Title: Chaos and resonances in EMRI (extreme mass ratio inspiral) dynamics

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Series: Cosmology & Gravitation

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Abstract: EMRIs are one of the primary targets of spaceborne gravitational wave (GW) detectors and will be ideal GW sources for testing fundamental laws of gravity. In a generic non-Kerr spacetime, the EMRI system is non-integrable due to the lack of the Carter constant. As a result, chaos along with resonance islands arise in these systems leaving a non-Kerr signature in the EMRI waveform as proposed in many previous studies. In this work, we systematically analyze the dynamics of an EMRI system near orbital resonances and we have derived an effective resonant Hamiltonian that describes the dynamics of the resonant degree of freedom with the action-angle formalism. We have two major findings: (1) the chaotic orbits in general produce unique commensurate jumps in actions and (2) the EMRI orbits driven by radiation-reaction in general do not cross the resonance islands.

Zoom Link: https://pitp.zoom.us/j/95975225333?pwd=V05NZzQ3cE9neUZpN3RIOCt0UE5mZz09

Motivation

$10^{5-7} M_\odot$ LISA

M

m

$r \approx 10M$

$q = \frac{m}{M} = 10^{-4} - 10^{-6}$

EMRIs are ideal GW sources for testing fundamental laws of gravity.

1. Incremental effect in the waveform

$$g_{\text{kerr}} \rightarrow g_{\text{kerr}} + \epsilon h \Rightarrow \Psi(f) \rightarrow \Psi(f) + \epsilon \Psi_h(f)$$

2. 0-1 effect in the waveform

   e.g., no chaos in Kerr v.s. chaos in non-Kerr

Integrable systems and KAM theorem

Integrable system: # degrees of freedom = # conserved quantities

e.g. a test particle in the Kerr spacetime:

$$(t, r, \theta, \phi) \longleftrightarrow (H, E, L, C)$$

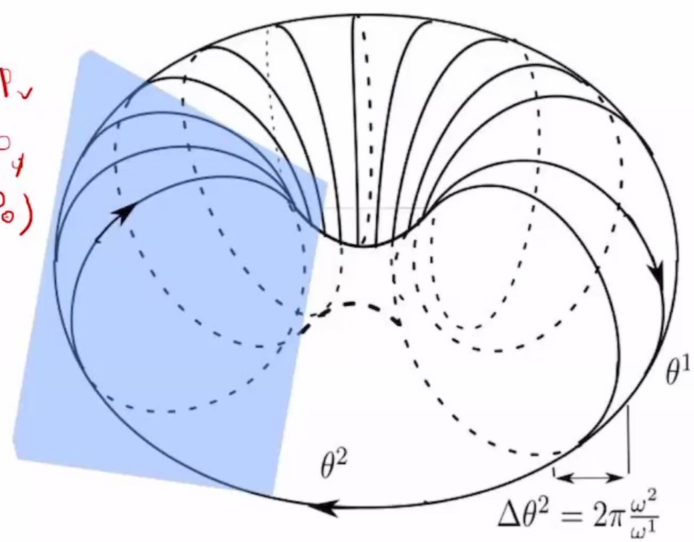
$H = \frac{1}{2} g^{\mu\nu}_{Kerr} P_\mu P_\nu$

$E = -P_t , \; L = +P_\phi$

$C = C(\theta, P_t, P_\phi, P_\theta)$

e.g. 2 d.o.f. Integrable system: orbit wraps on a 2-torus

(Cardenas-Avendano+2018)



Integrable systems and KAM theorem

KAM theorem: when an integrable system becomes non-integrable due to a perturbation, the trajectory torus in the phase space will be slightly deformed instead of being broken if the perturbation satisfies the two conditions

(1) the perturbation is small

(2) the perturbation contains ***no component of commensurate frequencies***.

Chaos can occur otherwise.

$$H_{int}(q^r, q^\theta \mid J^r, J^\theta) = \sum \underline{H_{n_r, n_\theta}(J)}\, e^{i(n_r q^r + n_\theta q^\theta)}$$

$$\rightarrow n_r \Omega^r + n_\theta \Omega^\theta = 0$$

$$\Omega^r \updownarrow \dot{q}^r \qquad \Omega^\theta \updownarrow \dot{q}^\theta$$

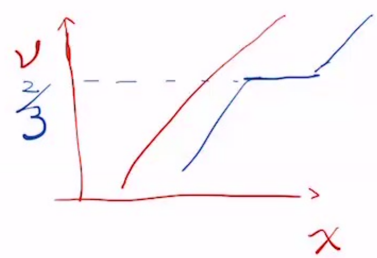
Poincare map and rotation number

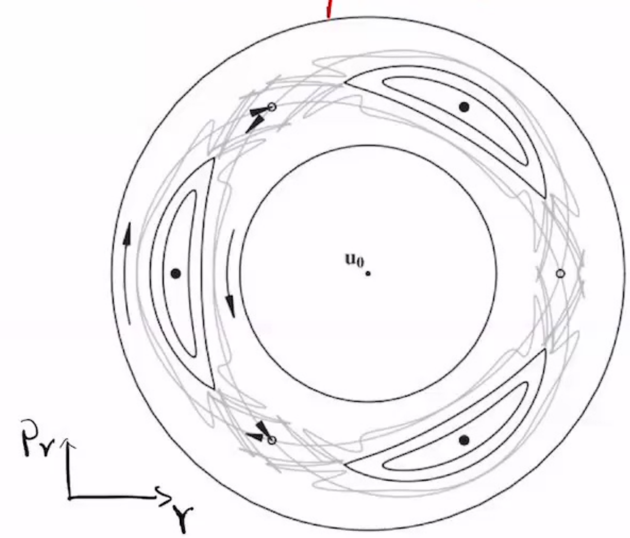
a 2 d.o.f. system as an example : (r, theta) + conjugate momenta (pr, p_theta)

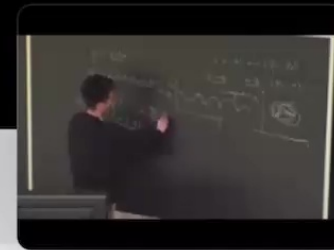
3 kinds of orbits

Poincare map: $(r, p_r)|_{\theta=\pi/2}$

Rotation number: $\nu := \frac{\langle \Omega^r \rangle}{\langle \Omega^\theta \rangle} = \frac{N_r}{N_\theta}$







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(1) the perturbation is small

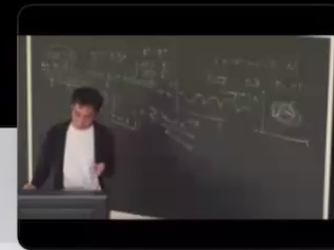
$$H_{int}(q^r, q^\theta \mid J^r, J^\theta) = \sum \underline{H_{n_r, n_\theta}(J)}\, e^{i(n_r q^r + n_\theta q^\theta)}$$

(2) the perturbation contains ***no component of commensurate frequencies***.

$$\rightarrow n_r \Omega^r + n_\theta \Omega^\theta = 0$$

$$\Omega^r \updownarrow \dot{q}^r \qquad \Omega^\theta \updownarrow \dot{q}^\theta$$

Chaos can occur otherwise.



