Title: Echoes and Entropy of Quantum Black Holes

Speakers: Naritaka Oshita

Collection: Quantum Spacetime in the Cosmos: From Conception to Reality

Date: May 11, 2023 - 9:00 AM

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Abstract: It has been proposed that quantum-gravitational effects may change the near-horizon structure of black holes, e.g. firewalls or ultra-compact objects mimicking black holes. Also, a Lorentz-violating theory as a candidate of quantum gravity, e.g. the Horava-Lifshitz theory, changes the causal structure of black holes due to the superluminal propagation of excited modes. The late-time part of the gravitational wave ringdown from a black hole is significantly affected by those effects, and the emission of gravitational wave echoes may be induced. The black hole quasi-normal (QN) modes are affected by the change of the horizon structure, which results in the drastic modification of the late-time signal of the gravitational wave. In this talk, I will discuss how the gravitational wave echo can be modeled and how the echo model is reasonable from an entropic point of view by counting QN modes to estimate the black hole entropy.

Zoom Link: https://pitp.zoom.us/j/99946149565?pwd=M2puMy9nSEtBZTg1MnRmSllHeUE0UT09

Quantum Spacetime in the Cosmos: Conception to Reality on May 11, 2023 @ Perimeter Inst. (virtual)

Echoes and Entropy of Quantum Black Holes

NO, N. Afshordi arXiv: 2302.08964
NO, S. Noda, H. Motohashi arXiv: 2205.15342
NO, D. Tsuna, N. Afshordi arXiv: 2001.11642
J. Abedi, N. Afshordi, NO, Q. Wang arXiv: 2001.09553
NO, Q. Wang, N. Afshordi arXiv: 1905.00464
NO and N. Afshordi arXiv: 1807.10287

Naritaka Oshita

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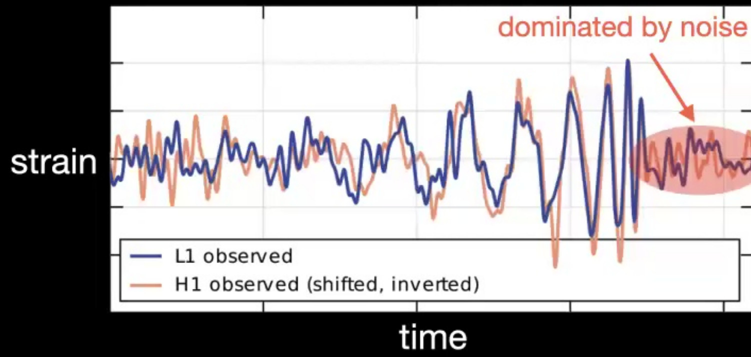
YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

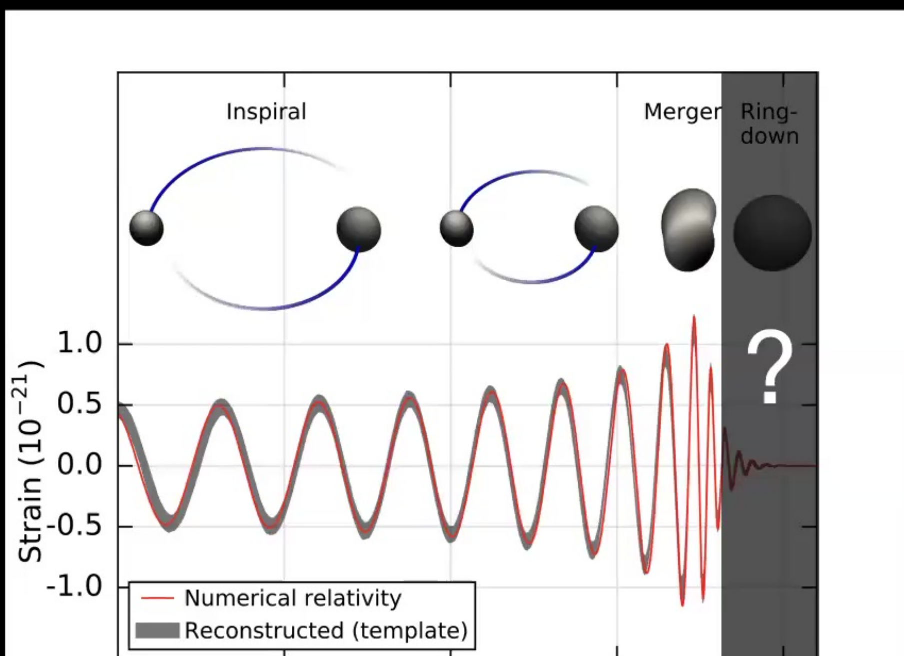
Hakubi Kyoto Univ.

