Title: Testing quantum states

Speakers: Mehdi Soleimanifar
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Abstract: In this talk, I will present three algorithms that address distinct variants of the problem of testing quantum states. First, I will discuss the problem of statistically testing whether an unknown quantum state is a matrix product state of certain bond dimension or it is far from all such states. Next, I will demonstrate a method for testing whether a bipartite quantum state, shared between two parties, corresponds to the ground state of a given gapped local Hamiltonian. Finally, I will present a scheme for verifying that a machine learning model of an unknown quantum state has high overlap with the actual state.

Zoom Link: https://pitp.zoom.us/j/99250127489?pwd=UCtXUi9zMzJZamppT29DbWtJcWU3Zz09

Testing quantum states

Mehdi Soleimanifar (Caltech)

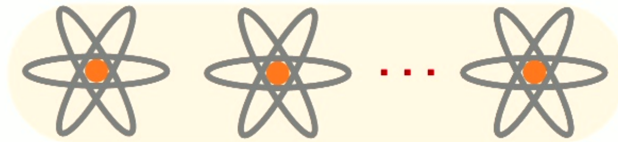
Based on joint works with

Anurag Anshu (Harvard) , Aram Harrow (MIT), and Robert Huang (Caltech)

1. Testing matrix product states

2. Testing machine learning models

3. Two-party testing of ground states



State of n qudits

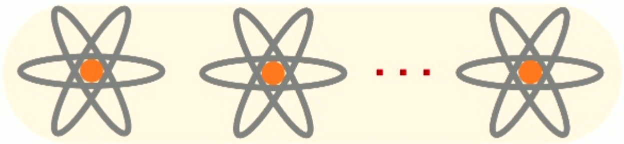
is a vector in tensor product space $\left(\mathbb{C}^{d}\right)^{\otimes n}$

$$|\psi\rangle = \sum_{i_1,\dots,i_n=1}^{d} a_{i_1\dots i_n} |i_1\rangle \cdots |i_n\rangle$$

$$a_{i_1\dots i_n} \in \mathbb{C} \qquad \| |\psi\rangle \|_2 = 1$$

$|\psi\rangle$ can be product or entangled:

Product state: $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \cdots \otimes |\psi_n\rangle$



Questions we can ask:

Is a state $|\psi\rangle$ entangled or product?

How entangled is a state $|\psi\rangle$?

Long history in quantum information:

Bell test or quantum games
Tensor optimization
Quantum cryptography
Hamiltonian complexity
Quantum many-body physics

Questions we can ask:

Is a state $|\psi\rangle$ entangled or product?

How entangled is a state $|\psi\rangle$?

Long history in quantum information:

Bell test or quantum games **Quantum cryptography**
Tensor optimization **Hamiltonian complexity**
Quantum many-body physics

This talk:
Statistical theory of testing many-body entanglement

Property testing model

Entanglement tester is an algorithm $\mathcal{A}$ such that

1. If $|\psi\rangle$ has at most certain amount of entanglement

$$\Pr\left[\mathcal{A} \text{ accepts given } |\psi\rangle^{\otimes m}\right] \geq 2/3$$ *Completeness*

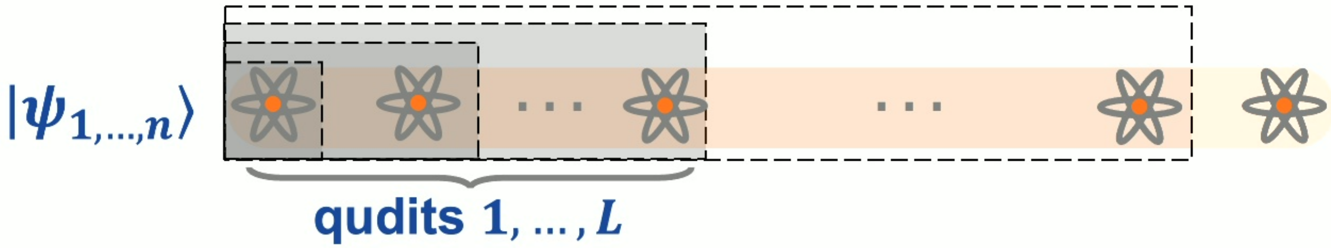
2. If $|\psi\rangle$ is far from states with at most certain amount of entanglement

$$\Pr\left[\mathcal{A} \text{ accepts given } |\psi\rangle^{\otimes m}\right] \leq 1/3$$ *Soundness*

What is the fewest number of copies m needed for entanglement testing?

$\mathrm{MPS}(r)$: Matrix product states with bond dimension r

Reduced state $\rho_{1,\ldots,L} = \mathrm{tr}_{L+1,\ldots,n} |\psi_{1,\ldots,n}\rangle\langle\psi_{1,\ldots,n}|$



$$\mathrm{rank}(\rho_{1,\ldots,L}) \leq r \quad \text{for} \quad 1 \leq L \leq n$$

If $r \sim d^n$, any state $|\psi\rangle$ can be written as an MPS

Bond dim limits the amount of entanglement

Let's go back to testing entanglement

Property testing model

MPS tester is an algorithm $\mathcal{A}$ such that

1. If $|\psi\rangle \in \mathbf{MPS}(r)$ then

$$\Pr\left[\mathcal{A} \text{ accepts given } |\psi\rangle^{\otimes m}\right] \geq 2/3$$ ***Completeness***

2. If $\mathrm{Dist}_r(|\psi\rangle) \geq \delta$ then

$$\Pr\left[\mathcal{A} \text{ accepts given } |\psi\rangle^{\otimes m}\right] \leq 1/3$$ ***Soundness***

What does it mean for $|\psi\rangle$ to be far from $\mathbf{MPS}(r)$?

$$\mathrm{Overlap}_r(|\psi\rangle) = \max_{|\phi\rangle \in \mathrm{MPS}(r)} |\langle\psi|\phi\rangle|^2$$

$$\mathrm{Dist}_r(|\psi\rangle) = \min_{|\phi\rangle \in \mathrm{MPS}(r)} \mathrm{D}_{\mathrm{trace}}(\psi, \phi) = \min_{|\phi\rangle \in \mathrm{MPS}(r)} \sqrt{1 - |\langle\psi|\phi\rangle|^2}$$

$$\mathrm{Dist}_r(|\psi\rangle) = \sqrt{1 - \mathrm{Overlap}_r(|\psi\rangle)}$$

Property testing model

MPS tester is an algorithm $\mathcal{A}$ such that

1. If $|\psi\rangle \in \mathrm{MPS}(r)$ then

$$\Pr[\mathcal{A}\text{ accepts given } |\psi\rangle^{\otimes m}] \geq 2/3$$ ***Completeness***

2. If $\mathrm{Dist}_r(|\psi\rangle) \geq \delta$ then

$$\Pr[\mathcal{A}\text{ accepts given } |\psi\rangle^{\otimes m}] \leq 1/3$$ ***Soundness***

Goal: Finding the smallest number of copies m

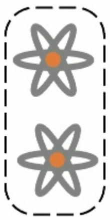
for a given
- number of qudits n
- bond dimension r
- distance δ

Product test (testing $\mathrm{MPS}(r)$ with $r = 1$)

Accept → Reject

$$\{\Pi_{\mathrm{SWAP}}, \mathrm{I} - \Pi_{\mathrm{SWAP}}\}$$

[Mintert, Kuś, Buchleitner]
[Harrow and Montanaro]