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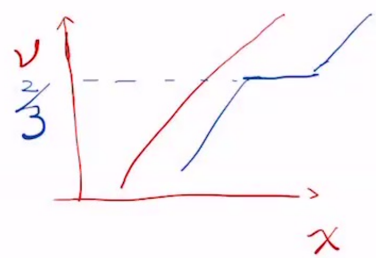
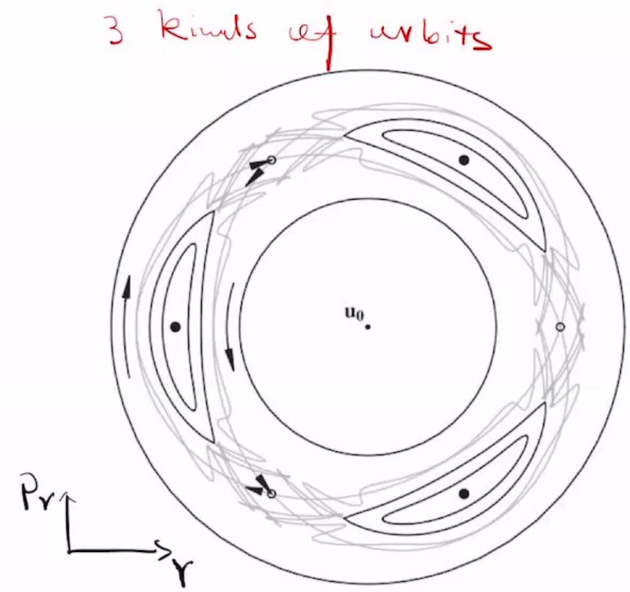
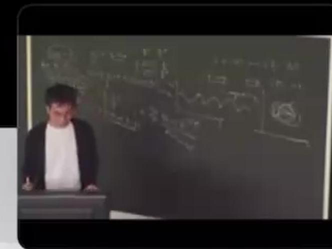






Kerr (integrable) → Perturbed Kerr (non-integrable)

Quadratic Gravity as an example Donoghue+2021

$$S_{\rm quad} = \int d^4x\sqrt{-g}\left[\frac{2}{\kappa^2}R + \frac{1}{6f_0^2}R^2 - \frac{1}{2\xi^2}C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}\right]$$

Spinning BH solution

$$g^{\mu\nu}(r,\theta) = g^{\mu\nu}_{\rm Kerr}(r,\theta) + \epsilon h^{\mu\nu}(r,\theta)$$

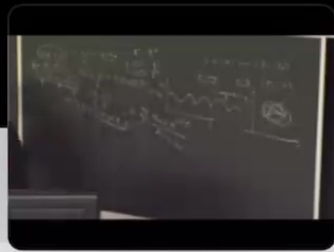
(H, E, L, C)

$\Downarrow$

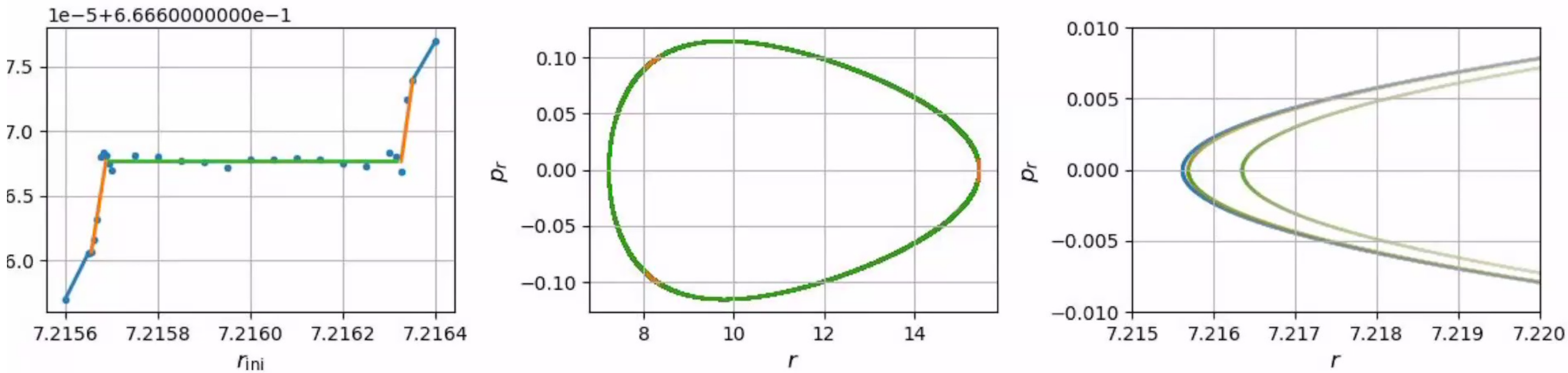
(H, E, L)

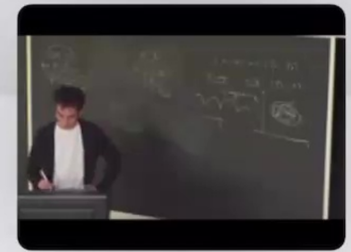
Hamiltonian

$$H(x^\mu, p_\nu) = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu = H_{\rm Kerr} + \epsilon H_{\rm int}$$



Near-resonance orbits : phenomenology



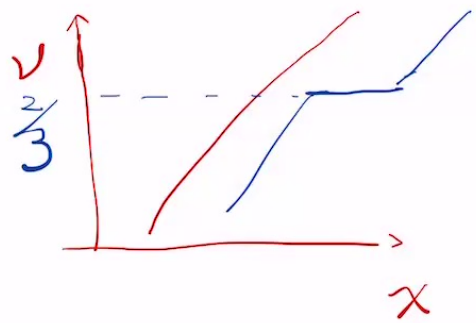


ap and rotation number

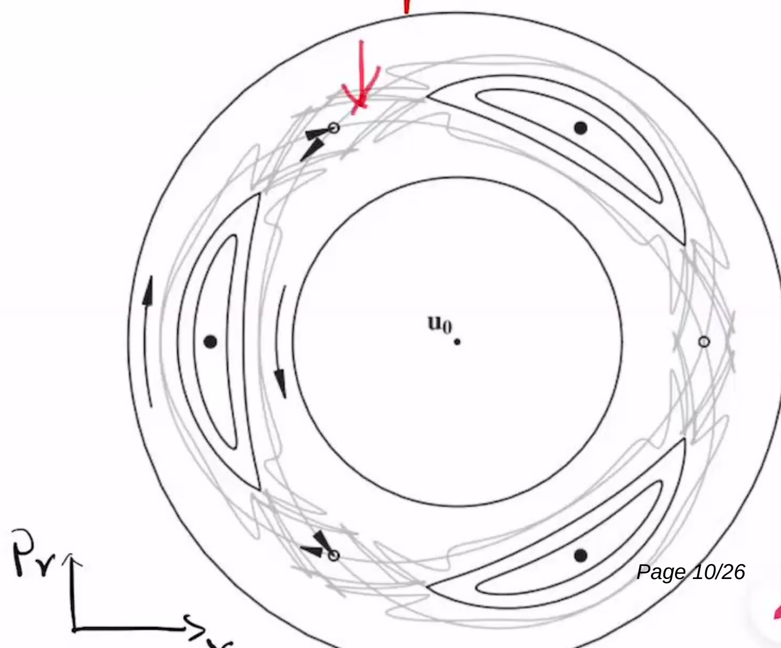
an example : (r, theta) + conjugate momenta (pr, p_theta)

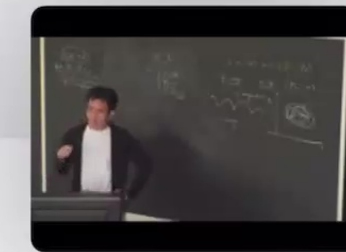
$$(r, p_r)|_{\theta=\pi/2}$$

$$\nu := \frac{\langle \Omega^r \rangle}{\langle \Omega^\theta \rangle} = \frac{N_r}{N_\theta}$$



