Title: Homological Link Invariants from Floer Theory

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Series: Mathematical Physics

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Abstract: In recent work, Aganagic proposed a categorification of quantum link invariants based on a category of A-branes, which is solvable explicitly. The theory is a generalization of Heegaard-Floer theory from gl(1|1) to arbitrary Lie algebras. I will describe in the detail the two simplest cases: the su(2) theory, categorifying the Jones polynomial, and the gl(1|1) theory, categorifying the Alexander polynomial. I will give an explicit algorithm for computing link homologies in these cases. I will also briefly describe the generalization to other simple Lie algebras and to Lie superalgebras of type gl(m|n). This talk is based on work to appear with Mina Aganagic and Miroslav Rapcak.

Zoom Link: TBD

Homological Link Invariants from Floer Theory

Elise LePage with Mina Aganagic and Miroslav Rapcak

Perimeter Institute, May 11, 2023

UC Berkeley

Knot invariants from CS theory

- In 1989, Witten showed that the Jones polynomial comes from Chern-Simons theory at level k with gauge group $SU(2)$
- Knots are Wilson lines colored by the fundamental rep of $SU(2)$
- The partition function with a Wilson line inserted coincides with the Jones polynomial after identifying

$$\mathfrak{q} = \exp\left(\frac{2\pi i}{2+k}\right)$$

Knot invariants from CS theory

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- By varying the gauge group and representation, one gets many different link invariants, known as quantum group invariants

Khovanov homology

- A categorification of the Jones polynomial due to Khovanov (1999)
- A link invariant
- The first categorification of a link polynomial
- Khovanov homology assigns to a link K a collection of bi-graded vector spaces

$$\mathcal{H}_K = \bigoplus_{i,j} \mathcal{H}_K^{i,j}$$

such that the graded Euler characteristic coincides with the Jones polynomial:

$$J_K(\mathfrak{q}) = \sum_{i,j\in\mathbb{Z}} (-1)^i \mathfrak{q}^{j/2} \dim_{\mathbb{C}} \mathcal{H}_K^{i,j}$$

Link Homology

- Should be a link invariant
- The Euler characteristic should be a link polynomial invariant
- Known examples:
  - Khovanov homology categorifies the Jones polynomial
  - Khovanov-Rozansky homology categorifies the quantum group invariants associated with the fundamental rep of $SU(N)$
  - Knot Floer homology categorifies the Alexander polynomial
  - Webster proposed a general categorification of quantum group invariants (2013)

The categorification problem

- A longstanding problem is to find a construction of link homology that works uniformly for all Lie algebras
- One would also like to find a construction with a physical/geometric origin
- This problem is solved in recent work of Aganagic (2020, 2021) using homological mirror symmetry
- The solution is based on insights from various other authors: Webster (2013), Auroux (2010), Seidel-Smith (2008), Ozsvath-Szabo (2004, 2008), Rasmussen (2003), Gaiotto-Moore-Witten (2015)...

Outline

1. Floer theory and the A-model
2. Homological link invariants
3. An algebraic approach to solving the theory
   - $\mathfrak{gl}_{1|1}$ examples
   - $\mathfrak{su}_2$ examples
4. Other Lie (super)algebras
5. Topological invariance

The Floer complex

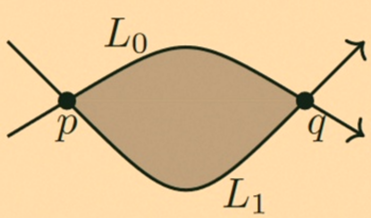
- Let (Y, ω) be a $2n$-dimensional symplectic manifold
- A Lagrangian submanifold L is an n-dimensional submanifold of M such that $\omega|_L = 0$
- Given two Lagrangians L_0 and L_1, one can associate a chain complex $CF(L_0, L_1)$ freely generated by the intersection points of L_0 with L_1 equipped with a differential Q
- Originally defined for compact L, but we will see later there is a generalization for noncompact L

The Floer differential

- Counts certain pseudo-holomorphic maps y from a disk to Y:

$$Qp = \sum_{\substack{q \in L_0 \cap L_1 \\ M(y)=1}} \#\mathcal{M}(p,q,y)\, q$$

- Example of a map that contributes to Q:



Floer homology

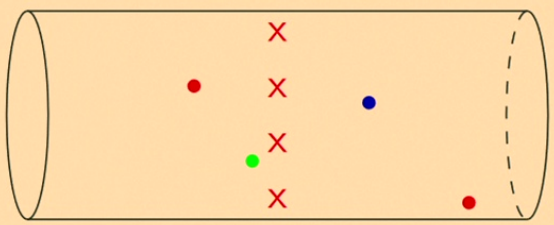
- Simply the homology of the Floer complex:

$$HF(L_0, L_1) = H^*(CF(L_0, L_1), Q)$$

- Invariant under Hamiltonian isotopy of L_0 or L_1

The A-model

- Let $Y = \bigotimes_{a=1}^{rk\mathfrak{g}} Sym^{d_a}\mathcal{A}$ where $\mathcal{A}$ is an infinite cylinder
  - $\mathfrak{g}$ is a Lie algebra
  - $d_a \in \mathbb{Z}^{\geq 0}$ are parameters
- One can think of Y as the configuration space of colored points on a (punctured) infinite cylinder
- Example:



The potential

- Equip Y with a potential W
- Let $y_{\alpha,a}$ parametrize Y with a corresponding to $Sym^{d_a}\mathcal{A}$ and $\alpha = 1, \ldots, d_a$ labelling the different copies of $\mathcal{A}$

$$W = \lambda_0 W^0 + \sum_{a=1}^{rk\mathfrak{g}} \lambda_a W^a$$

$$W^a = \sum_{\alpha} \ln y_{\alpha,a}$$

$$W^0 = \sum_{a=1}^{rk\mathfrak{g}} \ln f_a(y)$$

The potential

$$W^0 = \sum_{a=1}^{rk\mathfrak{g}} \ln f_a(y)$$

$$f_a(y) = \prod_{\alpha=1}^{d_a} \frac{\prod_i (1 - a_i/y_{\alpha,a})^{\langle e_a, \mu_i \rangle}}{\prod_{(b,\beta)\neq(a,\alpha)} (1 - y_{\beta,b}/y_{\alpha,a})^{\langle e_a, e_b \rangle/2}}$$

- a_i are special marked points on the cylinder ("punctures")
- μ_i is the highest weight of the representation coloring the corresponding puncture
- e_a are the roots of $\mathfrak{g}$

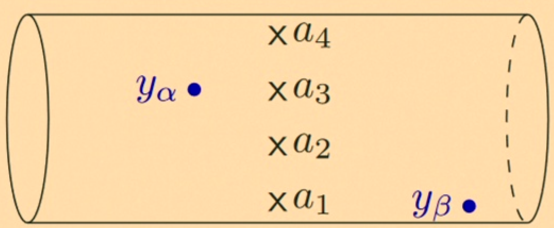
The potential: an example

Consider $\mathfrak{g} = \mathfrak{su}_2$ with 4 punctures labeled by w_1 and 2 marked points

In this case, $Y = Sym^2\mathcal{A}$ because $rk\,\mathfrak{su}_2 = 1$ and we chose $d_1 = 2$