SWAP test
[Buhrman, Cleve, Watrous, de Wolf]

$$\Pi_{\mathrm{SWAP}} = \frac{\mathrm{I+SWAP}}{2}, \quad \mathrm{I} - \Pi_{\mathrm{SWAP}} = \frac{\mathrm{I-SWAP}}{2}$$

$$\Pr[\text{SWAP test accepts } |\psi\rangle \otimes |\phi\rangle] = \frac{1}{2} + \frac{1}{2}|\langle\psi|\phi\rangle|^2$$

Product test (testing $\mathrm{MPS}(r)$ with $r = 1$)

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Accept ↓ Reject ↓

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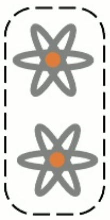

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$$\Pi_{\mathrm{SWAP}} = \frac{\mathrm{I+SWAP}}{2}, \quad \mathrm{I} - \Pi_{\mathrm{SWAP}} = \frac{\mathrm{I-SWAP}}{2}$$

$$\Pr[\text{SWAP test accepts } \rho \otimes \rho] = \frac{1}{2} + \frac{1}{2}\mathrm{tr}[\rho^2]$$ ← Purity

Product test (testing $\mathrm{MPS}(r)$ with $r = 1$)

[HM13]:

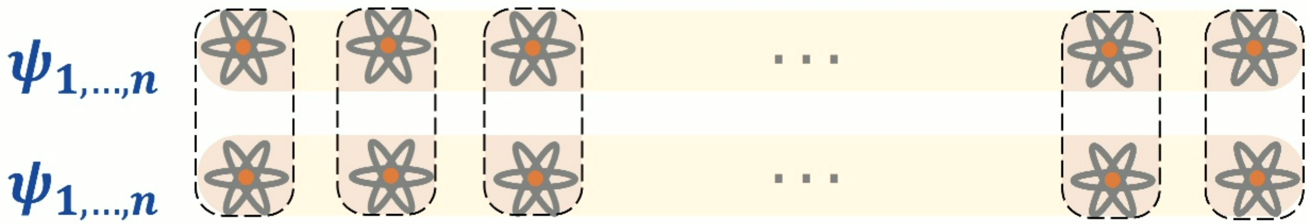
Product states pass this test with probability 1

States δ*-far from product fail this test with probability* $\Omega(\delta^2)$

Why?

Entangled $|\psi_{1,\ldots,n}\rangle$ means some mixed subsystems with $\mathrm{tr}[\rho^2] < 1$

$$\mathbf{Pr[SWAP\ test\ accepts}\ \rho \otimes \rho] = \frac{1}{2} + \frac{1}{2}\mathrm{tr}[\rho^2] < 1$$



Product test (testing $\mathrm{MPS}(r)$ with $r = 1$)

[HM13]:

Product states pass this test with probability 1

States δ*-far from product fail this test with probability* $\Omega(\delta^2)$

Rejection probability can be boosted to $2/3$ by repeating on $m = O\left(\frac{1}{\delta^2}\right)$ pairs

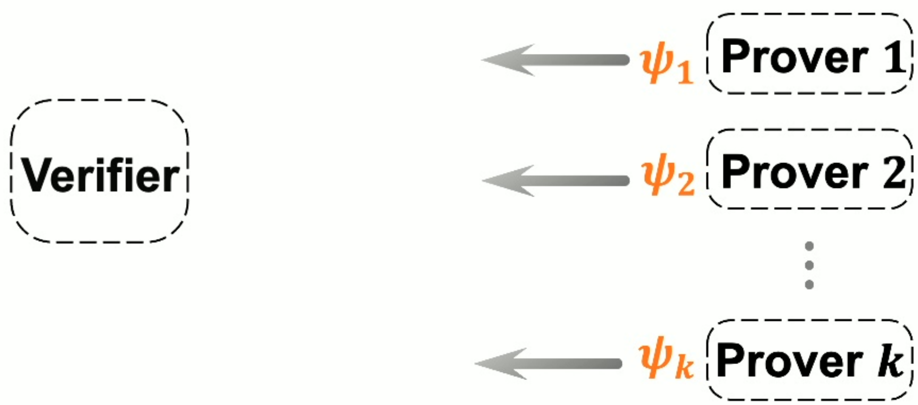
Product test (testing $\mathrm{MPS}(r)$ with $r = 1$)

[HM13]:

Product states pass this test
with probability 1

States δ***-far from product fail this test***
with probability $\Omega(\delta^2)$

This implies $\mathrm{QMA}(k) = \mathrm{QMA}(2)$ **for** $k \geq 2$ [HM13]



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With applications in hardness of tensor optimization problems

Result 1

Improved and simple analysis of product test

Proof sketch of product test

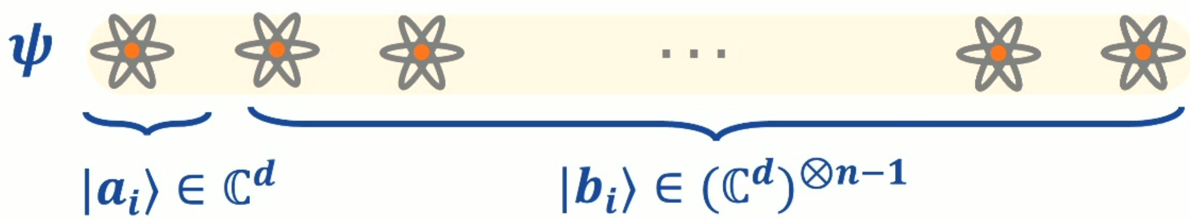
Schmidt decomposition

$$|\psi\rangle = \sqrt{\lambda_1}\,|a_1\rangle|b_1\rangle + \sqrt{\lambda_2}\,|a_2\rangle|b_2\rangle + \cdots + \sqrt{\lambda_d}\,|a_d\rangle|b_d\rangle$$

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$

$\lambda_1 + \lambda_2 + \cdots \lambda_d = 1$

$|a_i\rangle \in \mathbb{C}^d, |b_i\rangle \in (\mathbb{C}^d)^{\otimes n-1}$



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$|a_i\rangle \in \mathbb{C}^d, |b_i\rangle \in (\mathbb{C}^d)^{\otimes n-1}$

Suppose $|\psi\rangle$ is far from product.

If λ_1 *is large:*

$|\psi\rangle \approx |a_1\rangle \otimes |b_1\rangle$ ***and 1st SWAP test accepts***

But for **$|\psi\rangle$ to be far from product**

$|b_1\rangle$ ***has to be far from product***

Remaining SWAP tests reject **with high prob.**
(by induction)

Proof sketch of product test

Schmidt decomposition

$$|\psi\rangle = \sqrt{\lambda_1}\,|a_1\rangle|b_1\rangle + \sqrt{\lambda_2}\,|a_2\rangle|b_2\rangle + \cdots + \sqrt{\lambda_d}\,|a_d\rangle|b_d\rangle$$

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$

$\lambda_1 + \lambda_2 + \cdots \lambda_d = 1$

$|a_i\rangle \in \mathbb{C}^d, |b_i\rangle \in (\mathbb{C}^d)^{\otimes n-1}$

Given $|\psi\rangle$ that is δ-far from product states,

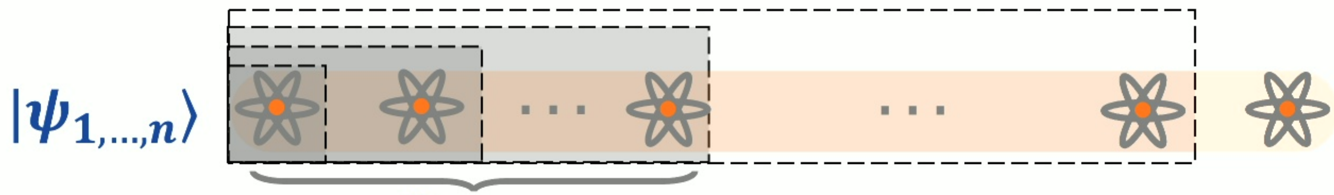
$$\Pr[\text{Product test rejects } |\psi\rangle^{\otimes 2}] \geq \begin{cases} \delta^2 - \delta^4 & \delta \leq \sqrt{1/2} \\ \frac{2}{3}\delta^2 - \frac{1}{3}\delta^4 & \text{otherwise} \end{cases}$$