3 kinds of orbits





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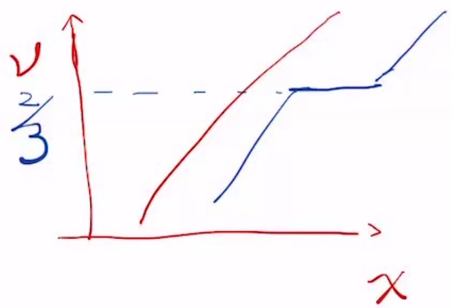
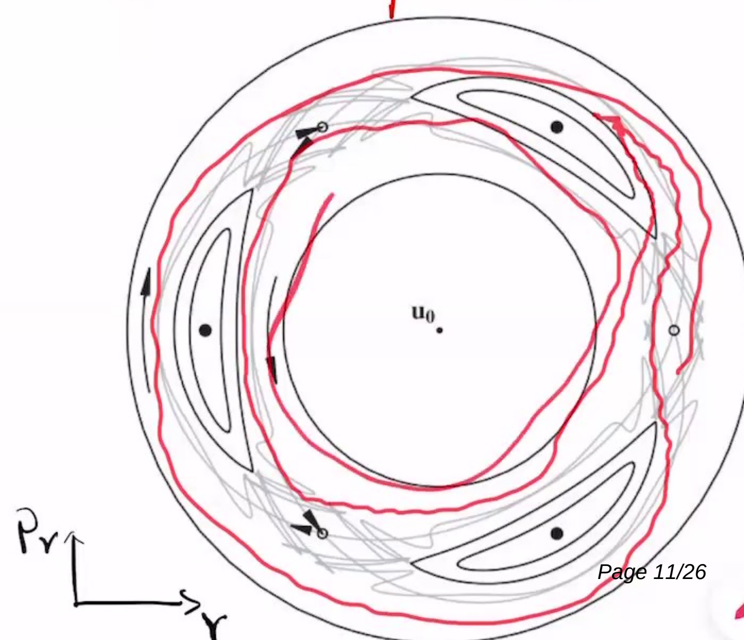
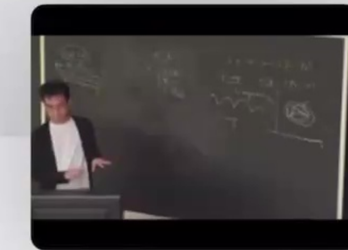
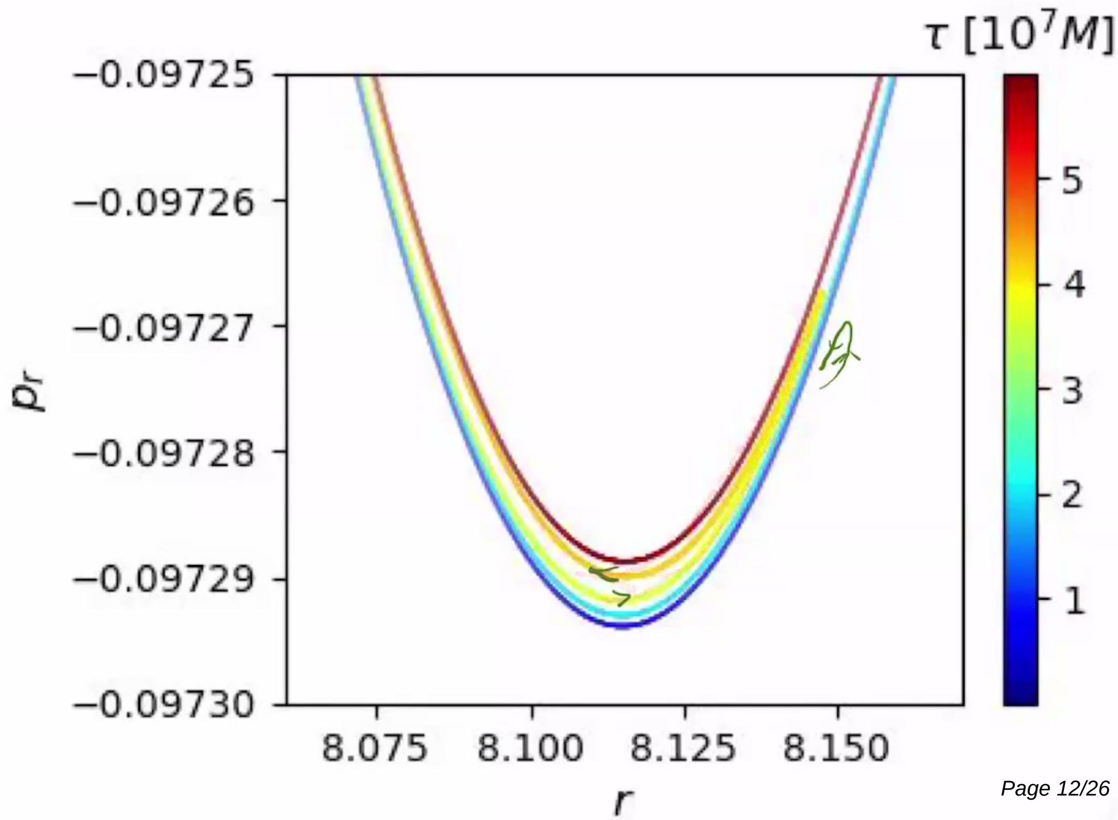






Near-resonance orbits : phenomenology



Action-Angle variables

Kerr spacetime

$$\mathcal{J}_\alpha = \mathcal{J}_\alpha(\{x^\beta, p_\beta\}), \quad q^\alpha = q^\alpha(\{x^\beta, p_\beta\})\,.$$

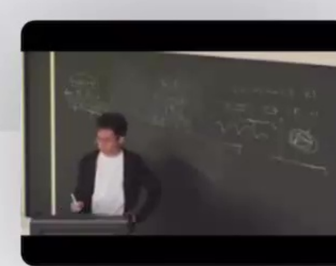
$$H_0(x^\mu, p_\nu) = \frac{1}{2} g^{\mu\nu}_{\mathrm{Kerr}} p_\mu p_\nu \longrightarrow H_0^{aa} = H_0^{aa}(\mathcal{J}_\alpha)$$

$$\dot{\mathcal{J}}_\alpha = -\frac{\partial H_0}{\partial q^\alpha} = 0, \quad \dot{q}^\alpha = \frac{\partial H_0}{\partial \mathcal{J}_\alpha} = \Omega^\alpha$$