RIKEN Interdisciplinary Theoretical and Mathematical Sciences Program

Is the remnant a BH?



GW150914
(the first detection of GWs by LIGO)



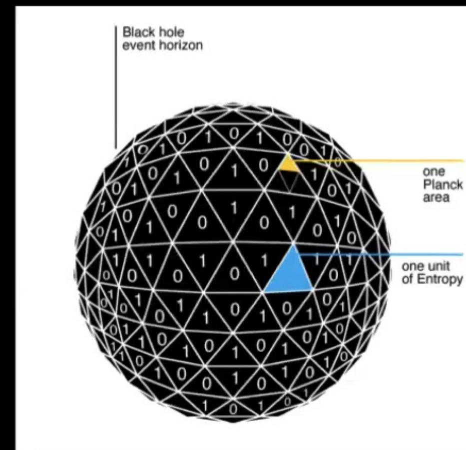
Ultra Compact Objects (UCOs) ?

- Wormholes Visser (1996)
- Gravastars Mazur, et al. (2002)
- Quantum BHs

firewalls	Almheiri, et al. (2014)
stretched horizons	Susskind, et al. (1993)
BH area quantization	Bekenstein (1974)
fuzzballs	Mathur, et al. (2002)

BH area quantization

$$(\text{entropy}) = \frac{(\text{area})}{4 \times (\text{Planck area})}$$

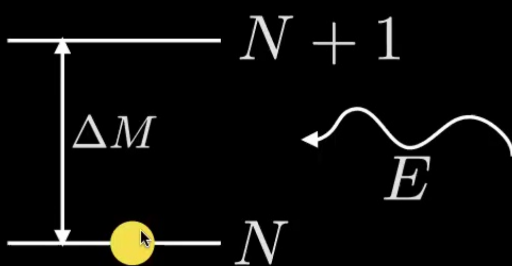


$$A = \alpha \ell_{\mathrm{Pl}}^2 N$$

const. integer

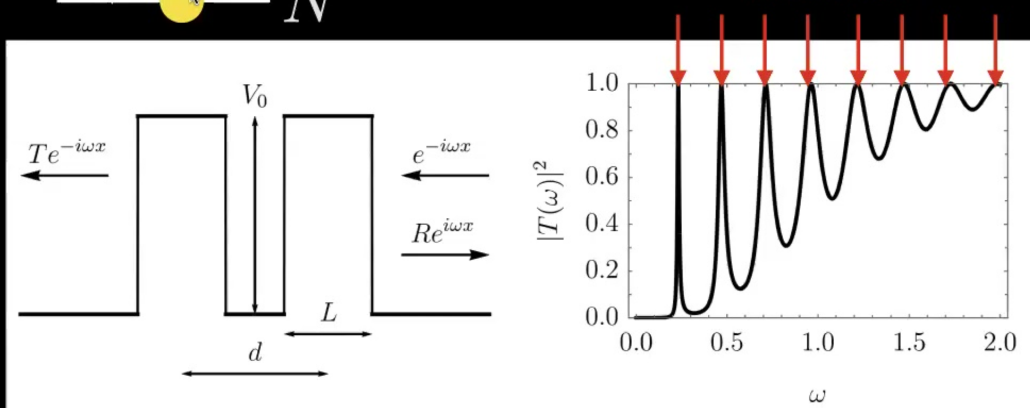
$$\Delta A \propto \Delta M$$

discretized mass



Cardoso, et al. (2019)