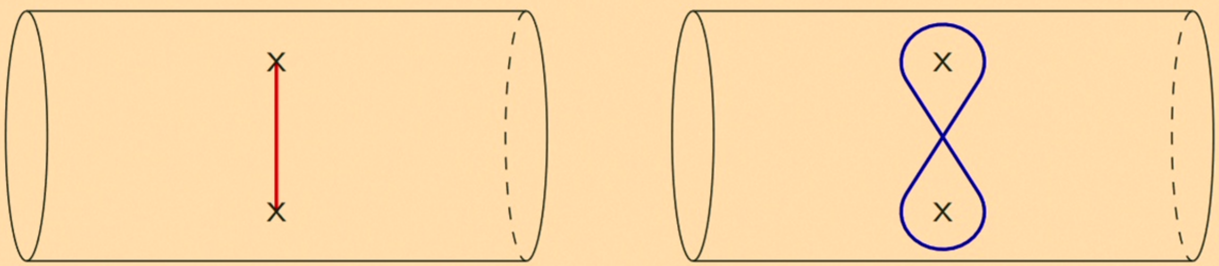
$$W(y) = \lambda_0 \ln f(y) + \lambda_1 \sum_{\alpha=1}^{2} \ln y_\alpha$$

$$f(y) = \prod_{\alpha=1}^{2} \frac{\prod_{i=1}^{4}(1 - a_i/y_\alpha)^{1/2}}{\prod_{\beta<\alpha}(1 - y_\beta/y_\alpha)}$$



A-branes

- A-branes are supported on Lagrangian submanifolds of Y
- They can be described as products of either
  - one-dimensional curves between a pair of punctures
  - closed one-dimensional curves (figure eights or ovals) encircling pairs of punctures
- Examples:



Morphisms between branes

- Morphisms between branes are defined by Floer theory

$$Hom^{*,*}_{\mathscr{D}_Y}(L_0, L_1) = HF^{*,*}(L_0, L_1)$$

- Two gradings:
  - the equivariant grading $\vec{J}$
  - the Maslov/homological grading M

The gradings: Maslov and equivariant

- Equivariant grading comes from non-singlevaluedness of W:

$$J^i(y) = -\frac{1}{2\pi i}\oint_{\partial D} y^* dW^i$$

- Maslov grading comes from the change of phase of the top holomorphic form on Y:

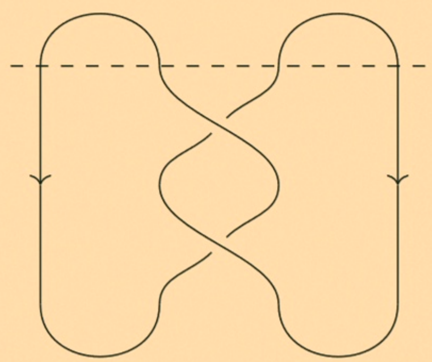
$$\Omega = \bigwedge_{a=1}^{rk\mathfrak{g}} \bigwedge_{\alpha=1}^{d_a} \frac{dy_{\alpha,a}}{y_{\alpha,a}}$$

$$\Omega = |\Omega| e^{i\varphi}$$

$$M(y) = \frac{1}{\pi}\oint_{\partial D} y^* d\varphi$$

Translating a link to A-branes

Start with a presentation of a link as a braid closure:



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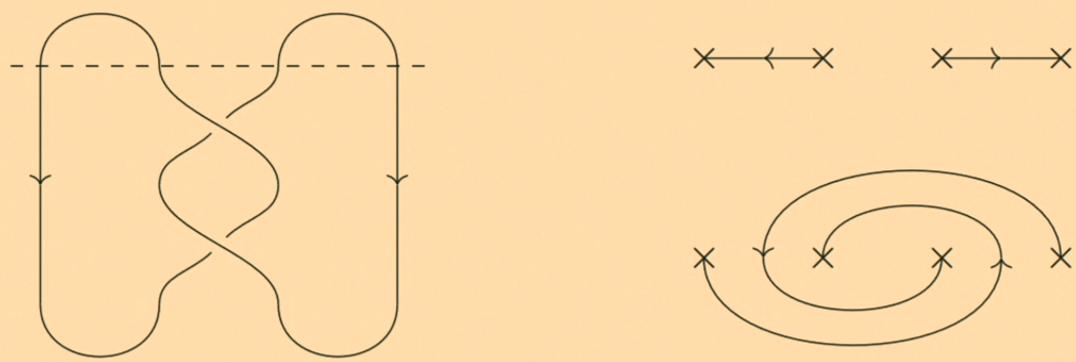
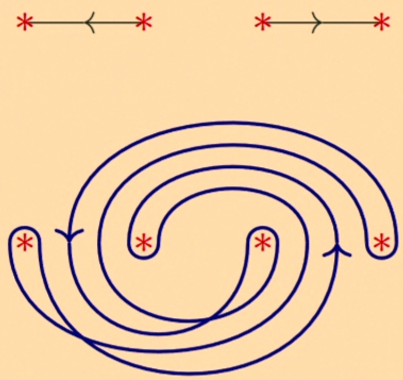
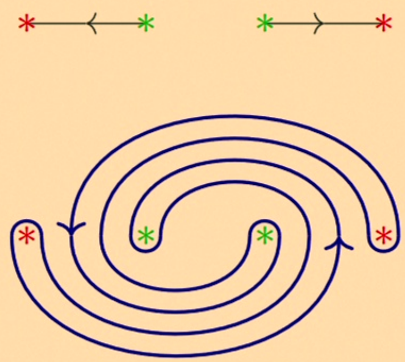


Translating a link to A-branes

- $\mathfrak{su}_2$



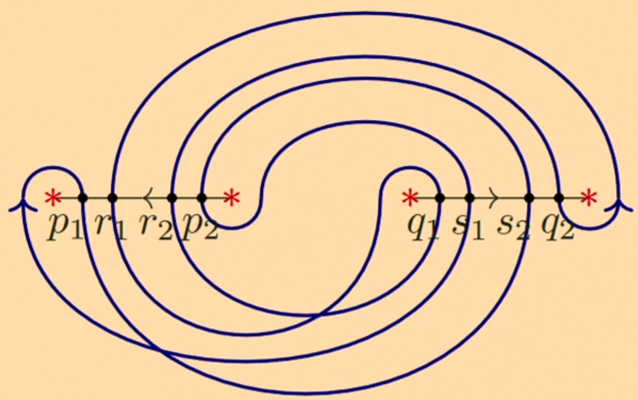
- $\mathfrak{gl}_{1|1}$



Homological link invariants

Theorem
$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$ is a link invariant.