Non-Kerr spacetime

$$H(x^\mu, p_\nu) = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu = H_0 + \epsilon H_{\mathrm{int}} \longrightarrow H_0(\mathcal{J}_\alpha) + \epsilon H_{\mathrm{int}}(\mathcal{J}_\alpha, q^\beta)$$

$$\frac{d\mathcal{J}_\alpha}{d\tau} = -\epsilon \frac{\partial \mathcal{H}_{\mathrm{int}}}{\partial q^\alpha}\,, \qquad \frac{dq^\alpha}{d\tau} = \Omega_\alpha + \epsilon \frac{\partial \mathcal{H}_{\mathrm{int}}}{\partial \mathcal{J}_\alpha}\,,$$



gle variables

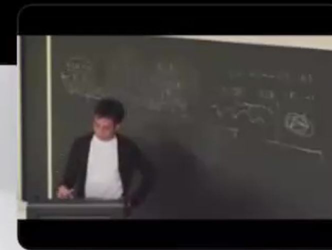
$$\mathcal{J}_\alpha = \mathcal{J}_\alpha(\{x^\beta, p_\beta\}), \quad q^\alpha = q^\alpha(\{x^\beta, p_\beta\}) \,.$$

$${}^{\mu\nu}_{\mathrm{Kerr}} p_\mu p_\nu \longrightarrow H_0^{aa} = H_0^{aa}(\mathcal{J}_\alpha) \qquad \dot{\mathcal{J}}_\alpha = -\frac{\partial H_0}{\partial q^\alpha} = 0, \quad \dot{q}^\alpha = \frac{\partial H_0}{\partial \mathcal{J}_\alpha} = \Omega^\alpha$$

e

$$H(x^\mu, p_\nu) = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu = H_0 + \epsilon H_{\mathrm{int}} \longrightarrow \underline{H_0(\mathcal{J}_\alpha)} + \epsilon \underline{H_{\mathrm{int}}(\mathcal{J}_\alpha, q^\beta)}$$

$$\underline{\frac{d\mathcal{J}_\alpha}{d\tau} = -\epsilon \frac{\partial \mathcal{H}_{\mathrm{int}}}{\partial q^\alpha}}\,, \qquad \frac{dq^\alpha}{d\tau} = \Omega_\alpha + \epsilon \frac{\partial \mathcal{H}_{\mathrm{int}}}{\partial \mathcal{J}_\alpha},$$



Perturbative sols and resonance as a singularity

$$\frac{dq^\alpha}{d\tau} = \Omega_\alpha + \epsilon \sum_{n_r,n_\theta} \frac{\partial H_{n_r,n_\theta}}{\partial \mathcal{J}_\alpha} e^{i(n_r q^r + n_\theta q^\theta)},$$

$$\frac{d\mathcal{J}_\alpha}{d\tau} = -i\epsilon \sum_{n_r,n_\theta} n_\alpha H_{n_r,n_\theta} e^{i(n_r q^r + n_\theta q^\theta)}.$$

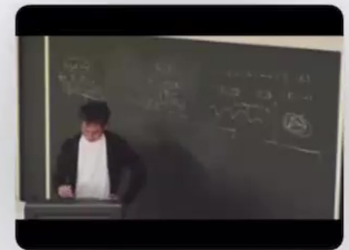
If

$$\tilde{q}^\alpha = q^\alpha + \epsilon \sum_{n_\alpha \in R} \frac{1}{n_\alpha \Omega^\alpha} \frac{\partial H_{n_j}}{\partial \mathcal{J}_\alpha} e^{i(n_r q^r + n_\theta q^\theta)},$$

$$\tilde{J}_\alpha = J_\alpha + \epsilon \sum_{n_\alpha \in R} \frac{n_\alpha}{n_\alpha \Omega^\alpha} H_{n_j} e^{i(n_r q^r + n_\theta q^\theta)}.$$

We have $\dot{\tilde{\mathcal{J}}}_\alpha = -\dfrac{\partial H_0}{\partial q^\alpha} = 0, \quad \dot{\tilde{q}}^\alpha = \dfrac{\partial H_0}{\partial \mathcal{J}_\alpha} = \Omega^\alpha$

But Resonance : $\sum_\alpha n_\alpha \Omega^\alpha = 0$



nance → Reduced d.o.f

$$\frac{1}{{}_\alpha\Omega^\alpha}\frac{\partial H_{n_j}}{\partial \mathcal{J}_\alpha} e^{i(n_r q^r + n_\theta q^\theta)} ,$$

$$\frac{n_\alpha}{{}_\alpha\Omega^\alpha} H_{n_j} e^{i(n_r q^r + n_\theta q^\theta)} .$$

We have

$$\frac{d\tilde{q}^\alpha}{d\tau} = \Omega_\alpha + \epsilon \sum_{k\in\mathbb{Z}} \frac{\partial H_{kN_j}}{\partial \mathcal{J}_\alpha} e^{ikN_j q^j} ,$$

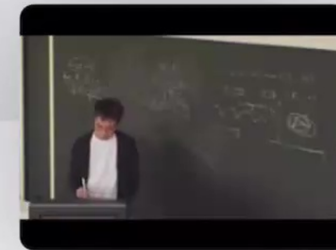
$$\frac{d\tilde{\mathcal{J}}_\alpha}{d\tau} = -i\epsilon N_\alpha \sum_{k\in\mathbb{Z}} k H_{kN_j} e^{ikN_j q^j} .$$

angle variables

nt

$$\frac{dQ}{d\tau} := \frac{d}{d\tau}(N_r\tilde{q}^r + N_\theta \tilde{q}^\theta) = N_\alpha \Omega^\alpha + O(\epsilon) := \Delta\omega ,$$

$$\frac{d\Theta}{d\tau} := \frac{1}{N_r}\frac{d\tilde{J}_r}{d\tau} = \frac{1}{N_\theta}\frac{d\tilde{J}_\theta}{d\tau} = O(\epsilon)$$



Near-resonance → Reduced d.o.f

$$\tilde{q}^\alpha = q^\alpha + \epsilon \sum_{n_\alpha \Omega^\alpha \neq 0} \frac{1}{n_\alpha \Omega^\alpha} \frac{\partial H_{n_j}}{\partial \mathcal{J}_\alpha} e^{i(n_r q^r + n_\theta q^\theta)},$$

$$\tilde{J}_\alpha = J_\alpha + \epsilon \sum_{n_\alpha \Omega^\alpha \neq 0} \frac{n_\alpha}{n_\alpha \Omega^\alpha} H_{n_j} e^{i(n_r q^r + n_\theta q^\theta)}.$$

$n^r\Omega^r + n^\theta\Omega^\theta \simeq 0$

We have

$$\frac{d\tilde{q}^\alpha}{d\tau} = \Omega_\alpha + \epsilon \sum_{k\in\mathbb{Z}} \frac{\partial H_{kN_j}}{\partial \mathcal{J}_\alpha} e^{ik}$$

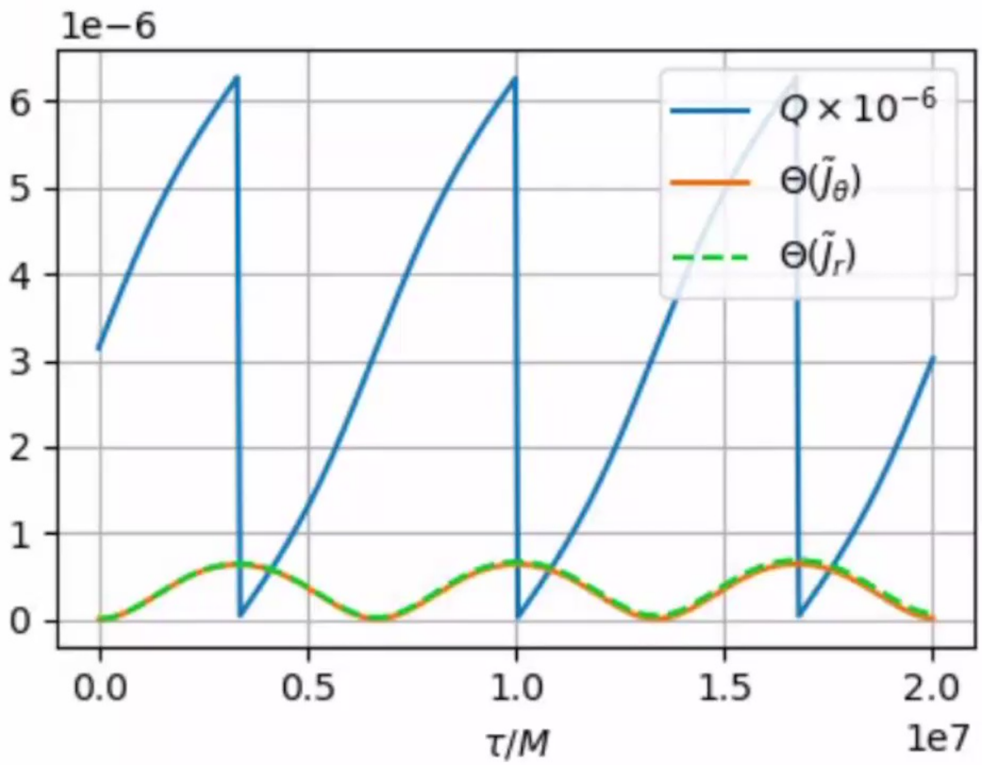
$$\frac{d\tilde{\mathcal{J}}_\alpha}{d\tau} = -i\epsilon N_\alpha \sum_{k\in\mathbb{Z}} k H_{kN_j} e^{ikN}$$