Our bound in tight for $n \geq 2, \delta \leq \sqrt{1/2}$ as shown by

$$|\psi\rangle = \sqrt{1-\delta^2}\,|1\rangle|1\rangle + \delta\,|2\rangle|2\rangle$$

$\mathbf{MPS}(r)$: Matrix product states with bond dimension r

Reduced state $\rho_{1,\dots,L} = \mathrm{tr}_{L+1,\dots,n} \, |\psi_{1,\dots,n}\rangle\langle\psi_{1,\dots,n}|$



$$\mathrm{rank}(\rho_{1,\dots,L}) \le r \quad \text{for} \quad 1 \le L \le n$$

[O'Donnell and Wright'15]:

An algorithm to test if

$\mathrm{rank}(\rho) \le r$ or ρ is ϵ-far from rank-r states

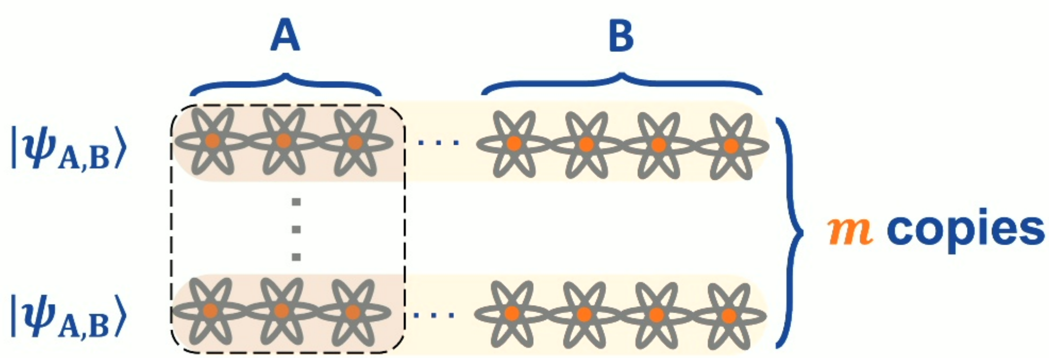
using $m = \Theta(r^2/\epsilon)$ copies

Testing $\mathrm{MPS}(r)$ *with* $r \geq 2$

Upper bound: Our **MPS tester** requires $m = O(nr^2/\delta^2)$

Proof relies on

1) $\exists$Cut (A, B) where ρ_A is $\Omega(1/n)$-far from being rank r



2) The rank tester projectors mutually commute

Testing $\mathbf{MPS}(r)$ *with* $r \geq 2$

Upper bound: Our **MPS tester** requires $m = O(nr^2/\delta^2)$

Proof relies on

Can this analysis of be improved

to show $m = O(1)$ copies are sufficient?

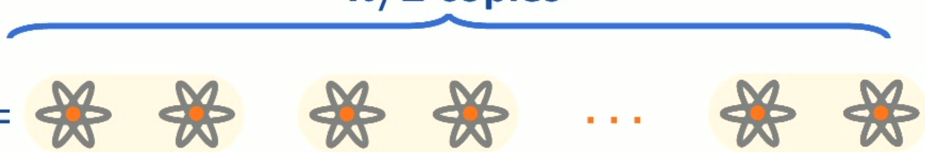
Can't be done for general states!

Lower bound: Any MPS tester requires $m = \Omega(\sqrt{n}/\delta^2)$

The hard example: $|\psi\rangle$ and its random local rotations

where $|\phi\rangle$ is $1/\sqrt{n}$-far from $\mathrm{MPS}(r)$

$n/2$ copies

$|\psi\rangle = |\phi\rangle^{\otimes n/2} =$ 

$|\phi\rangle$ $|\phi\rangle$ $|\phi\rangle$

Summary

Developed algorithms for testing matrix product states

1) Simple and improved analysis of the product test

2) Upper bound of $O(n)$ for MPS testing with bond dim ≥ 2

3) Lower bound of $\Omega(\sqrt{n})$ for MPS testing with bond dim ≥ 2

Setup

Suppose we have

1. Representation $R: \{0,1\}^n \to \mathbb{C}$ for state

$$|\psi\rangle = \sum_{x_1,\dots,x_n \in \{0,1\}^n} \psi(x_1, \dots, x_n)\, |x_1\rangle \cdots |x_n\rangle$$

Such that

$$R(x_1, \dots, x_n) \propto \psi(x_1, \dots, x_n)$$

Setup

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1. Representation $R: \{0,1\}^n \to \mathbb{C}$ for state

$$|\psi\rangle = \sum_{x_1,\ldots,x_n \in \{0,1\}^n} \psi(x_1, \ldots, x_n)\, |x_1\rangle \cdots |x_n\rangle$$

Such that

$$R(x_1, \ldots, x_n) = c \cdot \psi(x_1, \ldots, x_n)$$

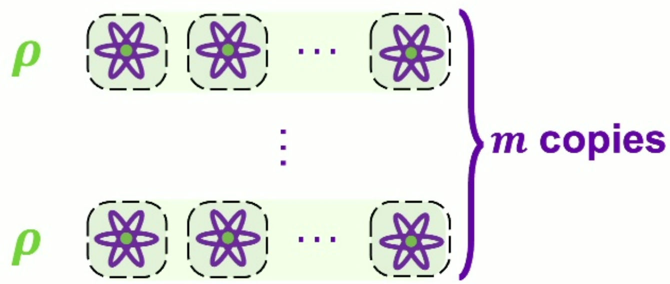
$$\frac{R(x_1, \ldots, x_n)}{R(y_1, \ldots, y_n)} = \frac{\psi(x_1, \ldots, x_n)}{\psi(y_1, \ldots, y_n)}$$

Setup

Goal:

1. Perform few single-qubit measurements on ρ

2. Query representation R of $|\psi\rangle$ a few times



Some applications:

- **Benchmarking** quantum devices

  ρ is prepared by a quantum device

  $|\psi\rangle$ is the **ideal state** given e.g. by classical simulation

- **Verifying ML models** of quantum states

  ρ is an unknown quantum state

  $|\psi_\theta\rangle$ is an **ML model** with **parameters** θ
  that allows query access to $\psi_\theta(x_1, \dots, x_n)$

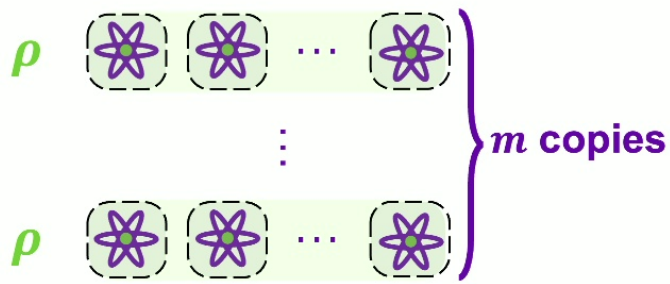
Setup

Goal:

1. Perform few single-qubit measurements on ρ

2. Query representation R of $|\psi\rangle$ a few times

And verify that $\langle\psi|\rho|\psi\rangle$ is large



Our test

Part 1

Measuring ρ

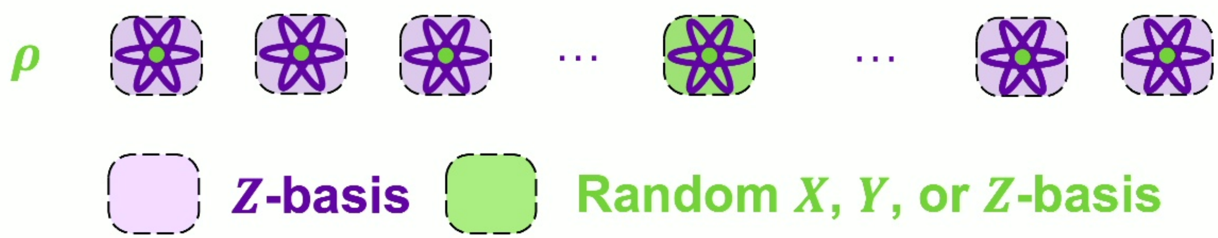
Our test

Choose $i \in \{1, \dots, n\}$ uniformly at random

Perform these measurements:

- **Z-basis on all qubits but qubit i**

- **Random X, Y, or Z-basis on qubit i**



Our test

Part 2

Querying representation R

Our test

Query representation R twice

$$|v\rangle = \frac{R\left(z^{(0)}\right)|0\rangle + R\left(z^{(1)}\right)|1\rangle}{\left(\left|R(z^{(0)})\right|^2 + \left|R(z^{(1)})\right|^2\right)^{1/2}}$$

$z^{(0)}$ = **string** $z \in \{0,1\}^{n-1}$ **padded with 0 on** i**'th bit**
$z^{(1)}$ = **string** $z \in \{0,1\}^{n-1}$ **padded with 1 on** i**'th bit**

Our test

Query representation R twice

$$|v\rangle = \frac{R\left(z^{(0)}\right)|0\rangle + R\left(z^{(1)}\right)|1\rangle}{\left(\left|R(z^{(0)})\right|^2 + \left|R(z^{(1)})\right|^2\right)^{1/2}}$$

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$z^{(1)}$ = **string** $z \in \{0,1\}^{n-1}$ **padded with 1 on i'th bit**

Compute and return overlap

$$\langle v|\ (3|s\rangle\langle s| - I)\ |v\rangle$$

Claim

$$\mathbb{E}\left[\langle v|\left(3|s\rangle\langle s|-I\right)|v\rangle\right] \geq 1-\delta$$
$$\Downarrow$$
$$\langle \psi|\rho|\psi\rangle \geq 1-\delta\cdot T$$

$T \sim$ **mixing time of certain random walk**

Claim

$$\mathbb{E}\left[\langle v|\,(3|s\rangle\langle s|-I)\,|v\rangle\right] \geq 1-\delta$$
$$\Downarrow$$
$$\langle\psi|\rho|\psi\rangle \geq 1-\delta\cdot T$$

T ~ mixing time of certain random walk

This random walk

- **is on Boolean hypercube $\{0,1\}^n$**
- **transitions defined by $\frac{\psi(x)}{\psi(y)}$**

Claim

$$\mathbb{E}\left[\langle v|\left(3|s\rangle\langle s|-I\right)|v\rangle\right] \geq 1-\delta$$
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$$\langle \psi|\rho|\psi\rangle \geq 1-\delta \cdot T$$

$T \sim$ mixing time of certain random walk

This random walk

- **is on Boolean hypercube $\{0,1\}^n$**
- **transitions defined by $\frac{\psi(x)}{\psi(y)}$**
- **stationary distribution is $|\psi(x)|^2$**

Claim

$$\mathbb{E}\left[\langle v|\ (3|s\rangle\langle s| - I)\ |v\rangle\right] \geq 1 - \delta$$

$$\Downarrow$$

$$\langle \psi|\rho|\psi\rangle \geq 1 - \delta \cdot T$$

$T \sim$ **mixing time of certain random walk**

When $|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} e^{i\alpha(x)} |x\rangle$ $\quad T = O(n)$

When $|\psi\rangle$ **is a Haar-random state** $\quad T = \text{poly}(n)$

Claim

$$\mathbb{E}\left[\langle v|\,(3|s\rangle\langle s|-I)\,|v\rangle\right] \geq 1-\delta$$
$$\Downarrow$$
$$\langle\psi|\rho|\psi\rangle \geq 1-\delta\cdot T$$

$T \sim$ mixing time of certain random walk

When $|\psi\rangle = \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n} e^{i\alpha(x)}|x\rangle$ $\quad T = O(n)$

When $|\psi\rangle$ **is a Haar-random state** $\quad T = \mathrm{poly}(n)$

$\mathrm{poly}(n)$ ***sample complexity as long as*** $T = \mathrm{poly}(n)$

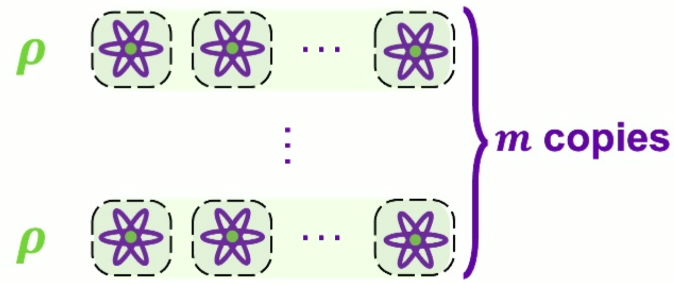
We can achieve

Goal:

1. Perform few single-qubit measurements on ρ

2. Query representation R a few times

And verify that $\langle\psi|\rho|\psi\rangle$ is large



2. Two-party testing of ground states

Joint work with Anurag Anshu (Harvard) and Aram Harrow (MIT)