Example:

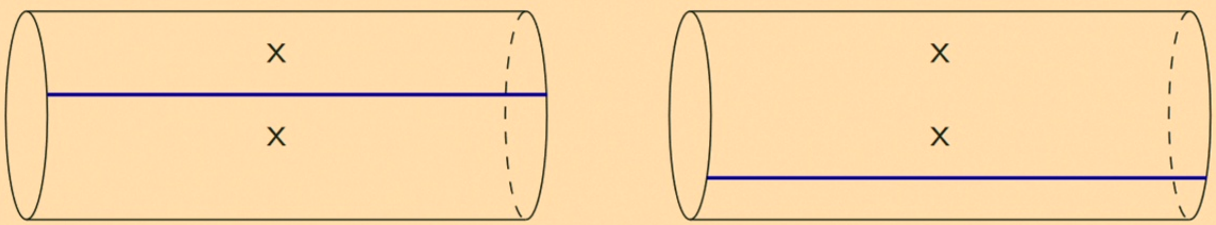


$CF(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$ is spanned by 8 points: $p_i q_j$, $r_i s_j$ for $i, j \in \{1, 2\}$

An algebraic approach to solving the theory

Thimbles

- Every A-brane can be written in terms of special branes known as Lefshetz thimbles
- Left Lefshetz thimbles $T_{\mathcal{C}}$ are downward gradient flows of $Re(W)$
- Each thimble passes through a critical point of W
- Left thimbles are products of real line Lagrangians

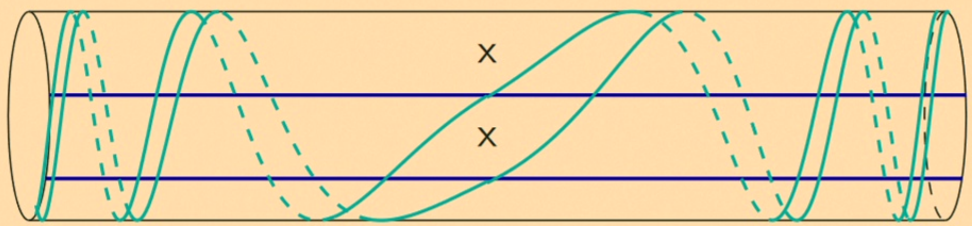


Morphisms between thimbles

- Morphisms between thimbles are given by intersection points after wrapping one thimble:

$$Hom_{\mathscr{D}_Y}(T_i, T_j) = HF^{0,0}(T_i^{\zeta}, T_j)$$

- Wrapped thimbles are gradient flows of $Re(e^{-i\zeta}W)$ for small, real ζ



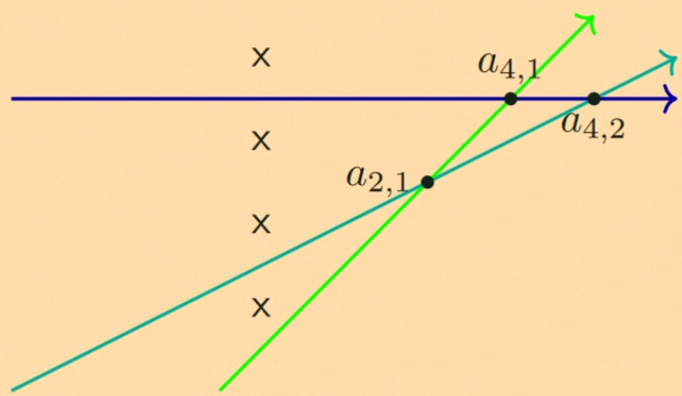
Composing morphisms

$$Hom_{\mathscr{D}_Y}(T_j, T_k) \otimes Hom_{\mathscr{D}_Y}(T_i, T_j) \to Hom_{\mathscr{D}_Y}(T_i, T_k)$$

$$HF^{0,0}(T_j^{\zeta}, T_k) \otimes HF^{0,0}(T_i^{2\zeta}, T_j^{\zeta}) \to HF^{0,0}(T_i^{2\zeta}, T_k)$$

Product of morphisms are given by counting holomorphic triangles

For example:



$$a_{4,2} \cdot a_{2,1} = a_{4,1}$$

The thimble algebra

- The thimble algebra can be described as a strand algebra on a cylinder
- This strand algebra is generated by certain "bits":

- Multiplication is given by stacking the diagrams, for example:

Composing morphisms

$$Hom_{\mathscr{D}_Y}(T_j, T_k) \otimes Hom_{\mathscr{D}_Y}(T_i, T_j) \to Hom_{\mathscr{D}_Y}(T_i, T_k)$$

$$HF^{0,0}(T_j^{\zeta}, T_k) \otimes HF^{0,0}(T_i^{2\zeta}, T_j^{\zeta}) \to HF^{0,0}(T_i^{2\zeta}, T_k)$$

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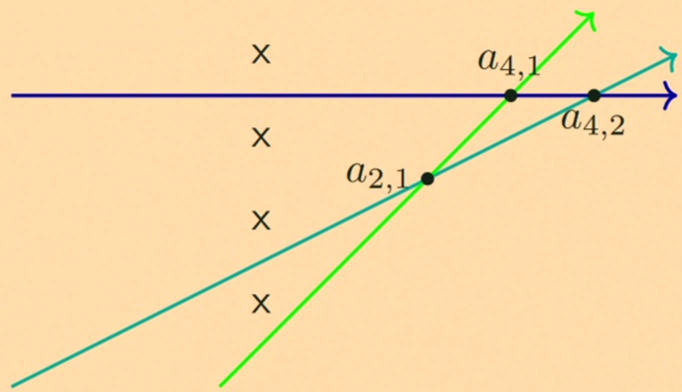


$$a_{4,2} \cdot a_{2,1} = a_{4,1}$$

Finding resolutions

- Resolutions of one-dimensional branes can be written down from geometry
- Resolutions of branes on $Y \backslash \Delta$ (where Δ is the diagonal in $Y = \bigotimes_{a=1}^{\mathrm{rk}\mathfrak{g}} Sym^{d_a}(A)$) are constructed from products of one-dimensional branes
- We then solve for the unique deformation extending the resolution from $Y \backslash \Delta$ to Y
- Passing from $Y \backslash \Delta$ to Y does not change the set of thimbles, only their algebra

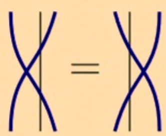
An algebraic approach to solving the theory

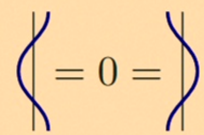
$\mathfrak{gl}_{1|1}$ examples

Relations for $\mathfrak{gl}_{1|1}$

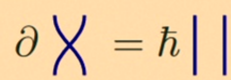
- There are two colors of punctures, so the "bits" are as follows:

- There are two relations:





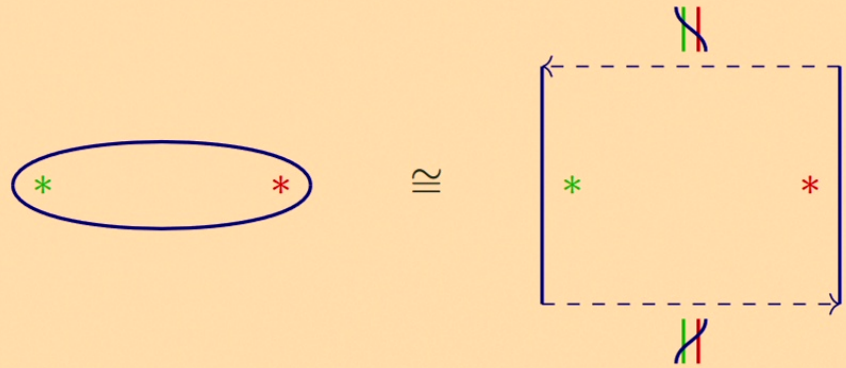
- There is also a differential:



The parameter $\hbar$

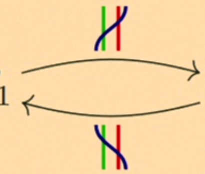
- $\hbar$ keeps track of the intersection with the diagonal Δ in $Sym^d(\mathcal{A})$
- It may be set to 1 by rescaling the algebra generators but it's useful to leave in for computational purposes
- Setting $\hbar = 0$ gives the algebra on $Sym^d(\mathcal{A})\backslash\Delta$

$\mathfrak{gl}_{1|1}$ cap



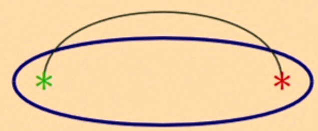
Projective resolution:

$$E_2 \quad \cong \quad T_1 \rightleftarrows T_3$$



$\mathfrak{gl}_{1|1}$ unknot

Naively, one would expect to recover $\widehat{HFK}$ of the unknot from the diagram

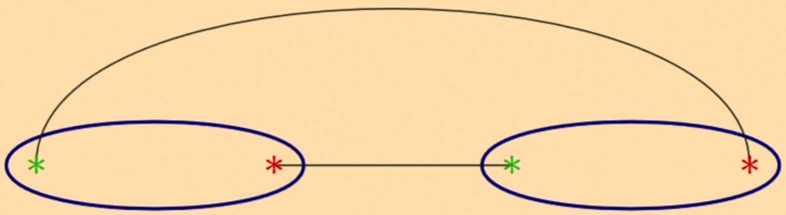


but it turns out that $Hom^{*,*}_{\mathscr{D}_Y}(E_2, I_2) = 0$.

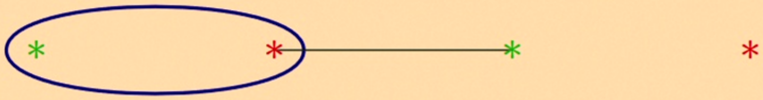
Instead, one must cut a strand to obtain a nonzero invariant.

$\mathfrak{gl}_{1|1}$ unknot

Another diagram for the unknot is



Then we can cut a strand to get



This second diagram gives

$$Hom^{*,*}_{\mathscr{D}_Y}(E_2, I_3) = \mathbb{C}$$

Twisted complexes

- Twisted complexes are necessary because not all maps have Maslov degree 0
- In an ordinary chain complex, the differential is made of maps

$$\delta_{k,k-1} : V_k \to V_{k-1}$$

- In a twisted complex, the differential has maps

$$\delta_{k,\ell} : V_k \to V_\ell$$

where $M(\delta_{k,\ell}) = \ell + 1 - k$

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Twisted complexes

- The condition that the differential square to 0 is

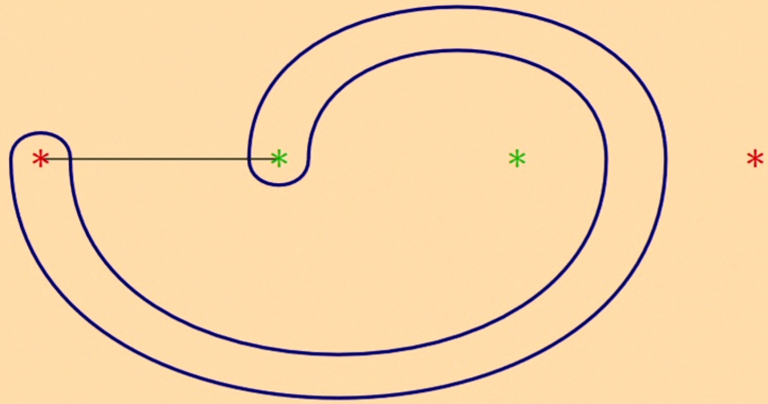
$$\sum_k m_k(\delta, \ldots, \delta) = 0$$

where $m_k : A^{\otimes k} \to A$

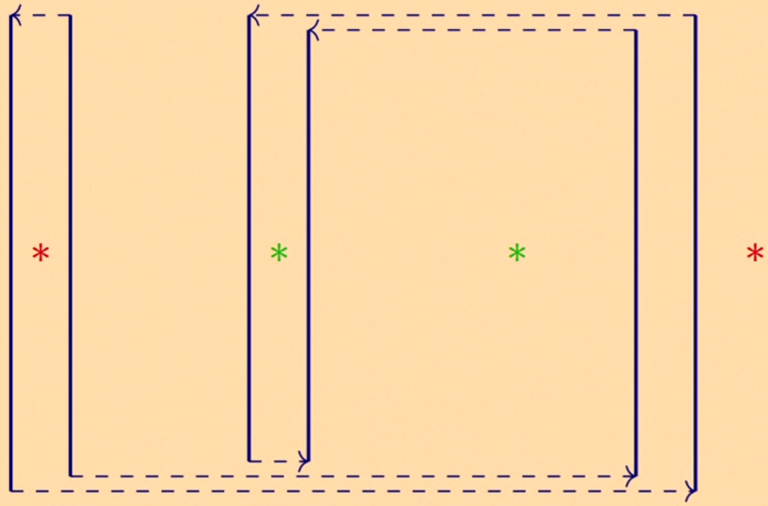
- For $\mathfrak{g} = \mathfrak{gl}_{1|1}$, the algebra is a differential graded algebra with differential $\partial \sim m_1$, product m_2, and all higher products vanishing
- In this case, the condition becomes

$$\partial\delta + \delta^2 = 0$$

$\mathfrak{gl}_{1|1}$ Hopf link



$\mathfrak{gl}_{1|1}$ Hopf link resolution



$$T_1 \longrightarrow T_4[-1] \longrightarrow T_2[2]\{-2\} \longrightarrow T_3[3]\{-2\} \longleftarrow T_4[4]\{-1\} \longleftarrow T_2[1]\{-1\} \longrightarrow .$$

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E, I_2) = \mathbb{C}[2]\{-2\} \oplus \mathbb{C}[1]\{-1\}$$

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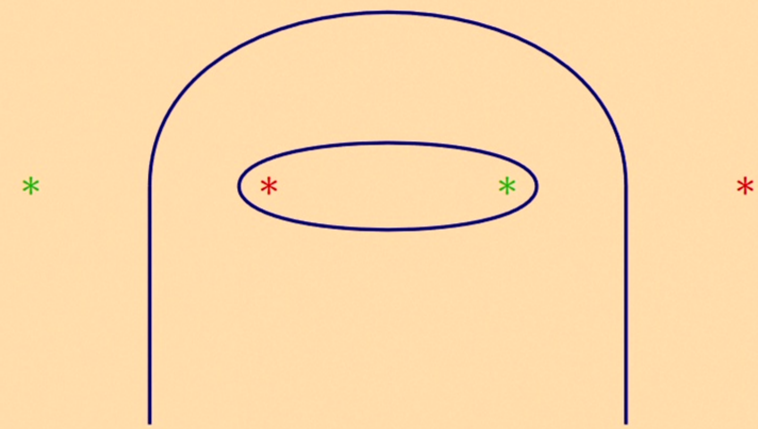
Examples with $d > 1$

- In the previous examples, there was only one oval brane, meaning ∂ acted trivially on the δ
- If we want to describe knots with bridge number greater than 2, we'll need more than one oval brane
- Those examples are quite complicated (the simplest having 8 crossings), so I will do a short toy example here:



$d = 2$ toy example

Then, we consider the product of L_1 with each of the other two thimbles: T_2 and T_4.