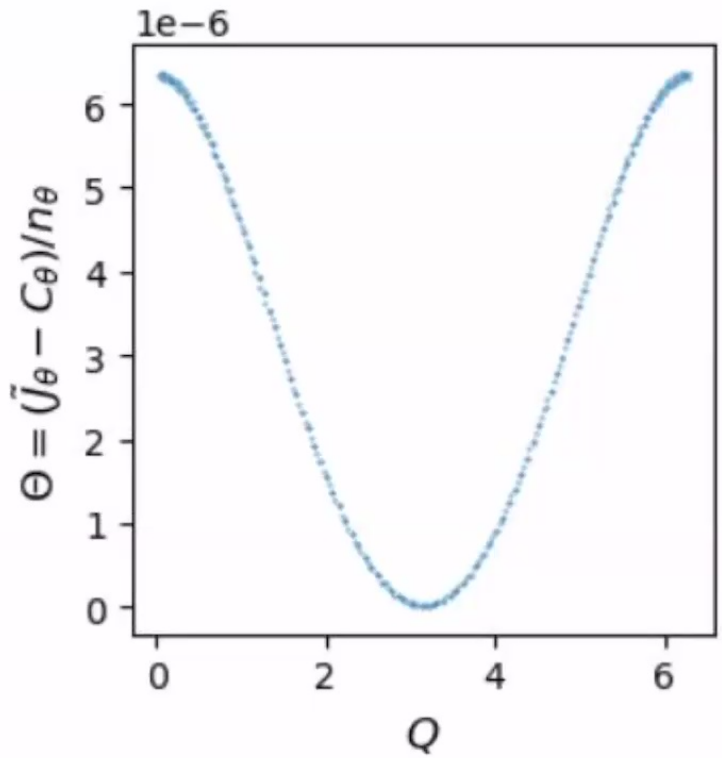
Not all action angle variables are independent

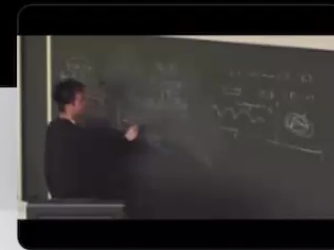
$$\frac{dQ}{d\tau} := \frac{d}{d\tau}(N_r \tilde{q}^r + N_\theta \tilde{q}^\theta) = N_\alpha \Omega^\alpha + \mathcal{O}(\epsilon) :=$$

$$\frac{d\Theta}{d\tau} := \frac{1}{N_r}\frac{d\tilde{J}_r}{d\tau} = \frac{1}{N_\theta}\frac{d\tilde{J}_\theta}{d\tau} = \mathcal{O}(\epsilon)$$

Near-resonance orbits: regular orbits







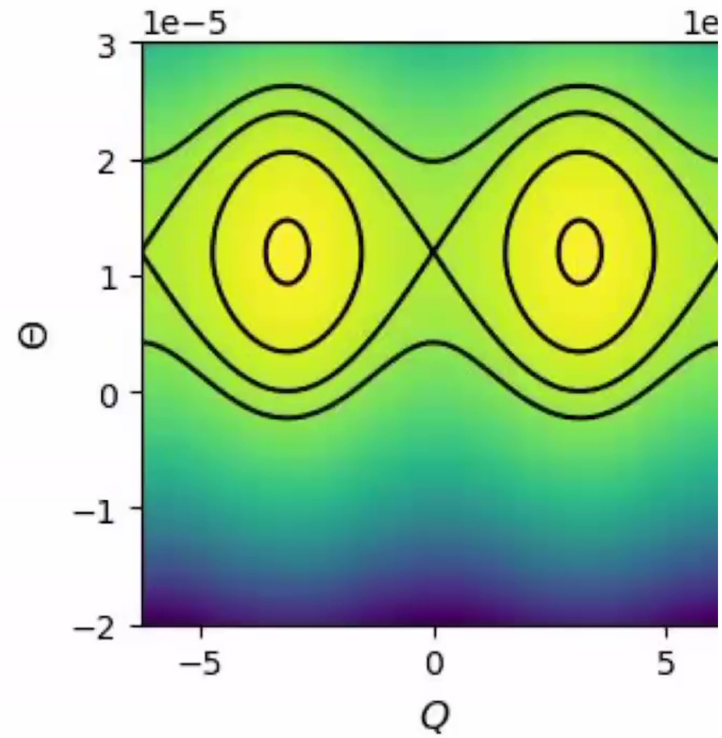
Near-resonance orbits : 1 d.o.f effective Hamiltonian

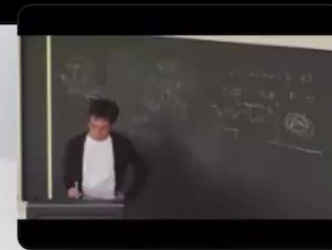
$$\frac{dQ}{d\tau} := \frac{d}{d\tau}(N_r\tilde{q}^r + N_\theta\tilde{q}^\theta) = N_\alpha\Omega^\alpha + \mathcal{O}(\epsilon) := \Delta\omega(\Theta)$$

$$\frac{d\Theta}{d\tau} := \frac{1}{N_r}\frac{d\tilde{J}_r}{d\tau} = \frac{1}{N_\theta}\frac{d\tilde{J}_\theta}{d\tau} = \mathcal{O}(\epsilon)$$

$$\mathcal{H}_{\text{eff}} = \int \Delta\omega d\Theta + 2\sum_{k\geq 1}\text{Re}(H_{k\mathbf{N}})\cos kQ - 2\sum_{k\geq 1}\text{Im}(H_{k\mathbf{N}})\sin kQ\,.$$