Nature Physics 2022
arxiv: 2004.15009

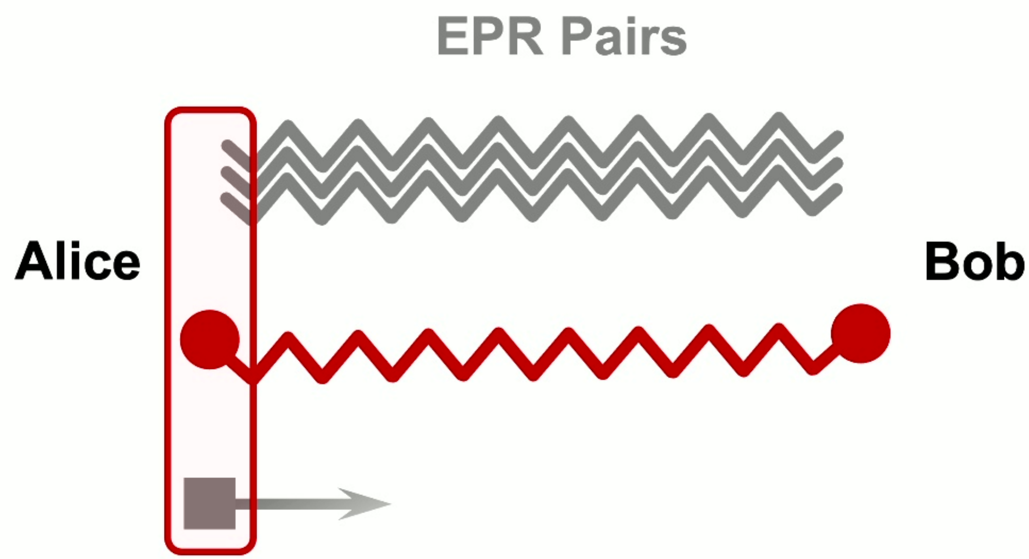
Testing bipartite states

YES NO

$$\{|\psi\rangle\langle\psi|_{AB}, I - |\psi\rangle\langle\psi|_{AB}\}$$

Two-Outcome Measurement

Testing bipartite states

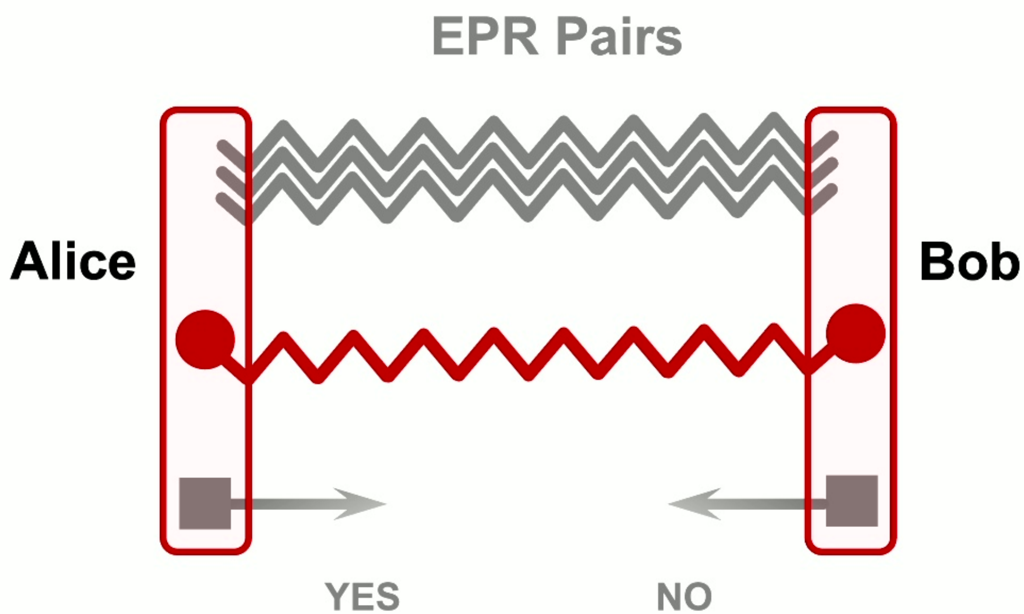


YES NO

$$\{|\psi\rangle\langle\psi|_{AB}, I - |\psi\rangle\langle\psi|_{AB}\}$$

Two-Outcome Measurement

Testing bipartite states

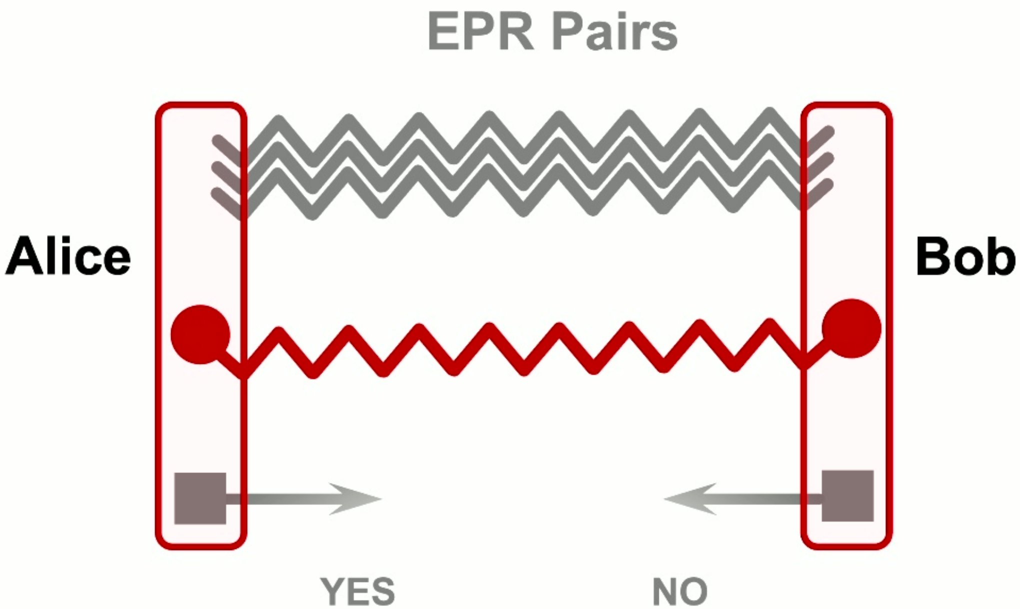


$\{|\psi\rangle\langle\psi|_{AB}, I - |\psi\rangle\langle\psi|_{AB}\}$

Two-Outcome Measurement

$C(\psi_{AB})$ = **Minimum # of exchanged qubits to perform** $\{|\psi\rangle\langle\psi|_{AB}, I - |\psi\rangle\langle\psi|_{AB}\}$

Testing bipartite states



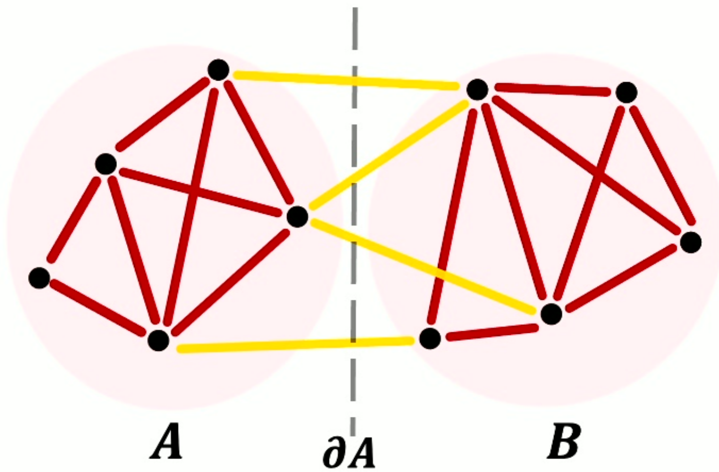
$$\{|\psi\rangle\langle\psi|_{AB}, I - |\psi\rangle\langle\psi|_{AB}\}$$

Two-Outcome Measurement

$C_\varepsilon(\psi_{AB}) =$ **Minimum # of exchanged qubits to perform ε approximation of** $\{|\psi\rangle\langle\psi|_{AB}, I - |\psi\rangle\langle\psi|_{AB}\}$

