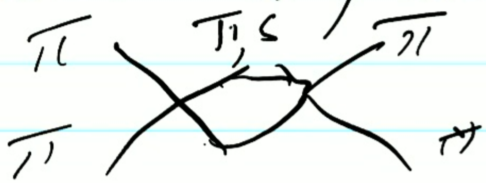
But EFT is full QFT

$$Z = \int [dS][d\pi]\, e^{i \mathcal{L}(S, \pi)}$$

$$= \int d$$

$$= \int d\pi \; e^{i\mathcal{L}_{eff}(\pi)} \quad \text{light DOF}$$

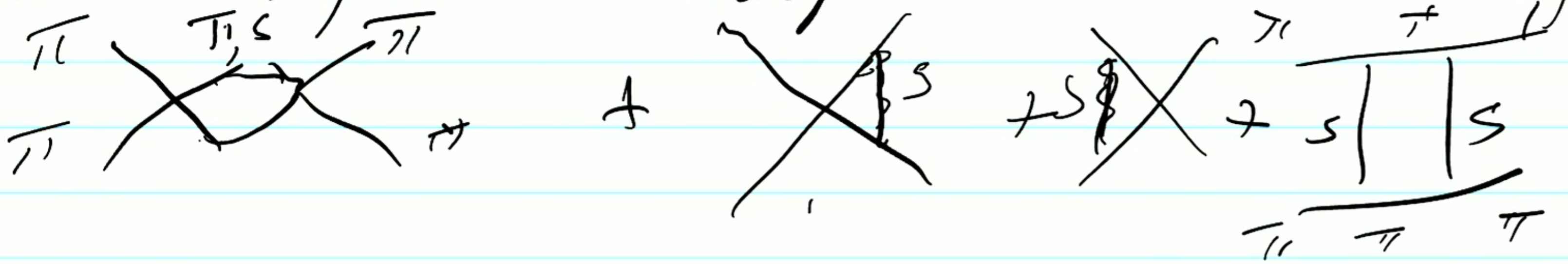
Full theory at one loop

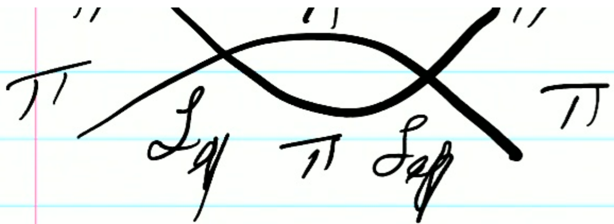


$+$

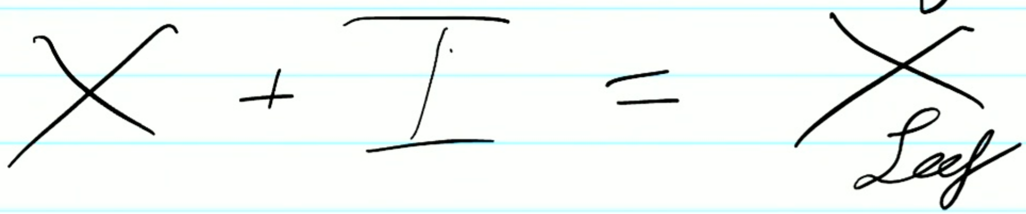
$$= \int d\pi \; e^{i \mathcal{L}_{eff}(\pi)}$$ light DOF