$$\Delta\omega(\Theta) \approx \alpha_0 + 2\beta_0\Theta$$





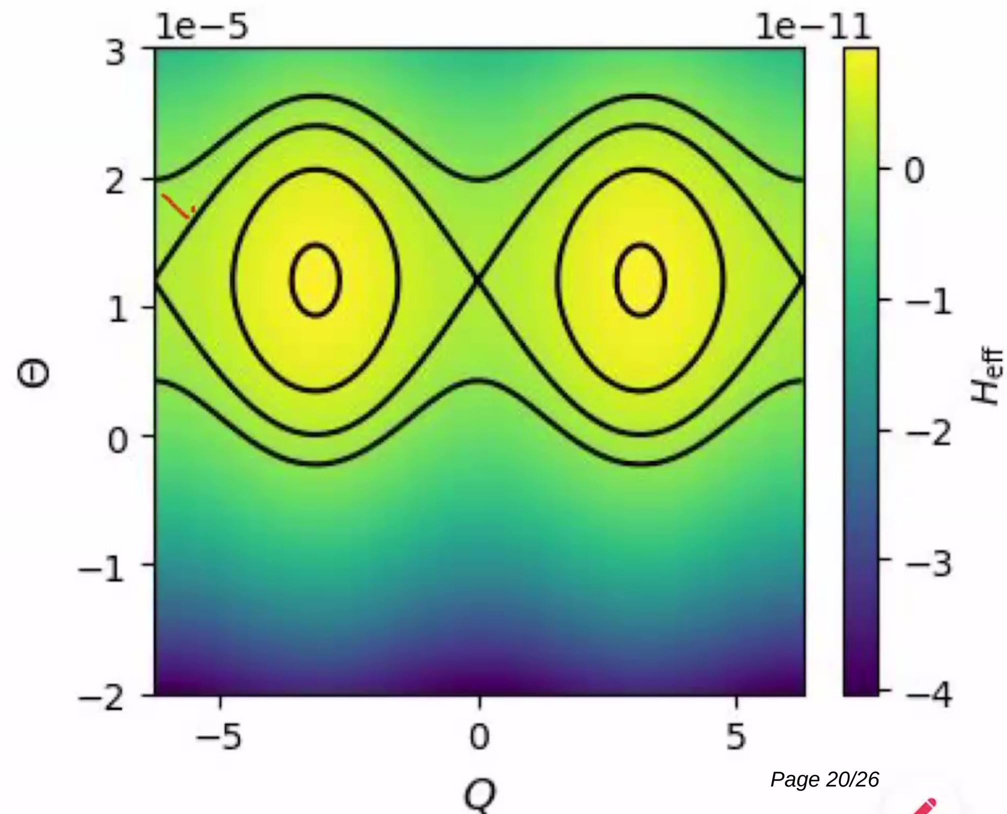
esonance orbits : 1 d.o.f effective Hamiltonian

$$N_r\tilde{q}^r + N_\theta\tilde{q}^\theta) = N_\alpha\Omega^\alpha + \mathcal{O}(\epsilon) := \Delta\omega(\Theta)$$

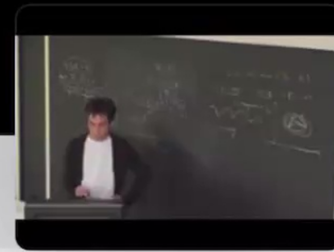
$$\frac{d\tilde{J}_r}{d\tau} = \frac{1}{N_\theta}\frac{d\tilde{J}_\theta}{d\tau} = \mathcal{O}(\epsilon)$$

$$) + 2\sum_{k\geq1}\mathrm{Re}(H_{k\mathbf{N}})\cos kQ - 2\sum_{k\geq1}\mathrm{Im}(H_{k\mathbf{N}})\sin kQ\,.$$

$$\Delta\omega(\Theta) \approx \alpha_0 + 2\beta_0\Theta$$

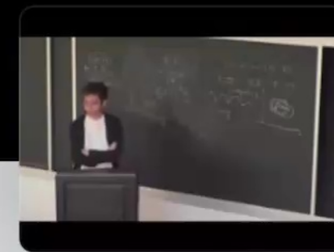


$\delta\psi\Big|_{\text{Kerr, self-force}} \sim 1/\sqrt{q}$



Summary

1. Large perturbation (h_res > q), adiabatic limit $\qquad \delta\Psi \sim \delta J/q \sim \sqrt{h_{\rm res}}/q$
   → no resonance crossing, only transitional circumventing
   → **commensurate jumps** in the two actions and large phase shift in the waveform
2. Small perturbation (h_res < q),
   → resonance crossing → small phase shift in the waveform $\qquad \delta\Psi \sim \delta J/q \sim h_{\rm res}/q^{3/2}$



Radiation dissipation and resonance crossing

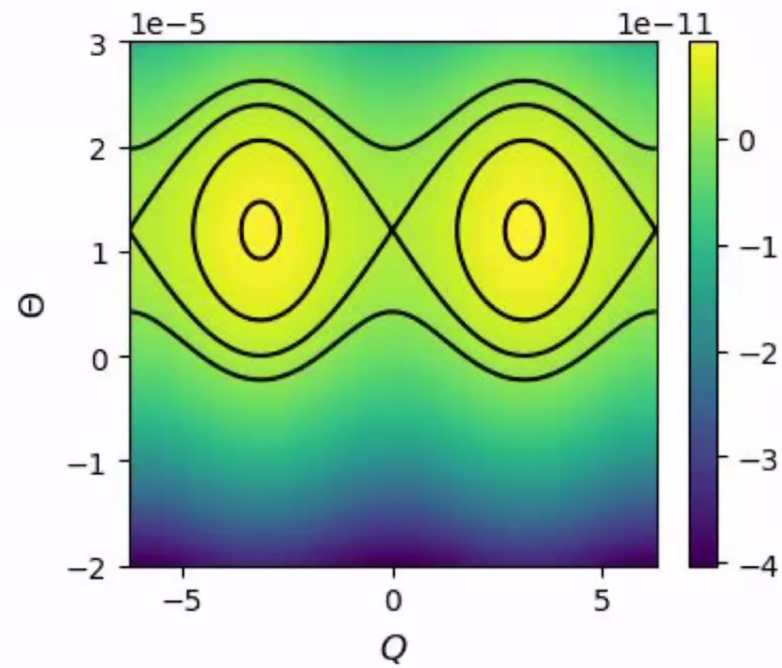
Adiabatic limit: t_cross > t_res (h_res > q)