absorbed



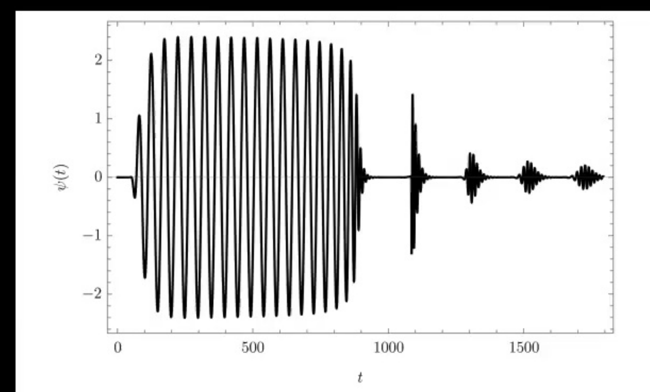
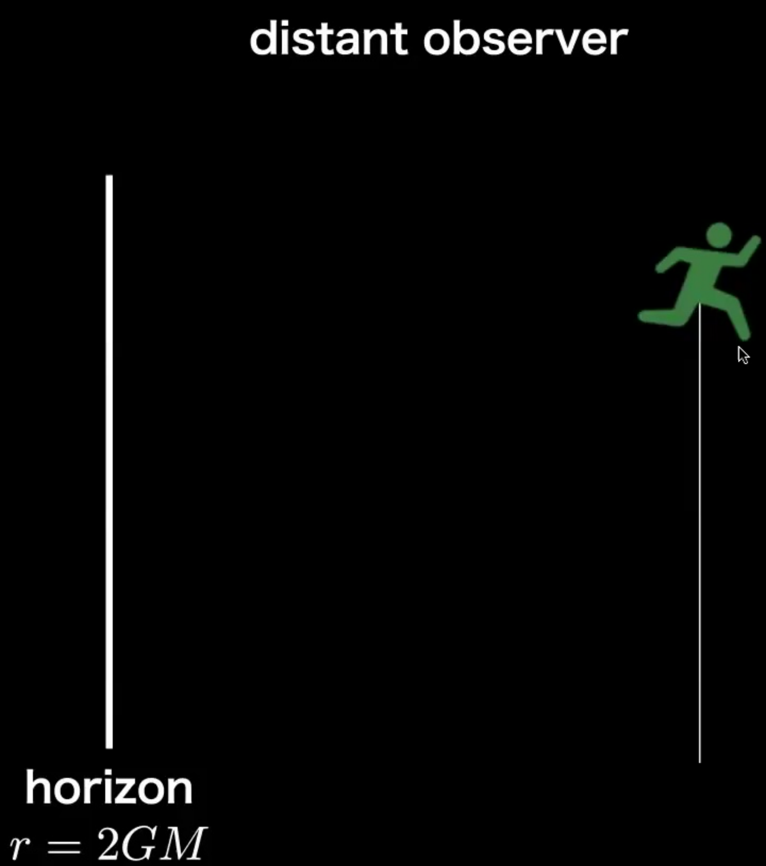


Figure 6. Inspiral, merger, and ringdown waveform of the numerical relativity simulation of an inspiral with mass ratio $M_1/M_2 = 10^3$, with Dirichlet boundary conditions near the horizon, and the first four of the resulting echoes.

Black hole complementarity

Susskind+ (1993)