Full theory at one loop

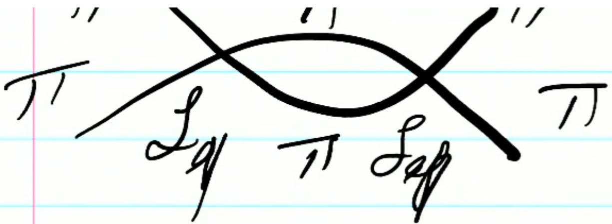




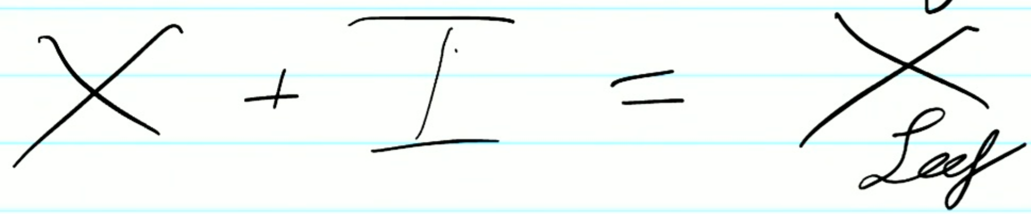
Equiv at low loop energie



High loop energie differ

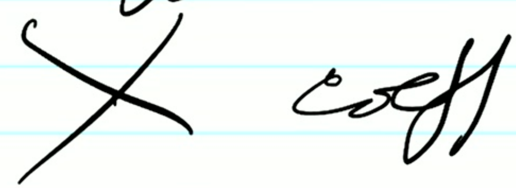


Equiv at low loop energies *



High loop energies differ

- local $\mathcal{L}$



EFT at work

Most general $\mathcal{L}_{eff}$

$$\mathcal{L} = \frac{v^2}{4} \mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \ell_1 \left[\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)\right]^2$$

$$+ \ell_2 \mathrm{Tr}\left(\partial_\mu U \partial_\nu U^\dagger\right) \mathrm{Tr}\left(\partial^\mu U \partial^\nu U^\dagger\right)$$

The full theory, at low energy

$$\mathcal{M}_{\text{full}} = \frac{t}{v^2} + \left[\frac{1}{m_\sigma^2 v^2} - \frac{11}{96\pi^2 v^4}\right] t^2$$
$$- \frac{1}{144\pi^2 v^4}[s(s-u) + u(u-s)]$$
$$- \frac{1}{96\pi^2 v^4}\left[3t^2 \ln\frac{-t}{m_\sigma^2} + s(s-u)\ln\frac{-s}{m_\sigma^2} + u(u-s)\ln\frac{-u}{m_\sigma^2}\right].$$

The EFT has diff High E divergences, but can be renormalized

$$\mathcal{M}_{\text{eff}} = \frac{t}{v^2} + \left[8\ell_1^r + 2\ell_2^r + \frac{5}{192\pi^2}\right]\frac{t^2}{v^4}$$
$$+ \left[2\ell_2^r + \frac{7}{576\pi^2}\right][s(s-u) + u(u-s)]/v^4$$
$$- \frac{1}{96\pi^2 v^4}\left[3t^2 \ln\frac{-t}{\mu^2} + s(s-u)\ln\frac{-s}{\mu^2} + u(u-s)\ln\frac{-u}{\mu^2}\right]$$

with identification $\ell_1^r = \ell_1 + \frac{1}{384\pi^2}\left[\frac{2}{4-d} - \gamma + \ln 4\pi\right]$ } renormalization

$+ \ell_2\, \mathrm{Tr}(\partial_\mu U \partial_\nu U^\dagger)\, \mathrm{Tr}(\partial^\mu U \partial^\nu U^\dagger)$

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$$\mathcal{M}_{\text{full}} = \frac{t}{v^2} + \left[\frac{1}{m_\sigma^2 v^2} - \frac{11}{96\pi^2 v^4}\right] t^2 - \frac{1}{144\pi^2 v^4}[s(s-u) + u(u-s)] - \frac{1}{96\pi^2 v^4}\left[3t^2 \ln\frac{-t}{m_\sigma^2} + s(s-u)\ln\frac{-s}{m_\sigma^2} + u(u-s)\ln\frac{-u}{m_\sigma^2}\right].$$

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$+ \ell_2 \, \mathrm{Tr}(\partial_\mu U \partial_\nu U^\dagger)\,\mathrm{Tr}(\partial^\mu U \partial^\nu U^\dagger)$