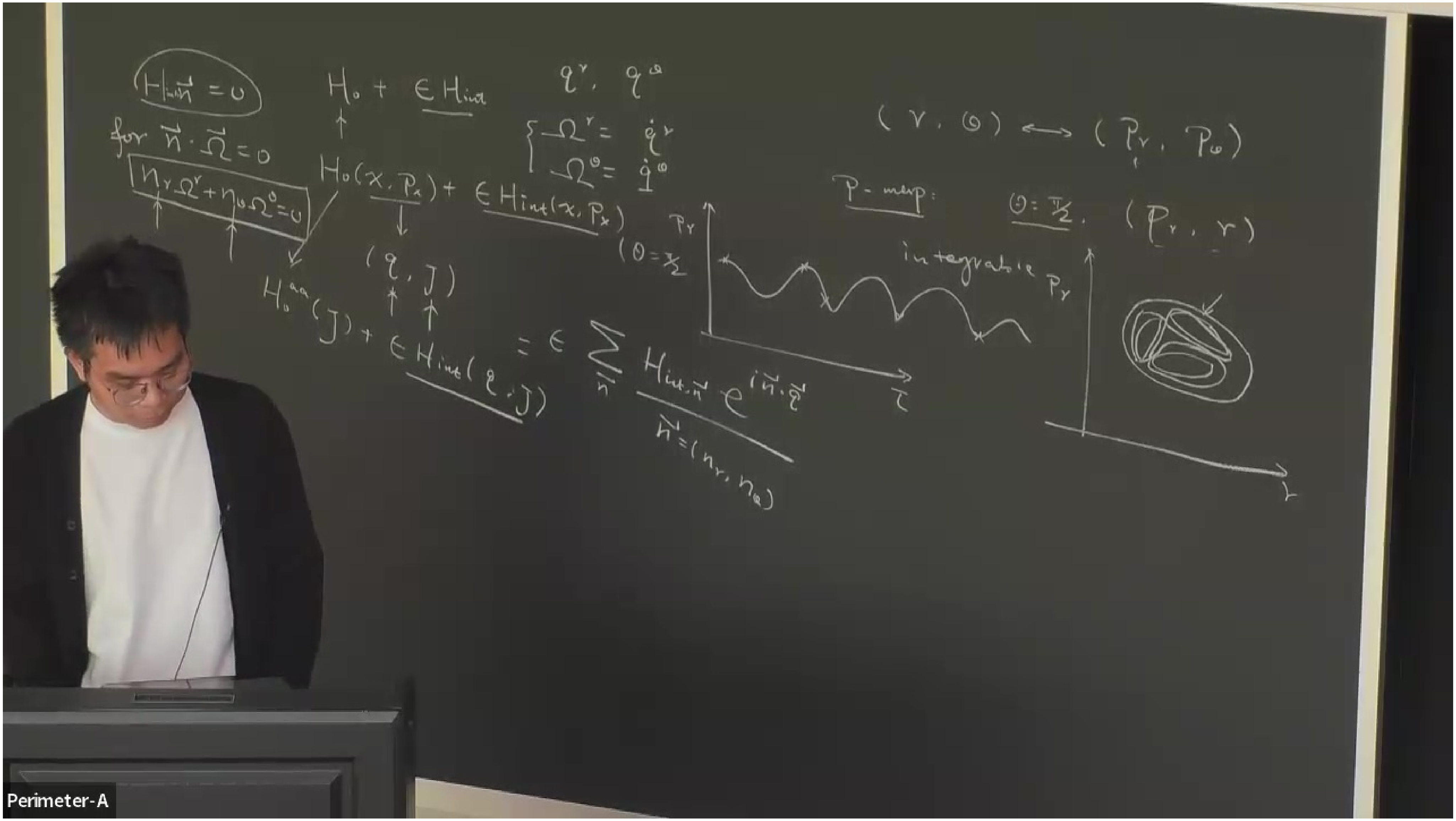
Area conserved as an adiabatic invariant. $\int \Theta dQ$

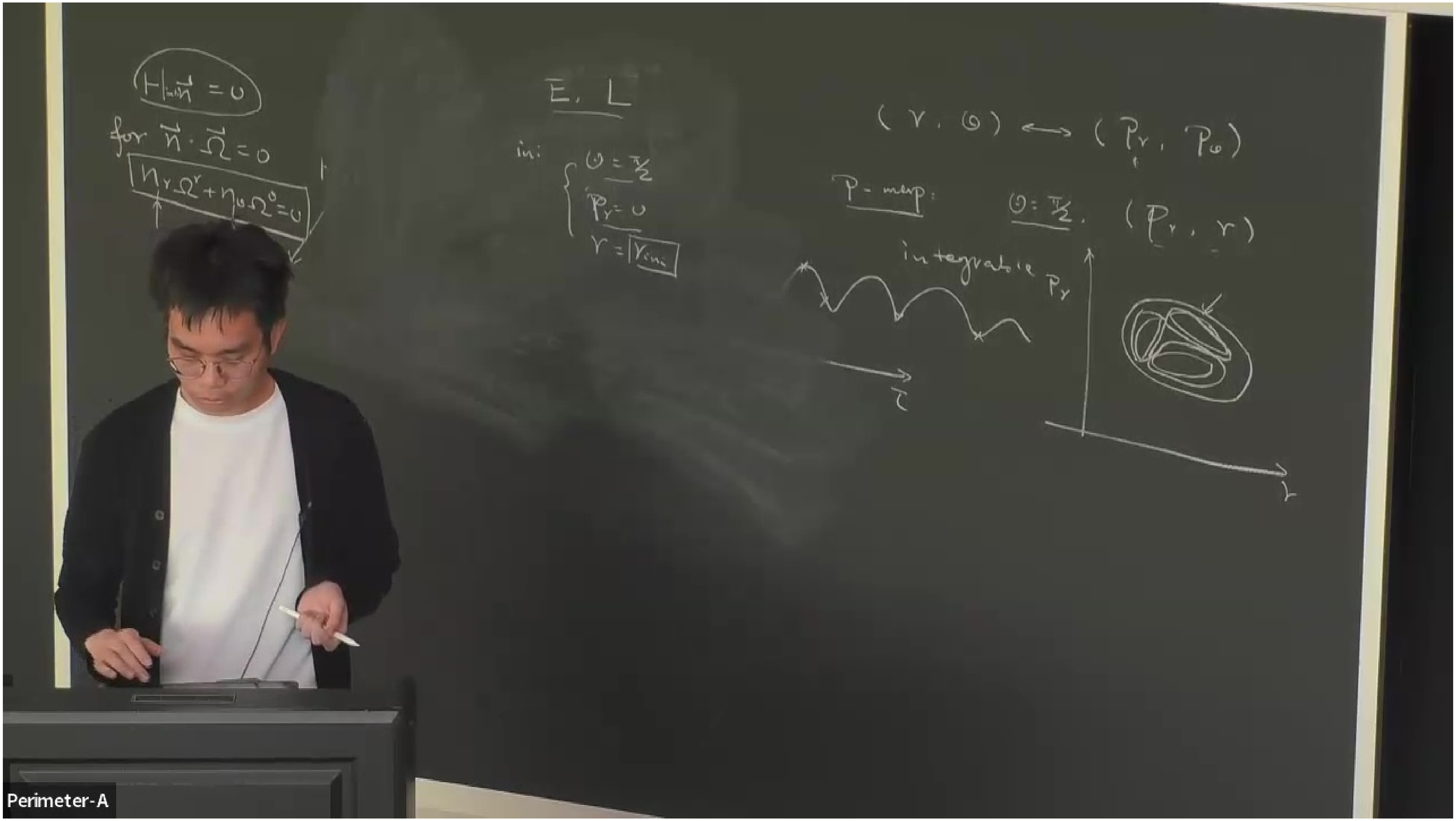
Circumventing resonance via transitional orbit.