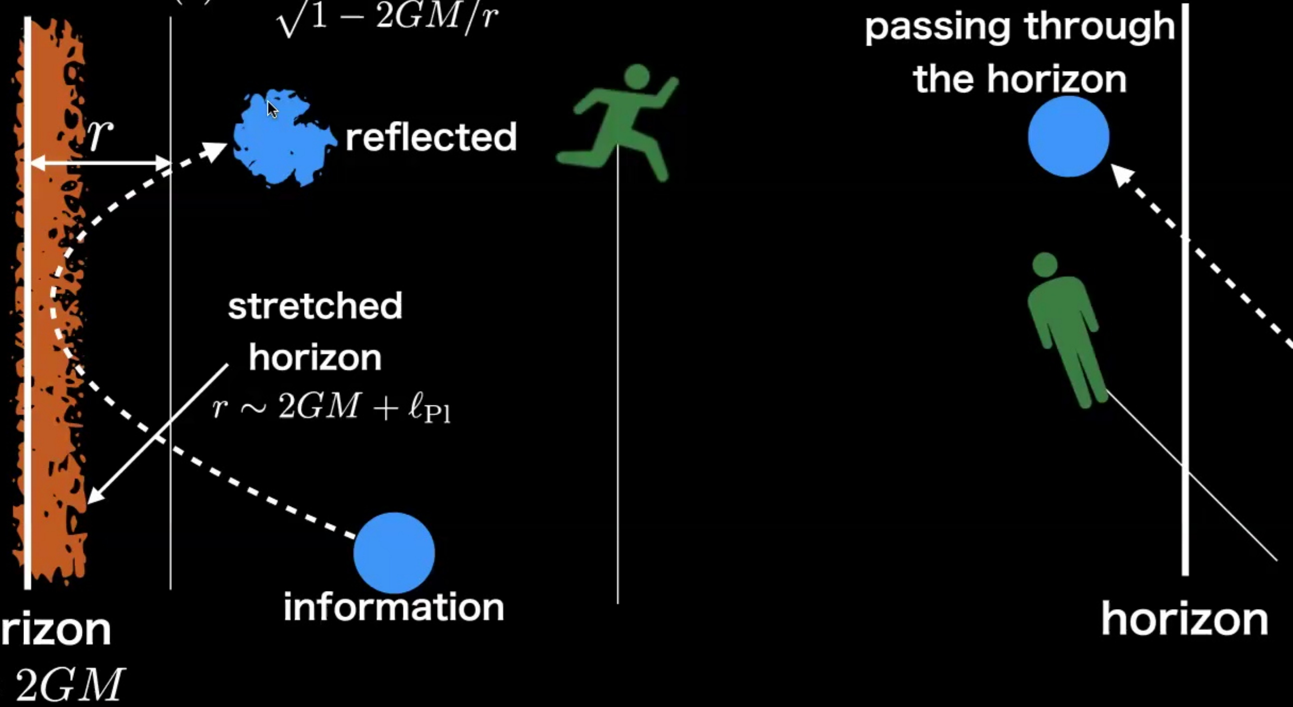
Black hole complementarity

$$T_H = \frac{1}{8\pi GM}$$

Susskind+ (1993)

distant observer **infalling observer**

$$T(r) = \frac{T_H}{\sqrt{1-2GM/r}}$$

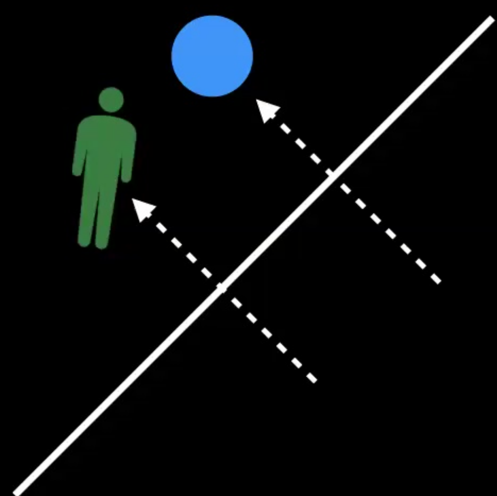


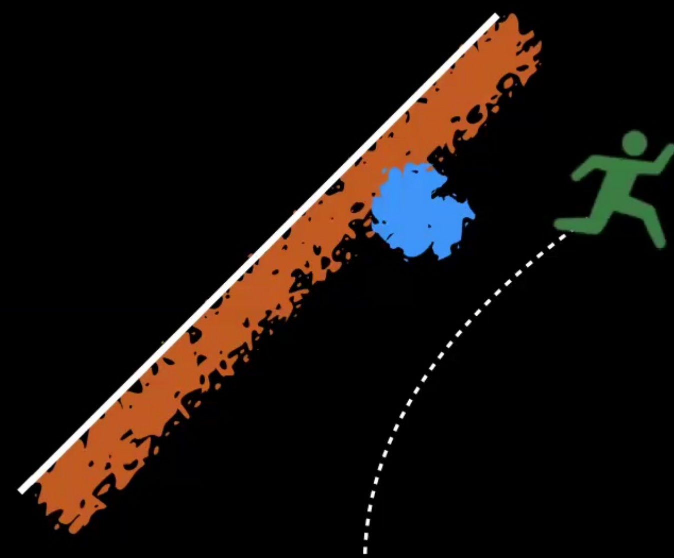
According to an infalling observer,

information causally disappears.