The full theory, at low energy

$$t = (p_1 - p_3)^2$$

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$$+ \ell_2 \, \mathrm{Tr}(\partial_\mu U \partial_\nu U^\dagger) \, \mathrm{Tr}(\partial^\mu U \partial^\nu U^\dagger)$$

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$$t = (p_1 - p_3)^2$$
$$s = (p_1 + p_2)^2$$
$$u = -s - t$$

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with identification
$$\ell_1^r = \ell_1 + \frac{1}{384\pi^2}\left[\frac{2}{4-d} - \gamma + \ln 4\pi\right]$$
$$\ell_2^r = \ell_2 + \frac{1}{192\pi^2}\left[\frac{2}{4-d} - \gamma + \ln 4\pi\right]$$
renormalization

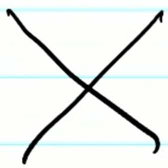
The theories <u>match</u> exactly if

$$\ell_1^r = \frac{v^2}{8m_\sigma^2} + \frac{1}{192\pi^2}\left[\ln\frac{m_\sigma^2}{\mu^2} - \frac{35}{6}\right]$$
$$\ell_2^r = \frac{1}{384\pi^2}\left[\ln\frac{m_\sigma^2}{\mu^2} - \frac{11}{6}\right].$$

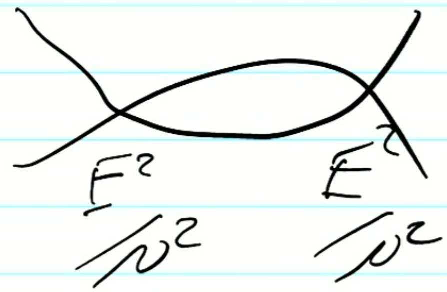
Match on measure

P_0

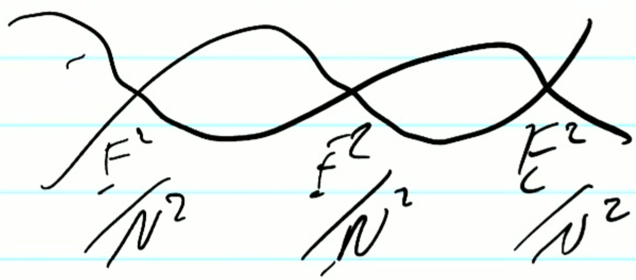
— Weinberg



$$\sim \frac{F^2}{N^2}$$



$$\sim \frac{F(F)}{N^4} \sim \frac{F^4}{N^4}$$



$$\sim \frac{F^6}{N^6}$$

Background Field Method

Game plan:

$$U(x) = \bar{U}(x)\, e^{i\Delta} \qquad \Delta = \tau^a \Delta^a$$

Δ ~ quantum field

$$\mathcal{L} = \mathcal{L}(\bar{U}) + \underbrace{\mathcal{L}_1(\bar{U}, \Delta)}_{\text{linear in } \Delta} + \mathcal{L}_2(\bar{U}, \Delta)$$

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Background Field Method

Game plan:

$$U(x) = \bar{U}(x)\, e^{i\underbrace{\Delta(x)}_{\text{quantum field}}} \qquad \Delta = \tau^a \Delta^a$$

$$\mathcal{L} = \mathcal{L}(\bar{U}) + \underbrace{\mathcal{L}_1(\bar{U}, \Delta)}_{\substack{\text{linear in } \Delta \\ \text{vanish by} \\ \text{eq of motion}}} + \underbrace{\mathcal{L}_2(\bar{U}, \Delta)}_{\text{2nd order}} + \cdots$$

quantum field

$$\mathcal{L} = \mathcal{L}(\bar{u}) + \underbrace{\mathcal{L}_1(\bar{u}, \Delta)}_{\text{linear in } \Delta} + \underbrace{\mathcal{L}_2(\bar{u}, \Delta)}_{\text{2nd order}} + \cdots$$

vanish by eq of motion

$$\mathcal{L}_2 = \frac{1}{2} \Delta^a \left[d_\mu d^\mu + \sigma(x) \right] \Delta^b$$