In doing so, we must keep track of the horizontal positions of T-branes relative to each other and to the punctures. We find the complexes

$$L_1 \times T_2 \cong T_{22} \rightleftarrows T_{24}$$

$$L_1 \times T_4 \cong T_{24} \rightleftarrows T_{44}$$

$d = 2$ toy example

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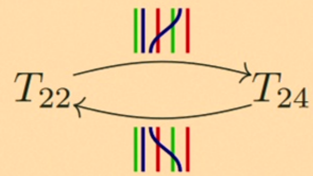
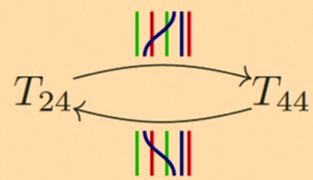
$L_1 \times T_2 \quad \cong$



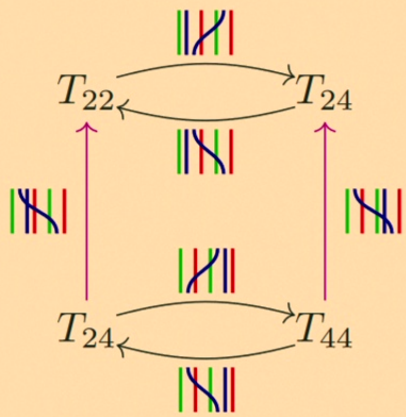
$L_1 \times T_4 \quad \cong$



$d = 2$ toy example

Next we glue $L_1 \times T_2$ to $L_1 \times T_4$, starting with their direct sum of the branes.

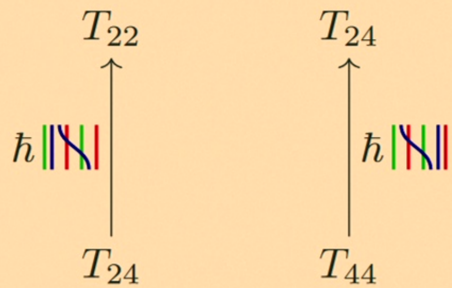
After turning on the map from L_2, we find



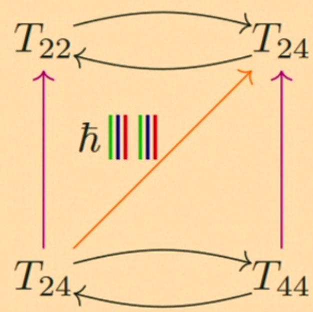
This is the geometric differential δ_0

$d = 2$ toy example

The A_∞ differential ∂ acts by uncrossing strands in δ_0. The resulting maps $\partial\delta_0$ are

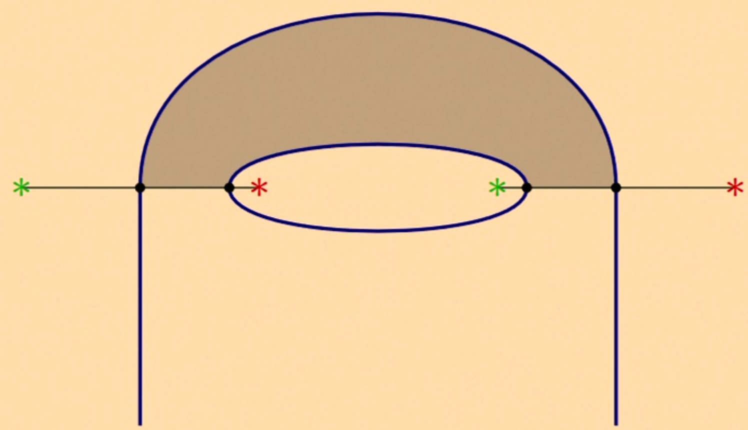


So to satisfy $\partial\delta + \delta^2 = 0$, we add an additional map δ_1 to δ_0, so that $\delta = \delta_0 + \delta_1$ satisfies the condition. The resulting complex is the correct projection resolution for our toy example:



$d = 2$ toy example

This additional map we turned on corresponds to the geometric disk:

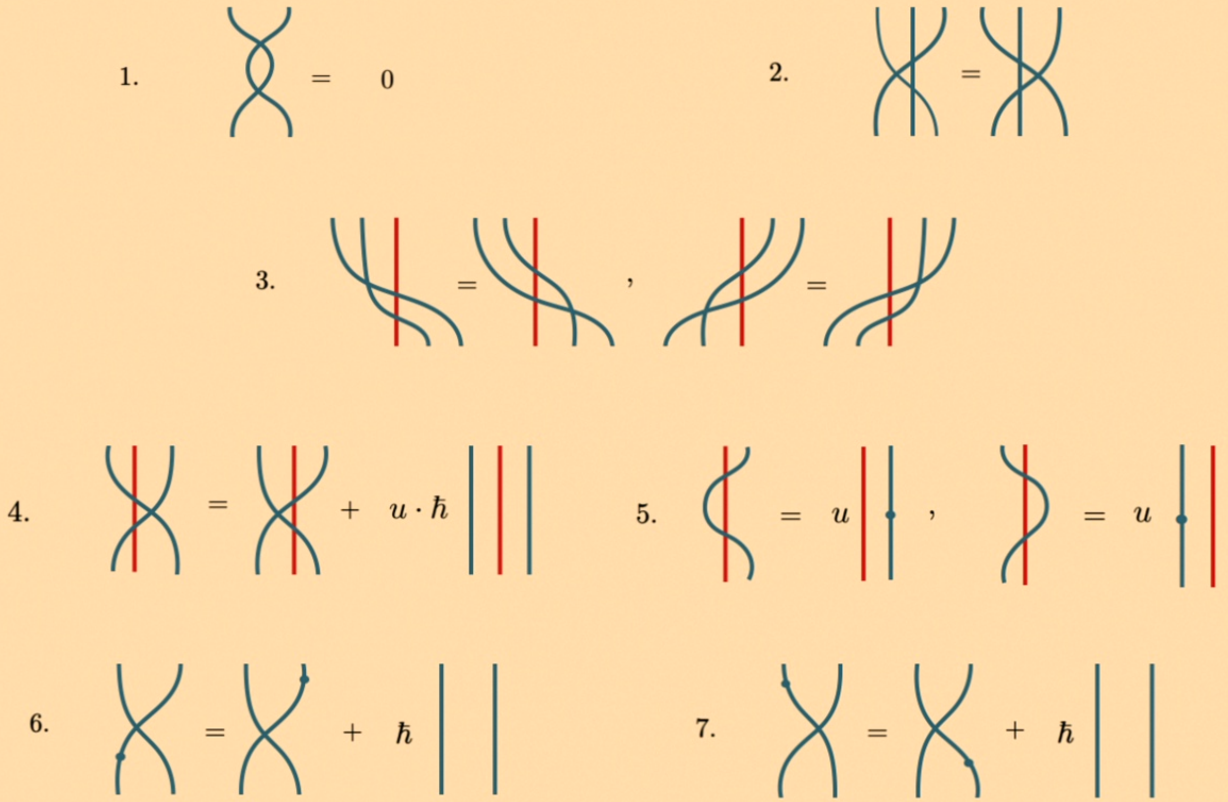


It is a general feature of the theory that the non-geometric terms in the complex corresponds to disks in $\widehat{HFK}$.