According to a distant observer,

information is dissipated due to the viscosity.





Modeling viscous membrane

NO and Afshordi (2019)

NO, Wang and Afshordi (2020)

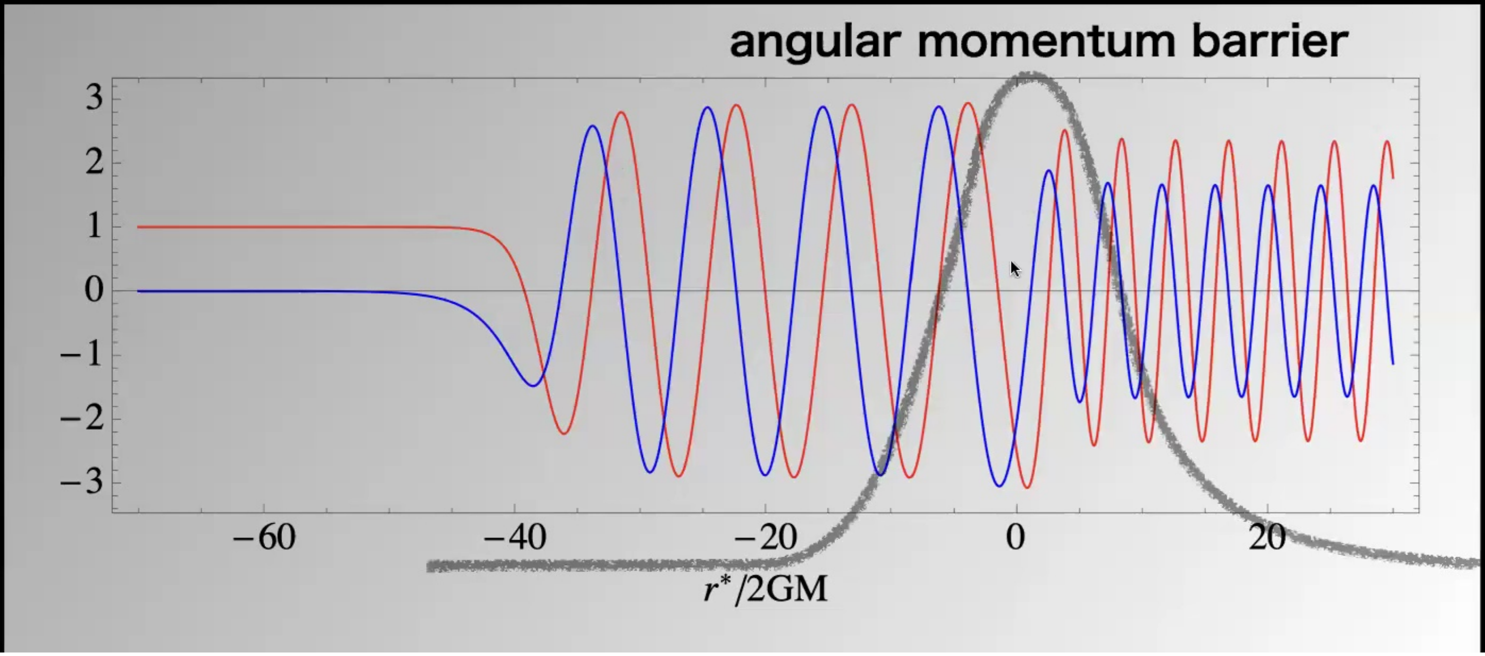
viscosity

blue-shifted frequency

classical general relativity

$$\left[-i\frac{\gamma\Omega}{E_{\rm Pl}}\frac{d^2}{dr^{*2}}+\frac{d^2}{dr^{*2}}+\omega^2-V_\ell(r^*)\right]\psi_\omega=0$$

Planck energy



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viscosity — blue-shifted frequency — classical general relativity