$\bar{u}$ $\bar{u}$

$$d_\mu = \partial_\mu + \Gamma_\mu$$

vanish by
eq of motion

2nd order

$$\mathcal{L}_2 = \frac{1}{2} \Delta^a \left[d_\mu d^\mu + \sigma(x) \right]_{ab} \Delta^b$$

$\overline{\Pi}$ $\overline{u}$

$$d_\mu = \partial_\mu + \Gamma_\mu$$

<u>Pull out divergences</u>

$$\mathcal{L}_{div} = \frac{1}{16\pi^2} \frac{1}{\varepsilon} \left[\frac{1}{12} \Gamma_{\mu\nu} \Gamma^{\mu\nu} + \frac{1}{2} \sigma^2 \right]$$

$$\Gamma_{\mu\nu} = [d_\mu , d_\nu]$$

$$\mathcal{L}_2 = \frac{1}{2} \Delta^a \left[d_\mu d^\mu + \sigma(x) \right]_{ab} \Delta^b$$

$$\uparrow \; \overline{u} \qquad \nwarrow \overline{u}$$

$$d_\mu = \partial_\mu + \Gamma_\mu$$

<u>Pull out divergences</u>

$$\mathcal{L}_{div} = \frac{1}{16\pi^2} \frac{1}{\varepsilon} \mathrm{Tr}\left(\left[\frac{1}{12} \Gamma_{\mu\nu} \Gamma^{\mu\nu} + \frac{1}{2} \sigma^2 \right] \right)$$

$$\nwarrow a$$

$$\Gamma_{\mu\nu} = [d_\mu , d_\nu]$$

Pull out divergences

$$\mathcal{L}_{div} = \frac{1}{16\pi^2}\frac{\underline{v}\,\mathrm{Tr}}{\varepsilon}\left(\left[\frac{1}{12}\Gamma_{\mu\nu}\Gamma^{\mu\nu} + \frac{1}{2}\sigma^2\right]\right)$$

"a_2 coeff"

$$\Gamma_{\mu\nu} = [d_\mu, d_\nu]$$

This local $\mathcal{L} \Rightarrow$ $l, -l$

Pull out divergences

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"a_2 coeff"

$$\Gamma_{\mu\nu} = [d_\mu, d_\nu]$$

This local $\mathcal{L} \Rightarrow \underline{\underline{l_1}}\,[\mathrm{Tr}\,1]^2 + \underline{\underline{l_2}}\,\mathrm{Tr}\,\mathrm{Tr}$

$$S_2^{(0)} = \int d^4x \left\{ \mathcal{L}_2(\bar{U}) - \frac{F_0^2}{2} \Delta^a \left(d_\mu d^\mu + \sigma \right)^{ab} \Delta^b + \cdots \right\}$$

$$d_\mu^{ab} = \delta^{ab} \partial_\mu + \Gamma_\mu^{ab},$$

$$\Gamma_\mu^{ab} = -\frac{1}{4} \operatorname{Tr} \left(\left[\lambda^a, \lambda^b \right] \left(\bar{U}^\dagger \partial_\mu \bar{U} + i \bar{U}^\dagger \ell_\mu \bar{U} + i r_\mu \right) \right),$$

$$\sigma^{ab} = \frac{1}{8} \operatorname{Tr} \left(\{\lambda^a, \lambda^b\} \left(\chi^\dagger \bar{U} + \bar{U}^\dagger \chi \right) + \left[\lambda^a, \bar{U}^\dagger D_\mu \bar{U} \right] \left[\lambda^b, \bar{U}^\dagger D^\mu \bar{U} \right] \right).$$

$$
\begin{aligned}
d_\mu^{ab} &= \delta^{ab}\partial_\mu + \Gamma_\mu^{ab}, \\
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\sigma^{ab} &= \frac{1}{8}\,\mathrm{Tr}\,\left(\{\lambda^a, \lambda^b\}\left(\chi^\dagger\bar{U} + \bar{U}^\dagger\chi\right) + \left[\lambda^a, \bar{U}^\dagger D_\mu\bar{U}\right]\left[\lambda^b, \bar{U}^\dagger D^\mu\bar{U}\right]\right).
\end{aligned}
$$