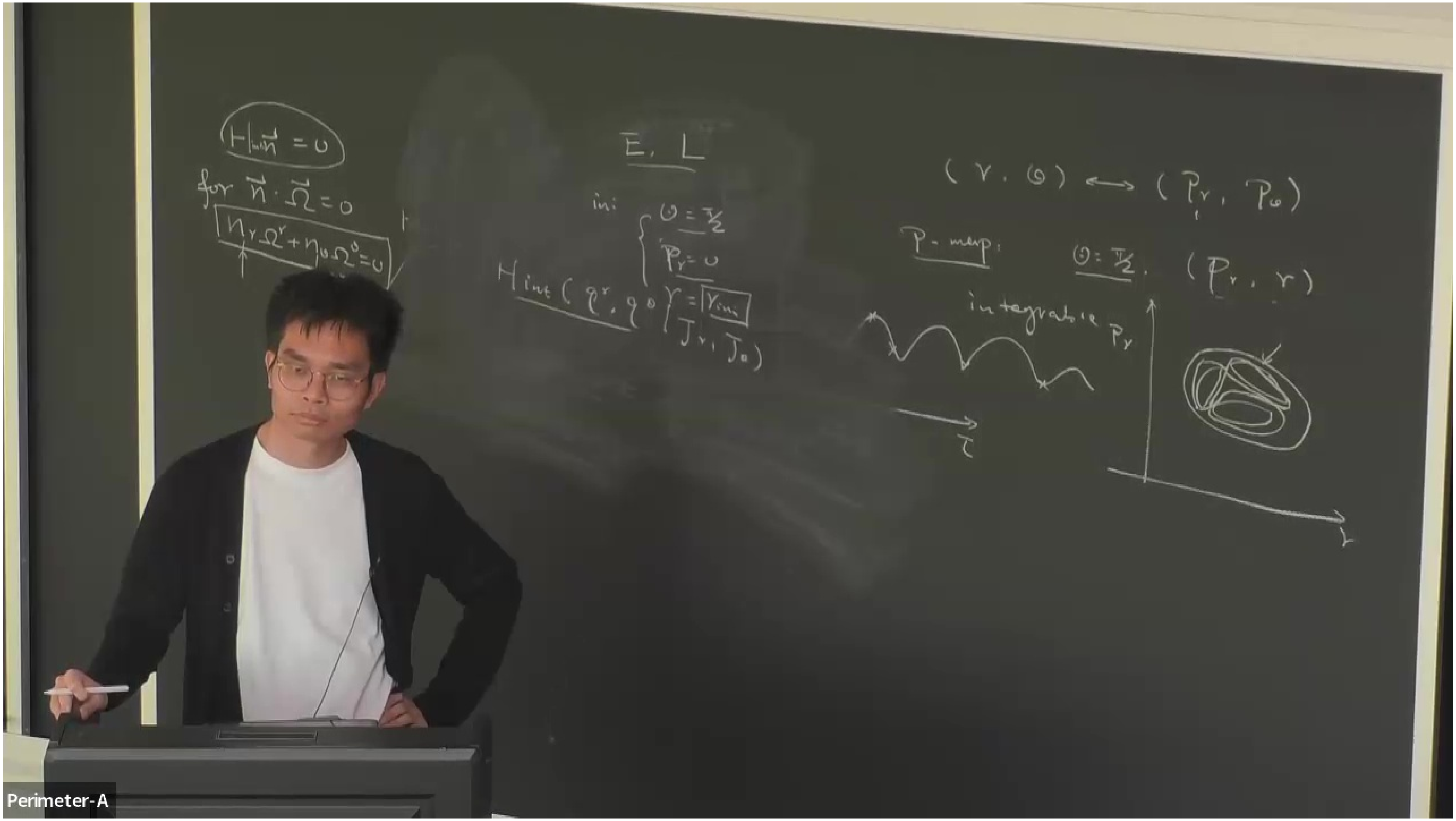
$$\delta J_r/\delta J_\theta = N_r/N_\theta = -3/2\,, \delta J \approx \delta\Theta \sim \sqrt{h_{\mathrm{res}}}$$

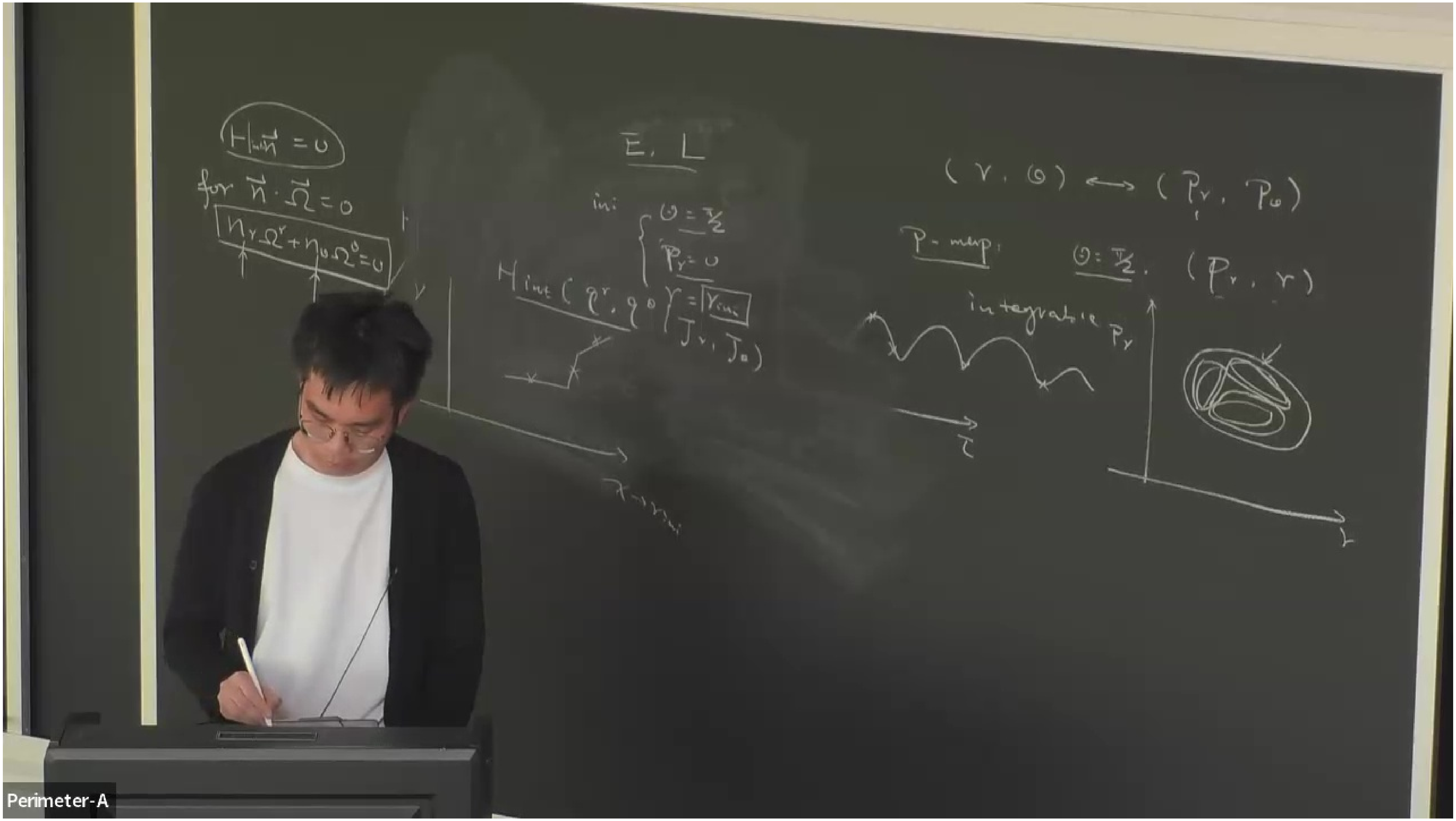
$$\delta\Psi \sim \delta J/q \sim \sqrt{h_{\mathrm{res}}}/q$$




$H_{int,\vec{n}} = 0$
for $\vec{n} \cdot \vec{\Omega} = 0$
$n_r \Omega^r + n_\theta \Omega^\theta = 0$
$H_0 + \epsilon H_{int}$
$q^r, \; q^\theta$
$\Omega^r = \dot{q}^r$
$\Omega^\theta = \dot{q}^\theta$
$H_0(x, p_x) + \epsilon H_{int}(x, p_x)$
(q, J)
$H_0^{aa}(J) + \epsilon H_{int}(q, J)$
$= \epsilon \sum_{\vec{n}} H_{int,\vec{n}} e^{i \vec{n} \cdot \vec{q}}$
$\vec{n} = (n_r, n_\theta)$
$(r, \theta) \longleftrightarrow (p_r, p_\theta)$
P-map:
$\theta = \pi/2$
(p_r, r)
p_r
$(\theta = \pi/2)$
r
integrable
Perimeter-A


Perimeter-A


Perimeter-A
E, L
for n · Ω = 0
in:
P-map:
integrable
(r, θ) ⟷ (p_r, p_θ)
θ = π/2
p_r = 0
(p_r, r)


Perimeter-A