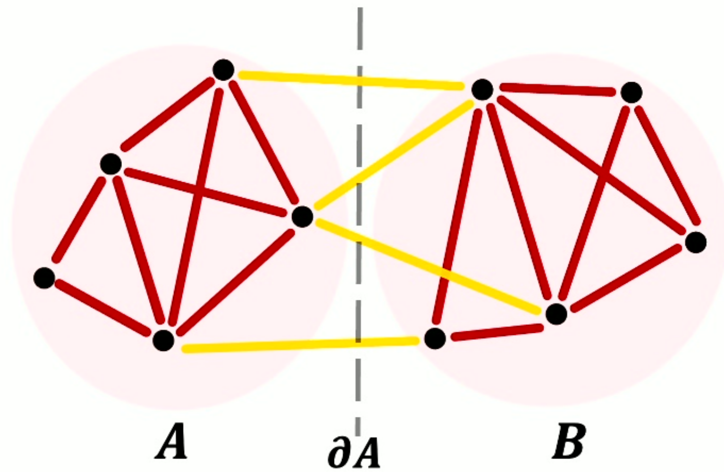
*Testing **gapped** ground states*

Testing ground states



$$H_{AB} = H_A + H_{\partial A} + H_B$$

Testing ground states



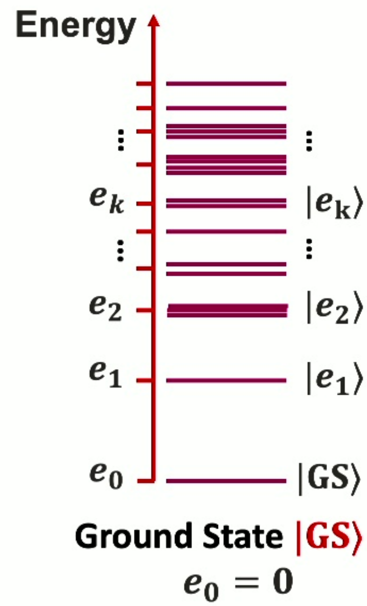
$$H_{AB} = H_A + H_{\partial A} + H_B$$

What is the communication complexity of testing the ground state?

Testing gapped ground states

Measure energy $\langle\psi|H|\psi\rangle$

- **Yes**: $\langle\psi|H|\psi\rangle \leq \mathbf{gap}/2$
- **No**: $\langle\psi|H|\psi\rangle > \mathbf{gap}/2$



Testing gapped ground states

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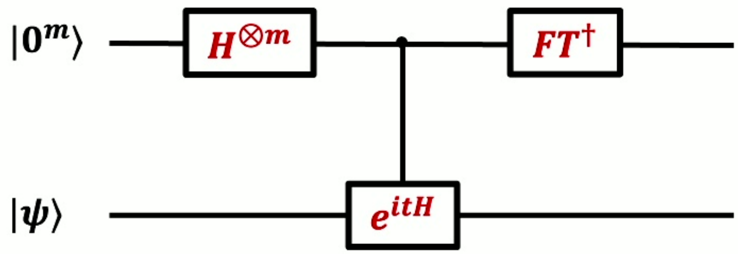
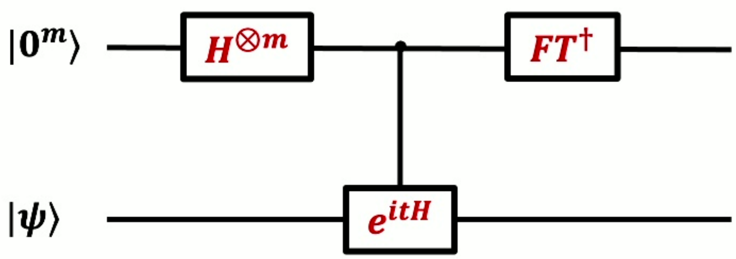
Quantum Phase Estimation

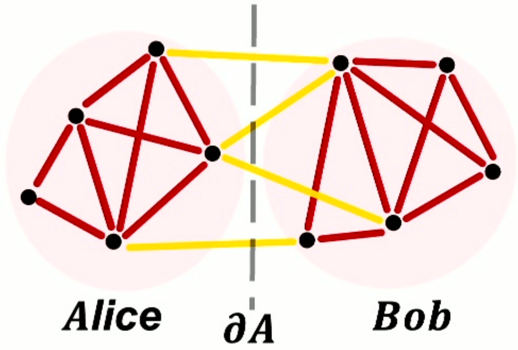
$$t = O\left(\frac{1}{\mathbf{gap}}\right)$$



Testing gapped ground states

Communication Protocol



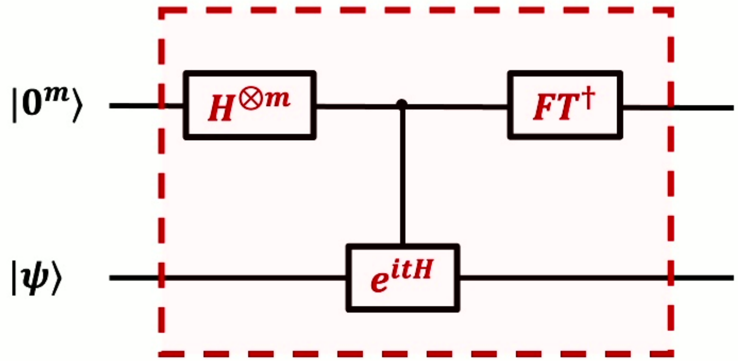


Testing gapped ground states

Measure energy $\langle\psi|H|\psi\rangle$
- **Yes**: $\langle\psi|H|\psi\rangle \leq \mathbf{gap}/2$
- **No**: $\langle\psi|H|\psi\rangle > \mathbf{gap}/2$

Quantum Phase Estimation

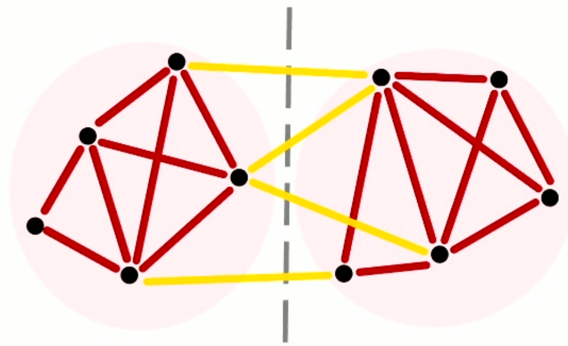
$$t = O\left(\frac{1}{\mathbf{gap}}\right)$$



$$\approx_\varepsilon \; |\mathbf{GS}\rangle\langle\mathbf{GS}|$$

Repeat for $O\left(\log\frac{1}{\varepsilon}\right)$ to get ε approximation

Hamiltonian Simulation (Performing $e^{itH_{AB}}$)

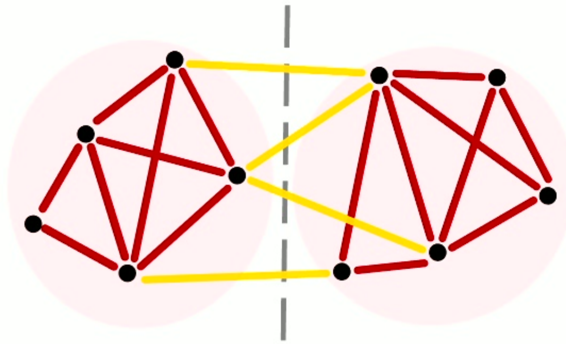


$$H_{AB} = H_A + H_{\partial A} + H_B$$

Depth of Hamiltonian simulation algorithms is $O(t\|H_{AB}\|)$

Communication cost of $e^{itH_{AB}}$ is $O(t\|H_{AB}\|)$

Hamiltonian Simulation (Performing $e^{itH_{AB}}$)

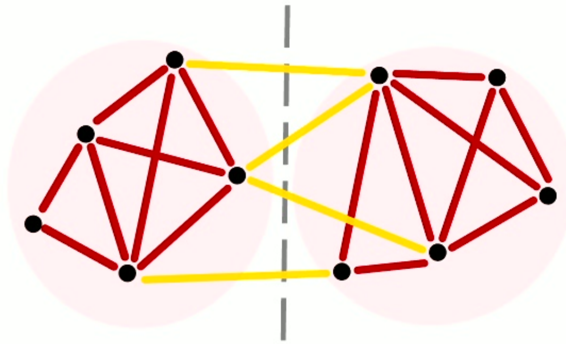


$$H_{AB} = H_A + H_{\partial A} + H_B$$

$$e^{itH_{AB}} = e^{itH_A} \cdot e^{itH_B} \cdot e^{itH_{\partial A}}$$ when $H_A, H_B, H_{\partial A}$ Commute