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$\sim \eta^a$

Evaluate divergence:

1) $\int [d\Delta] \, e^{i \int d^4x \, \Delta [d d + \sigma] \Delta}$

$= \left(\det[d_\mu d^\mu + \sigma]\right)^{-1/2} = e^{-\frac{1}{2} \mathrm{Tr} \ln \sigma}$

Evaluate divergence:

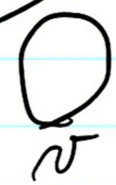
1) $$\int [d\Delta]\, e^{i\int d^4x\, \Delta [\partial \partial + \sigma] \Delta}$$

$$= \left(\det [\partial_\mu \partial^\mu + \sigma]\right)^{-1/2} = e^{-\frac{1}{2} \mathrm{Tr} \ln [\partial^2 + \sigma^2]}$$

$$\mathrm{Tr} \ln [\partial_\mu \partial^\mu + \sigma] = \mathrm{Tr} \ln \Box + \sigma$$

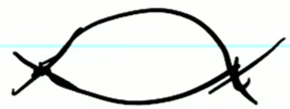
$$= \mathrm{Tr}\ln \Box + \mathrm{Tr}\frac{1}{\Box}v + \mathrm{Tr}\left(\frac{1}{\Box}v\frac{1}{\Box}v\right) + \dots$$

$$\langle x|\frac{1}{\Box}|y\rangle = D_F(x-y)$$



$$\sim \int d^4x\, D(x-x)\, v(x)$$

$$\left(\frac{1}{\Box}v\frac{1}{\Box}v\right) \sim$$



$$\int d^4x\, d^4y\; v(x)\, D(x-y)\, v(y)\, D(y-x)$$

$\Longrightarrow$ Feynman diagrams $\Rightarrow a_2$

Or

<u>Heat kernel</u>

$$H(x,\tau) = \langle x | e^{-\mathcal{D}\tau} | x \rangle$$

$$\mathcal{D} = \partial_\mu \partial^\mu + \sigma + m^2$$

$$= \left(\frac{1}{4\pi}\right)^{d/2} \frac{e^{-\tau m^2}}{\tau^{d/2}} \left[a_0 + a_1 \tau + a_2 \frac{\tau^2}{2} \right.$$

$$H(x,\tau) = \langle x | e^{-D\tau} | x \rangle$$

$$D = \partial_\mu \partial^\mu + \sigma + m^2$$

$$= \frac{1}{(4\pi)^{d/2}} \frac{e^{-\tau m^2}}{\tau^{d/2}} \left[a_0 + a_1 \tau + a_2 \frac{\tau^2}{2} + \cdots \right]$$

Seeley DeWitt

Seeley

DeWitt

$$\langle x | \ln \mathcal{D} | x \rangle = -\int \frac{d\tau}{\tau} \langle x | e^{-\tau \mathcal{D}} | x \rangle$$

$$= \frac{-i}{(4\pi)^{d/2}} \sum m^{d-2n} \Gamma(n - \tfrac{d}{2}) a_n(x)$$

$$\Gamma(2 - \tfrac{d}{2}) a_2$$

$$H(x,\tau) = \int \frac{d^d p}{(2\pi)^d}\, e^{-ip\cdot x} e^{-\tau\mathcal{D}} e^{ip\cdot x}, \tag{1.11}$$

where in d dimensions use is made of the relations

$$\begin{aligned}
\langle p|x\rangle &= \frac{1}{(2\pi)^{d/2}} e^{ip\cdot x},\\
\langle x|x'\rangle &= \int \frac{d^d p}{(2\pi)^d} e^{ip\cdot(x'-x)} = \delta^{(d)}(x-x'),\\
\langle p'|p\rangle &= \int \frac{d^d x}{(2\pi)^d} e^{i(p'-p)\cdot x} = \delta^{(d)}(p'-p).
\end{aligned} \tag{1.12}$$