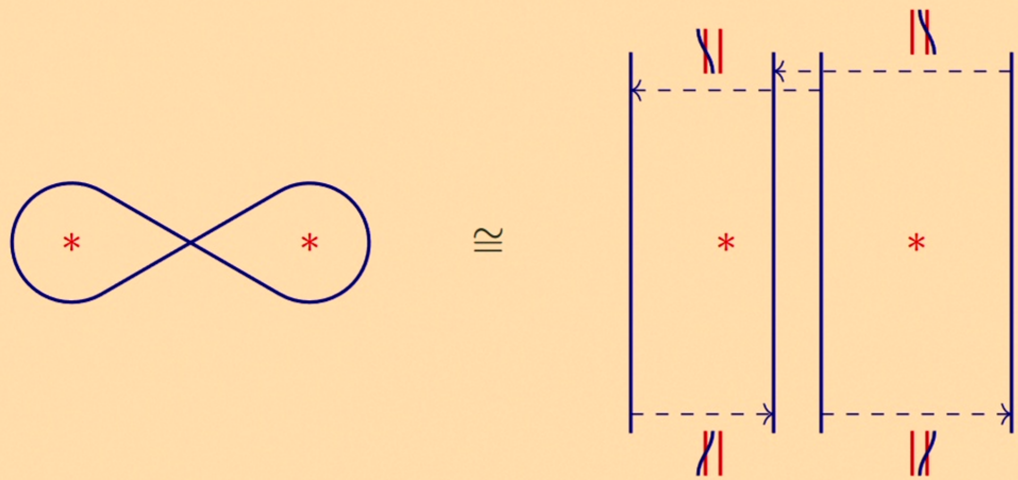
Adding dots

- For all Lie (super)algebras besides $\mathfrak{gl}_{1|1}$, Y is actually a subspace of a larger space $\mathcal{Y}$, which fibers over Y with $(\mathbb{C}^\times)^d$ fibers
- We want to consider morphisms of thimbles in $\mathcal{Y}$ rather than in Y
- We can account for the total space $\mathcal{Y}$ by adding dots to the strand algebra
- For Lie algebras, the strand algebra is known as the KLRW algebra and has been studied from a B-side perspective by Webster (2019)
- The calculation of the strand algebra on the A-side will appear in upcoming work of Aganagic-Danilenko-Li-Shende-Zhou

Relations for $\mathfrak{su}_2$



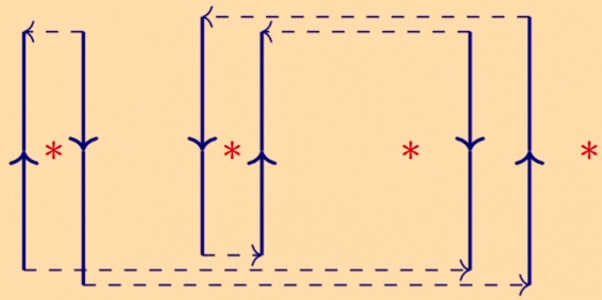
$\mathfrak{su}_2$ cup



Resolution:

$$E_2 \cong T_2\{-1\} \longrightarrow \begin{matrix} T_3\{-1\} \\ \oplus \\ T_1 \end{matrix} \longrightarrow T_2$$

$\mathfrak{su}_2$ Hopf link: $\mathscr{B}E_1$ resolution

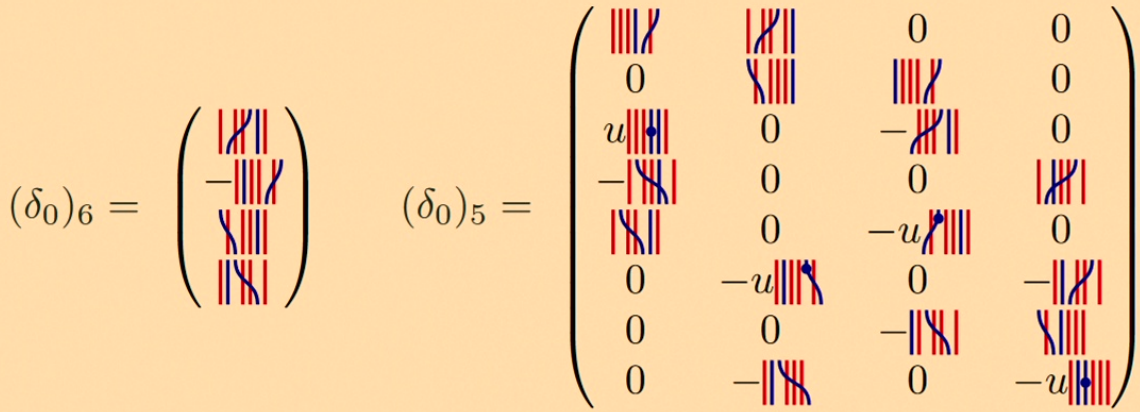


$$T_1[2]\{-1\} \xrightarrow{-} T_4[1]\{-1\} \xrightarrow{-} T_3 \longleftarrow T_2[1] \longleftarrow T_4[2]\{-2\} \longleftarrow T_2[3]\{-2\} \longrightarrow .$$

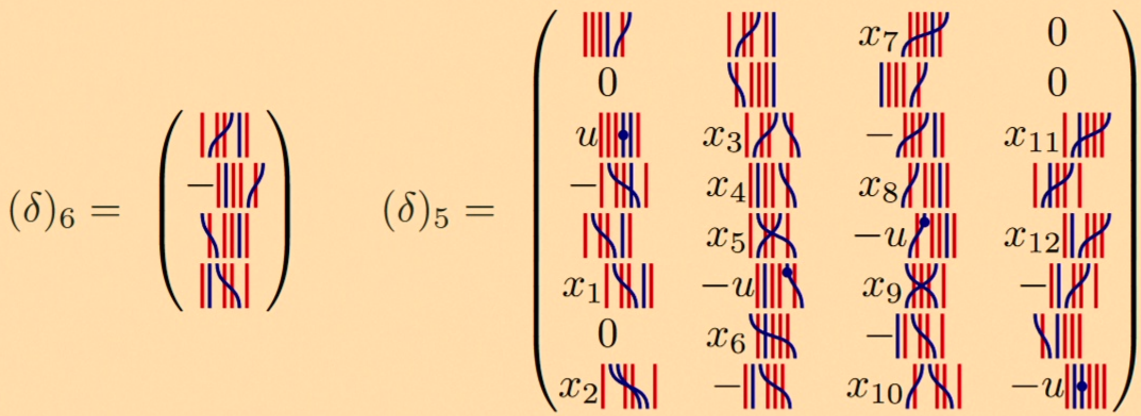
$\mathfrak{su}_2$ Hopf link: geometric complex