Interaction Picture: Time-dependent Hamiltonian [LW18]

Hamiltonian Simulation (Performing $e^{itH_{AB}}$)



$$H_{AB} = H_A + H_{\partial A} + H_B$$

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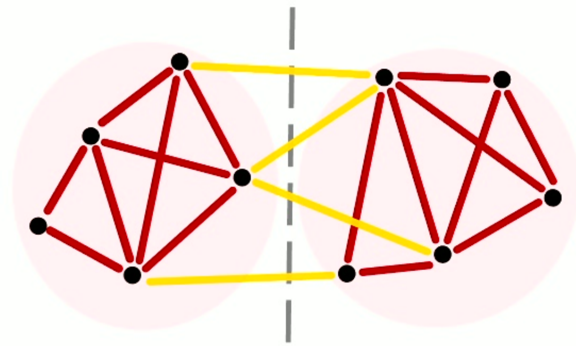
Interaction Picture: Time-dependent Hamiltonian [LW18]

$$H_I(t) = e^{-it(H_A+H_B)} \cdot H_{\partial A} \cdot e^{it(H_A+H_B)}$$

$$e^{itH_{AB}} = e^{itH_A} \cdot e^{itH_B} \cdot e^{\int_{\tau=0}^{t} iH_I(\tau)\, d\tau}$$

Hamiltonian Simulation (Performing $e^{itH_{AB}}$)



$$H_{AB} = H_A + H_{\partial A} + H_B$$

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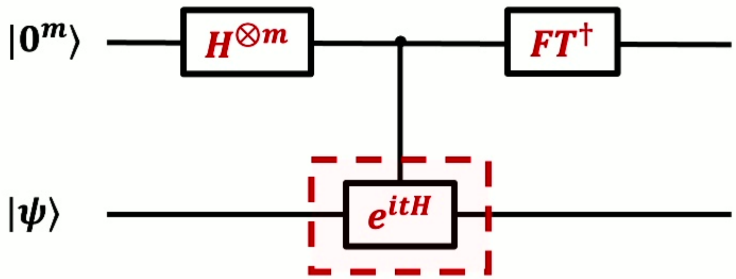
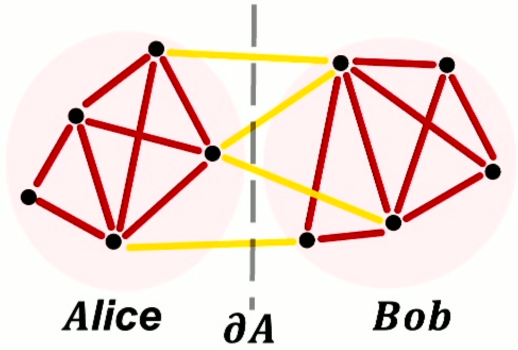
$$O(t\|H_I\|) = \mathrm{O}(t\|H_{\partial A}\|)$$

Testing gapped ground states

Communication Protocol





Overall Communication Cost: $\tilde{O}(|\partial A|/\text{gap} \cdot \log 1/\varepsilon)$

Our protocol gives an upper bound on $C_\varepsilon(\mathrm{GS}_{AB})$

$$\tilde{O}\left(\frac{|\partial A|}{\mathrm{gap}} \cdot \log\frac{1}{\varepsilon}\right) \geq C_\varepsilon(\mathrm{GS}_{AB})$$

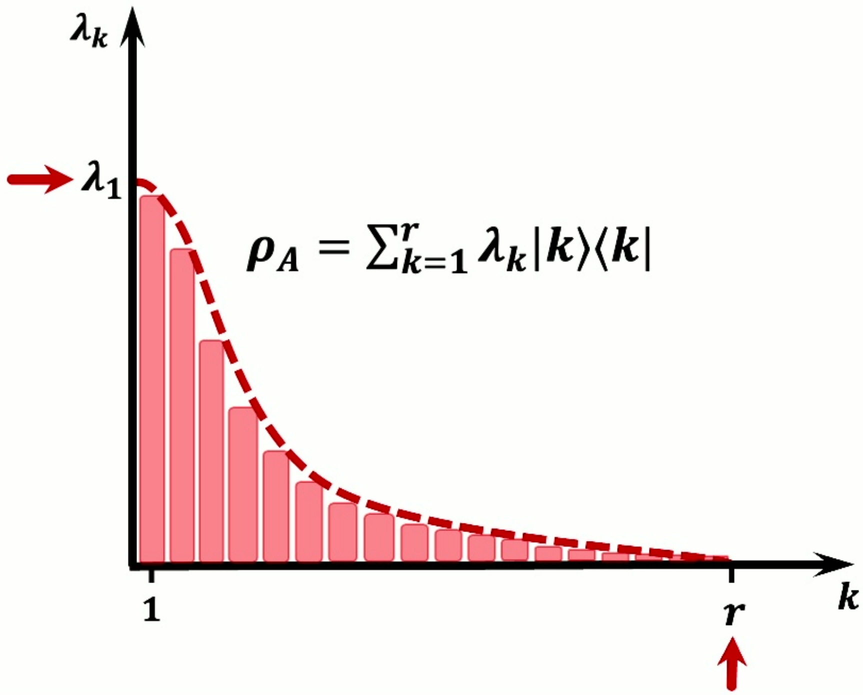
But there is also a known lower bound on $C_\varepsilon(GS_{AB})$:

Communication Complexity ≥ Entanglement Spread

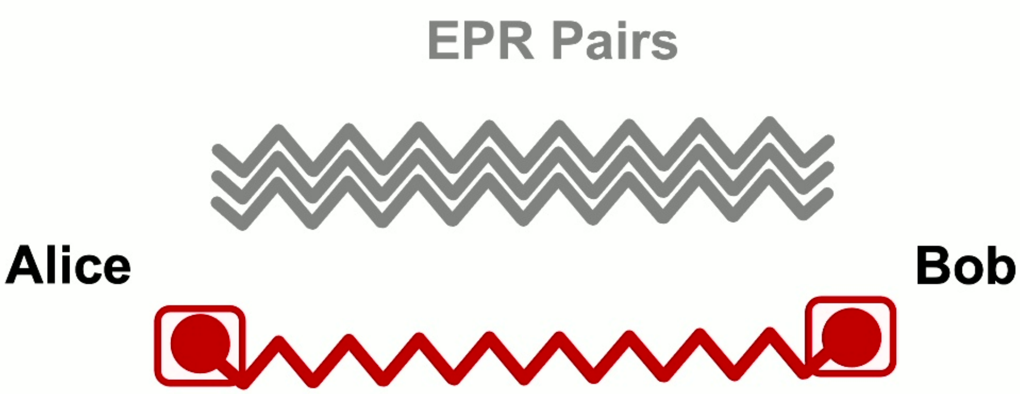
[Hayden, Winter'03]
[Coudron , Harrow'19]
[Harrow, Leung'11]