From the identities

$$\begin{aligned}
d_\mu e^{ip\cdot x} &= e^{ip\cdot x}(ip_\mu + d_\mu),\\
d_\mu d^\mu e^{ip\cdot x} &= e^{ip\cdot x}(ip_\mu + d_\mu)(ip^\mu + d^\mu),
\end{aligned} \tag{1.13}$$

we can then write

$$\begin{aligned}
H(x,\tau) &= \int \frac{d^d p}{(2\pi)^d}\, e^{-\tau\left[(ip_\mu + d_\mu)^2 + m^2 + \sigma\right]}\\
&= \int \frac{d^d p}{(2\pi)^d}\, e^{\tau[p^2 - m^2]} e^{-\tau[d\cdot d + \sigma + 2ip\cdot d]}.
\end{aligned} \tag{1.14}$$

The first exponential factor is simply the free field result, while all the interesting physics is in the second exponential. The latter can be Taylor expanded in powers

$$\langle x|x'\rangle = \int \frac{d^d p}{(2\pi)^d} e^{ip\cdot(x'-x)} = \delta^{(d)}(x-x'),$$
$$\langle p'|p\rangle = \int \frac{d^d x}{(2\pi)^d} e^{i(p'-p)\cdot x} = \delta^{(d)}(p'-p). \tag{1.12}$$

From the identities

$$d_\mu e^{ip\cdot x} = e^{ip\cdot x}(ip_\mu + d_\mu),$$
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we can then write

$$H(x,\tau) = \int \frac{d^d p}{(2\pi)^d}\ e^{-\tau\left[(ip_\mu + d_\mu)^2 + m^2 + \sigma\right]}$$
$$= \int \frac{d^d p}{(2\pi)^d}\ e^{\tau[p^2 - m^2]} e^{-\tau[d\cdot d + \sigma + 2ip\cdot d]}. \tag{1.14}$$

The first exponential factor is simply the free field result, while all the interesting physics is in the second exponential. The latter can be Taylor expanded in powers of τ, keeping those terms which contribute up to order τ^2 after the integration over momentum is performed. Note that each power of p^2 contributes a factor of $1/\tau$. Thus, we obtain the expansion

$$H(x,\tau) = \int \frac{d^d p}{(2\pi)^d}\ e^{\tau(p^2-m^2)}\Big[1 - \tau[d\cdot d + \sigma]$$
$$+ \frac{\tau^2}{2}[(d\cdot d + \sigma)(d\cdot d + \sigma) - 4p\cdot d\ p\cdot d]$$

Employing these relations to evaluate Eq. (1.14) gives (to second order in τ),

$$H(x,\tau) = \frac{i e^{-m^2\tau}}{(4\pi)^{d/2}\tau^{d/2}} \times \left[1 - \tau\sigma + \tau^2\left(\frac{1}{2}\sigma^2 + \frac{1}{12}[d_\mu, d_\nu][d^\mu, d^\nu] + \frac{1}{6}[d_\mu,[d^\mu,\sigma]]\right)\right], \tag{1.17}$$

or in the notation of Eq. (1.3),

Total deriv

$$a_0(x) = 1, \qquad a_1(x) = -\sigma,$$
$$a_2(x) = \frac{1}{2}\sigma^2 + \frac{1}{12}[d_\mu, d_\nu][d^\mu, d^\nu] + \frac{1}{6}\left[d_\mu,[d^\mu,\sigma]\right]. \tag{1.18}$$

Fermions are treated in a similar manner. For example, the identity

$$\ln \not{D} = \frac{1}{2}\ln(\not{D}\,\not{D}) \tag{1.19}$$

allows the same technique to be used for the operator $\not{D}\,\not{D}$. In particular let us consider the case where

$$\not{D} = \not{\partial} + i\not{V} + i\not{A}\gamma_5. \tag{1.20}$$