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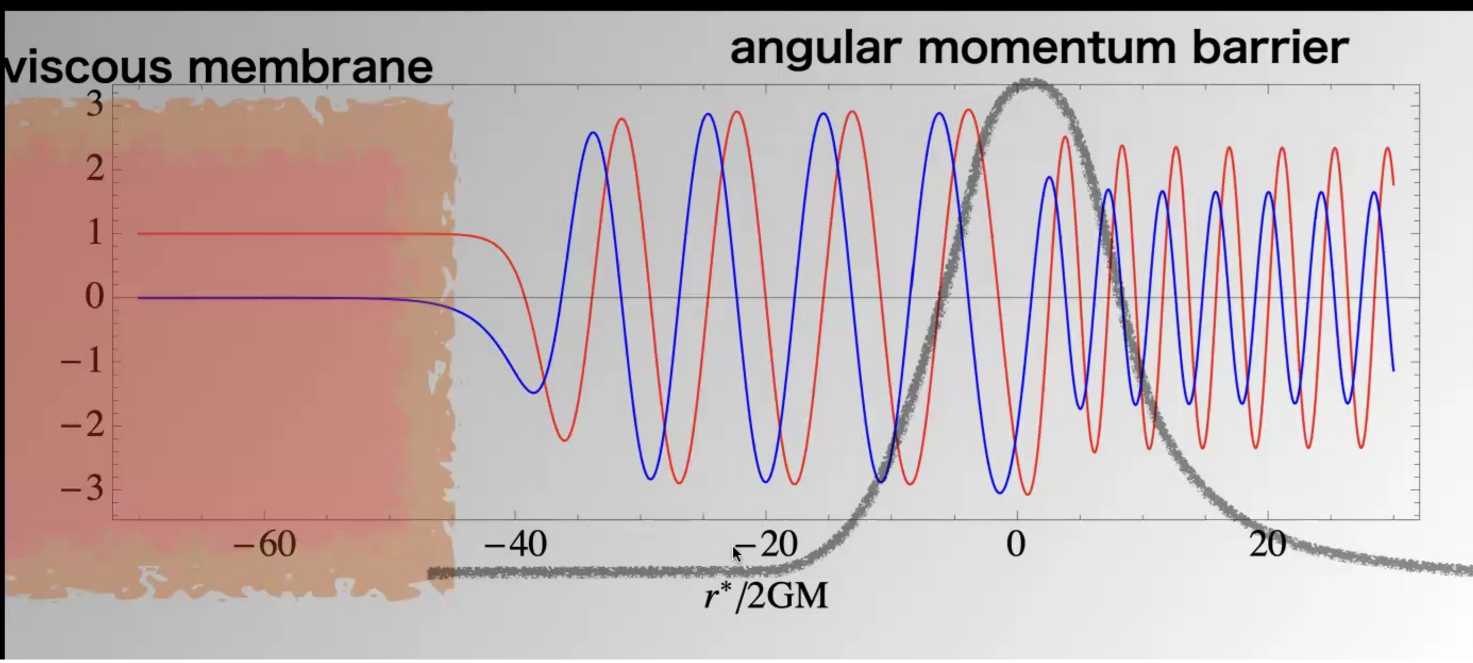