Entanglement spread

$$\mathrm{ES}(\rho_A) = \log(r\lambda_1)$$



Testing Bipartite States



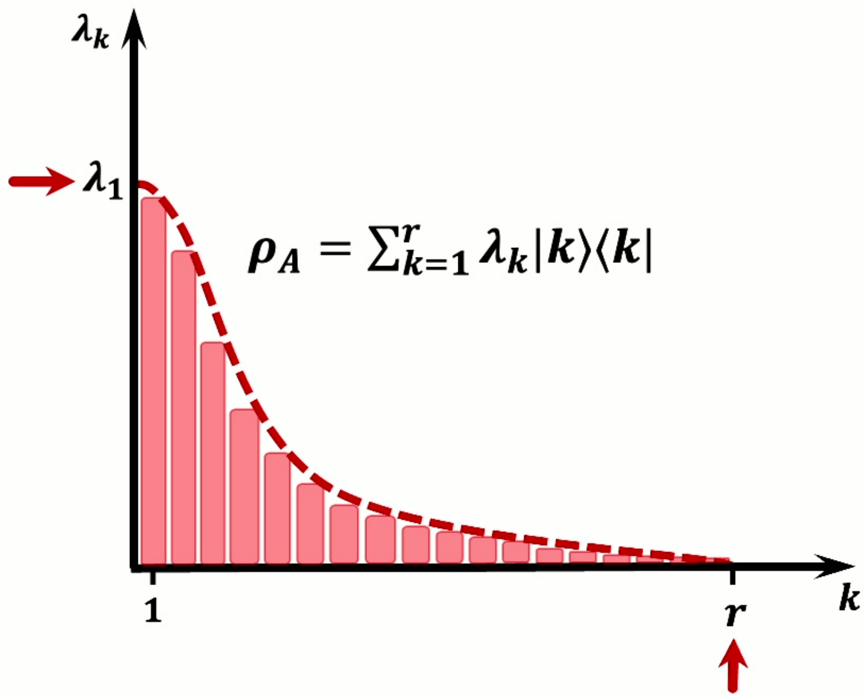
$$|\psi\rangle_{AB} = \sum_{k=1}^{r} \sqrt{\lambda_k} |k\rangle_A |k\rangle_B$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$$
$$\lambda_1 + \lambda_2 + \cdots + \lambda_r = 1$$

Schmidt Form

Entanglement spread

$$\mathrm{ES}(\rho_A) = \log(r\lambda_1)$$
$$= \mathrm{S}_{\max}(\rho_A) - \mathrm{S}_{\min}(\rho_A)$$
$$\approx \log(\lambda_1/\lambda_r)$$



Entanglement spread

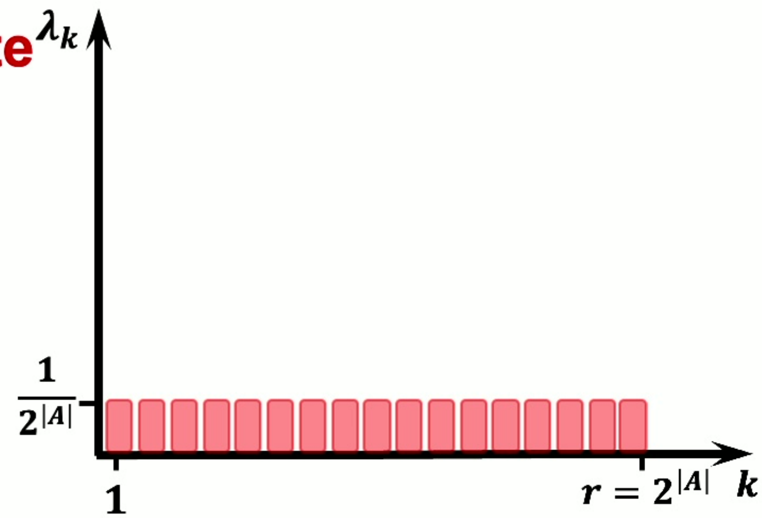
$$\mathrm{ES}(\rho_A) = \log(r\lambda_1)$$
$$= S_{\max}(\rho_A) - S_{\min}(\rho_A)$$
$$\approx \log(\lambda_1/\lambda_r)$$

Some examples:

Maximally entangled state

$$|\phi\rangle_{AB} = \sum_{k=1}^{2^{|A|}} \sqrt{1/2^{|A|}}\,|k\rangle_A |k\rangle_B$$

$$\mathrm{ES}(\rho_A) = 0$$



Entanglement spread

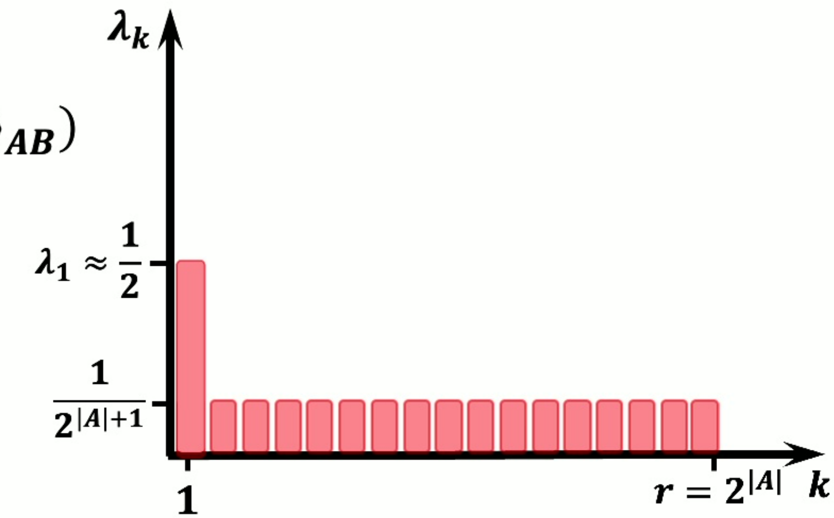
$$\mathrm{ES}(\rho_A) = \log(r\lambda_1)$$
$$= \mathrm{S}_{\max}(\rho_A) - \mathrm{S}_{\min}(\rho_A)$$
$$\approx \log(\lambda_1/\lambda_r)$$

Some examples:

or more exotic

$|\omega\rangle_{AB} \approx 1/\sqrt{2}\,(|\psi\rangle_{AB} + |\phi\rangle_{AB})$

$\mathrm{ES}(\rho_A) = O(|A|)$



Our protocol gives an **upper bound** on $C_\varepsilon(\mathrm{GS}_{AB})$

$$\tilde{O}\left(\frac{|\partial A|}{\mathrm{gap}} \cdot \log\frac{1}{\varepsilon}\right) \geq C_\varepsilon(\mathrm{GS}_{AB})$$

But there is also a known **lower bound** on $C_\varepsilon(GS_{AB})$:

$$\tilde{O}\left(\frac{|\partial A|}{\mathrm{gap}} \cdot \log\frac{1}{\varepsilon}\right) \geq \textbf{Entanglement Spread}$$

Our protocol gives an upper bound on $C_\varepsilon(\mathrm{GS}_{AB})$

$$\tilde{O}\left(\frac{|\partial A|}{\mathrm{gap}} \cdot \log\frac{1}{\varepsilon}\right) \geq C_\varepsilon(\mathrm{GS}_{AB})$$

But there is also a known lower bound on $C_\varepsilon(GS_{AB})$:

$$\tilde{O}\left(\frac{|\partial A|}{\mathrm{gap}} \cdot \log\frac{1}{\varepsilon}\right) \geq \text{Entanglement Spread}$$

Compared to area law: $O\left(\frac{|\partial A|}{\mathrm{gap}}\right) \geq$ *Entanglement Entropy*

1. Testing matrix product states

2. Testing machine learning models

3. Two-party testing of ground states