$$
\begin{array}{ccccccccccccc}
T_{12}[3] & \xrightarrow{-} & T_{13}[2] & \longleftarrow & T_{14}[3]\{-1\} & \longleftarrow & T_{12}[4]\{-1\} & \xleftarrow{-} & T_{14}[5]\{-3\} & \longrightarrow & T_{15}[4]\{-3\} & \longrightarrow & \cdot \\
{\scriptstyle -}\downarrow & & {\scriptstyle -}\downarrow & & {\scriptstyle -}\downarrow & & {\scriptstyle -}\downarrow & & {\scriptstyle -}\downarrow & & {\scriptstyle -}\downarrow & & \\
T_{24}[2]\{-1\} & \longrightarrow & T_{34}[1]\{-1\} & \xleftarrow{-} & T_{44}[2]\{-2\} & \xleftarrow{-} & T_{24}[3]\{-2\} & \longleftarrow & T_{44}[4]\{-3\} & \xrightarrow{-} & T_{45}[3]\{-3\} & \xrightarrow{-} & \cdot \\
{\scriptstyle -}\downarrow & & {\scriptstyle -}\downarrow & & {\scriptstyle -}\downarrow & & {\scriptstyle -}\downarrow & & {\scriptstyle -}\downarrow & & {\scriptstyle -}\downarrow & & \\
T_{23}[1] & \xrightarrow{-} & T_{33}\{-1\} & \longleftarrow & T_{34}[1]\{-2\} & \longleftarrow & T_{23}[2]\{-1\} & \xleftarrow{-} & T_{34}[3]\{-2\} & \longrightarrow & T_{35}[2]\{-2\} & \longrightarrow & \cdot \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
T_{22}[2] & \longrightarrow & T_{23}[1]\{-1\} & \xleftarrow{-} & T_{24}[2]\{-2\} & \xleftarrow{-} & T_{22}[3]\{-1\} & \longleftarrow & T_{24}[4]\{-2\} & \xrightarrow{-} & T_{25}[3]\{-2\} & \xrightarrow{-} & \cdot \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
T_{24}[3]\{-2\} & \xrightarrow{-} & T_{34}[2]\{-2\} & \longleftarrow & T_{44}[3]\{-3\} & \longleftarrow & T_{24}[4]\{-3\} & \xleftarrow{-} & T_{44}[5]\{-4\} & \longrightarrow & T_{45}[4]\{-4\} & \longrightarrow & \cdot \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
T_{22}[4]\{-1\} & \longrightarrow & T_{23}[3]\{-1\} & \xleftarrow{-} & T_{24}[4]\{-2\} & \xleftarrow{-} & T_{22}[5]\{-2\} & \longleftarrow & T_{24}[6]\{-4\} & \xrightarrow{-} & T_{25}[5]\{-4\} & \xrightarrow{-} & \cdot \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot
\end{array}
$$

$\mathfrak{su}_2$ Hopf link: dotted corrections

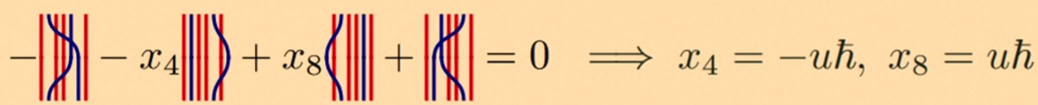


$\mathfrak{su}_2$ Hopf link: possible $\hbar$ corrections



We use $(\delta)_5 \cdot (\delta)_6 = 0$ to solve for the coefficients x_i

For example, the 4th row of $(\delta)_5 \cdot (\delta)_6$ gives



$\mathfrak{su}_2$ Hopf link: taking homology

Taking only the terms T_{24} from the complex gives $CF^{*,*}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$:

$$\begin{matrix}\mathbb{C}\{-2\}\\ \mathbb{C}\{-1\}\end{matrix} \xrightarrow{\begin{pmatrix} u\hbar & 0 \\ u\hbar & 0\end{pmatrix}} \begin{matrix}\mathbb{C}\{-2\}\\ \mathbb{C}\{-2\}\end{matrix} \xrightarrow{\begin{pmatrix} 0 & 0 \\ u\hbar & -u\hbar \\ u\hbar & -u\hbar\end{pmatrix}} \begin{matrix}\mathbb{C}\{-3\}\\ \mathbb{C}\{-2\}\\ \mathbb{C}\{-2\}\end{matrix} \to 0 \to \mathbb{C}\{-4\}$$

Taking the homology gives the $\mathfrak{su}_2$ homology of the Hopf link:

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \mathbb{C}[4]\{-1\} \oplus \mathbb{C}[2]\{-3\} \oplus \mathbb{C}[2]\{-2\} \oplus \mathbb{C}\{-4\}$$

Solving order by order in $\hbar$

- At order zero, we just have

$$\delta_0^2 = O(\hbar)$$

- At order one, we have

$$\hbar(\delta_0\delta_1 + \delta_1\delta_0) + \delta_0^2 = O(\hbar^2)$$

- At order k, the equation has the form

$$\hbar^k(\delta_0\delta_k + \delta_k\delta_0) + D_k(\delta_0, \ldots, \delta_{k-1}) = O(\hbar^{k+1})$$

  where D_k is an operator that depends only on δ_ℓ for $\ell < k$
- We see that at each order, we have a linear equation for δ_k

Simple Lie algebras

- Choose a representation V of $\mathfrak{g}$ with highest weight μ
- $Y = \bigotimes_{a=1}^{rk\mathfrak{g}} Sym^{d_a}\mathcal{A}$ where d_a are chosen to satisfy

$$\mu + \mu^* = \sum_{a=1}^{rk\mathfrak{g}} d_a e_a$$

- Punctures are colored by V and V^* such that cups and caps connect dual pairs of punctures
- Cups are products of figure-eights colored by roots according to d_a
- Caps are intervals with colored dots given by d_a

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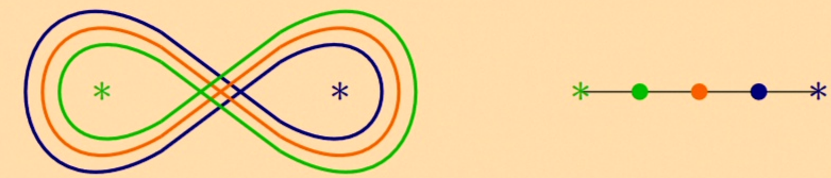
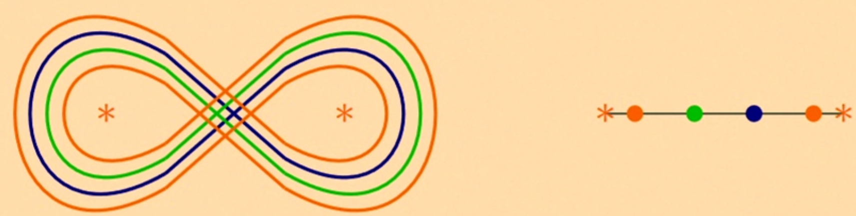
- Punctures are colored by V and V^* such that cups and caps connect dual pairs of punctures
- Cups are products of figure-eights colored by roots according to d_a
- Caps are intervals with colored dots given by d_a
- Ordering of the figure-eights and colored dots corresponds to the order one needs to substract roots e_a from μ to reach $-\mu^*$

Simple Lie algebras: examples

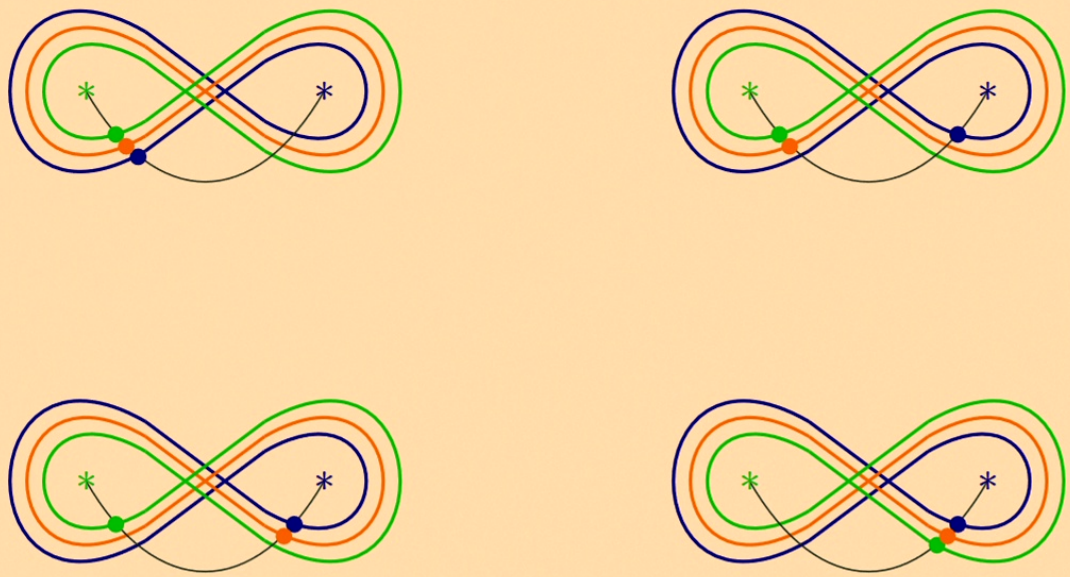
- $\mathfrak{su}_4$ with $\mu = w_1$



- $\mathfrak{su}_4$ with $\mu = w_2$



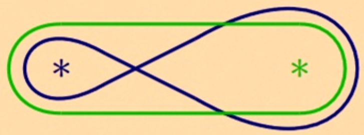
Unknot homology for $\mathfrak{su}_4$ with $\mu = w_1$



$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \mathbb{C}[6]\{-3\} \oplus \mathbb{C}[4]\{-2\} \oplus \mathbb{C}[2]\{-1\} \oplus \mathbb{C}$$

$\mathfrak{gl}_{\mathbf{m|n}}$

- Theory for $\mathfrak{gl}_{m|n}$ is similar to that of Lie algebras
- Main difference is that fermionic roots are associated to ovals rather than to figure-eights
- Example: cups and caps for $\mathfrak{gl}_{2|1}$

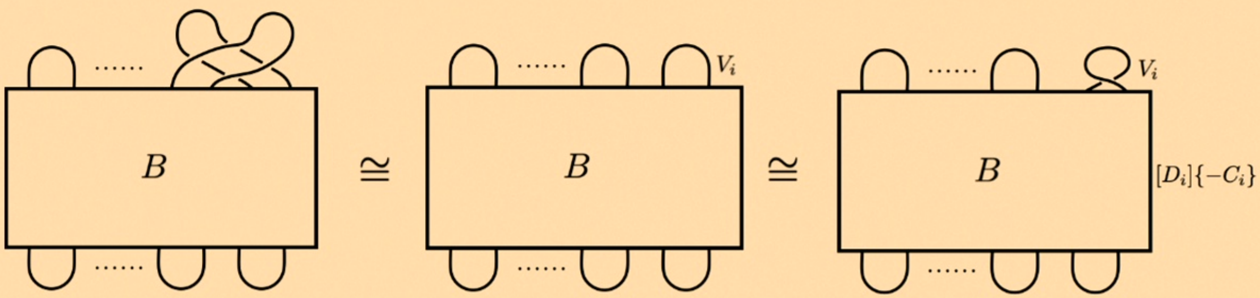


Computations for other Lie (super)algebras

- Computational methods developed for $\mathfrak{su}_2$ and $\mathfrak{gl}_{1|1}$ extend more or less directly to other cases
- Resolutions are more complicated, requiring more thimbles
- Algebra for simple Lie algebras never has a differential
- Algebra for $\mathfrak{gl}_{m|n}$ has a differential and a nontrivial product

Markov moves

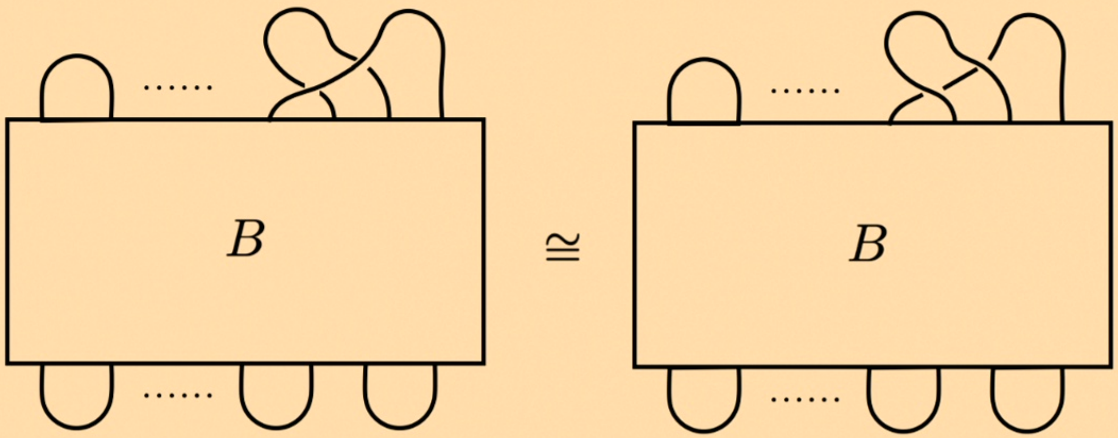
To prove that $Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$ are link invariants, we must prove they satisfy four relations, which are the analogues of Markov moves for plat closures



The first two relations, shown above, are manifest by construction

Markov moves

The next relation is the analogue of a Markov I move:



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Sketch of proof

In terms of cups and caps, the two relations we need to prove are

and

Sketch of proof

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and

Both are these of relations follow from

Sketch of proof

- Start with resolutions $L_1 \cong T_1 \times E_2$ and $L_2 \cong E_2 \times T_3\{-1\}$
- Explicitly show L_1 and L_2 are homotopy equivalent:
  1. Construct chain maps $f : L_1 \to L_2$ and $f' : L_2 \to L_1$
  2. Show $f' \cdot f \sim id_{L_1}$ and $f \cdot f' \sim id_{L_2}$
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- Same method works for $\mathfrak{gl}_{1|1}$ using definitions adjusted for twisted complexes
- Method should work for any Lie algebra or superalgebra limited by computational feasibility

Summary

- We reviewed Aganagic's A-model formulation of link homology and extended it to include Lie superalgebras
- We gave a method for computing link homology associated with a (minuscule) representation any Lie algebra or superalgebra
- We explicitly showed how this method works for fundamentel rep of $\mathfrak{su}_2$ and $\mathfrak{gl}_{1|1}$
- We gave a proof of topological invariance for the $\mathfrak{su}_2$ and $\mathfrak{gl}_{1|1}$ theories

Thank you