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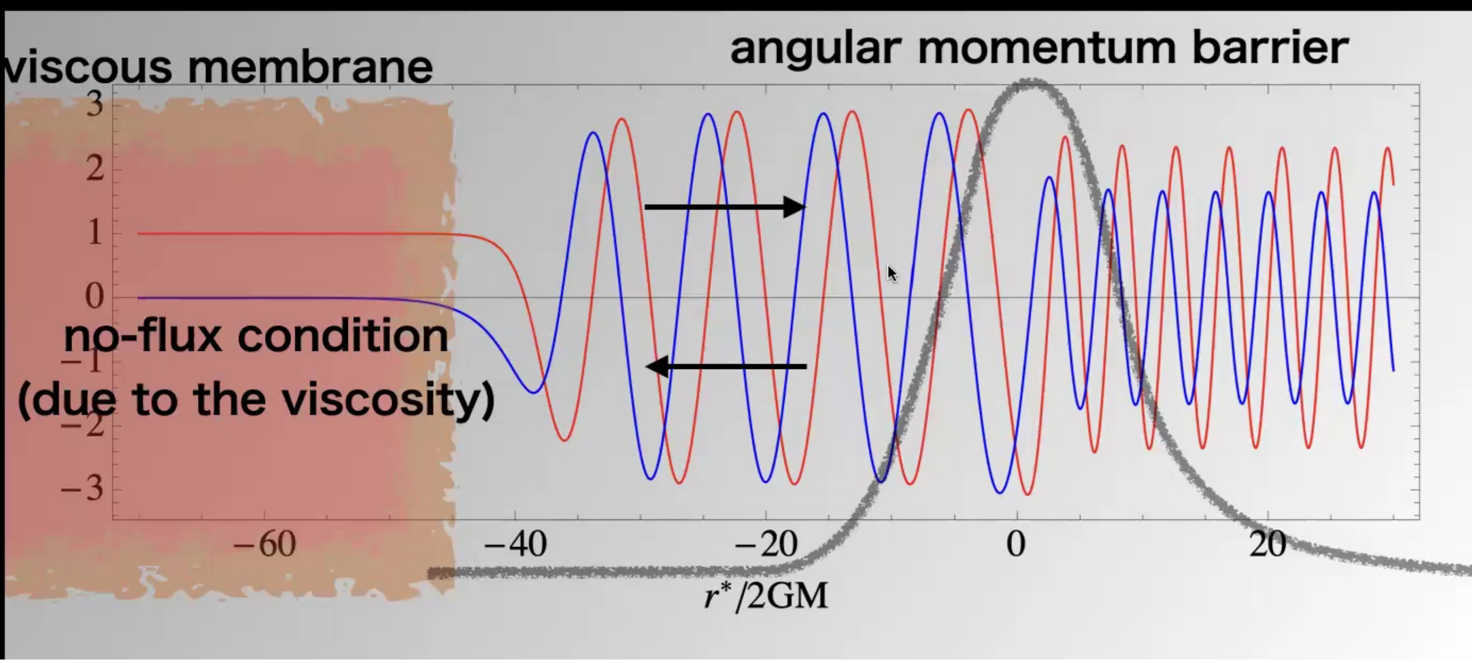


Boltzmann reflection from the membrane

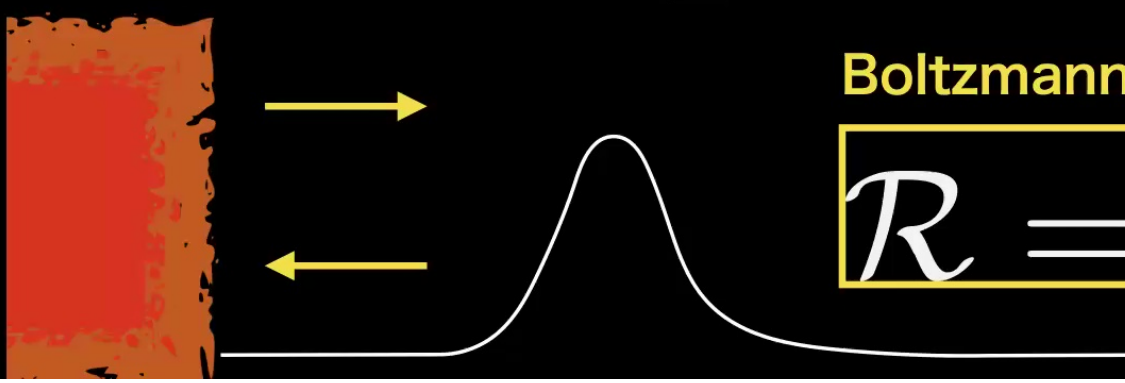
NO and Afshordi (2019)

NO, Wang and Afshordi (2020)

$$\left[-i\frac{\gamma\Omega}{E_{\rm Pl}}\frac{d^2}{dr^{*2}} + \frac{d^2}{dr^{*2}} + \omega^2 - V_\ell(r^*) \right] \psi_\omega = 0$$

$$\lim_{r^* \to -\infty} \psi_\omega(r^*) = \text{const.} \qquad \psi_\omega = {}_2F_1\left[-i\frac{\omega}{\kappa}, i\frac{\omega}{\kappa}, 1, -i\frac{E_{\rm Pl} e^{\kappa r^*}}{\gamma\omega} \right]$$

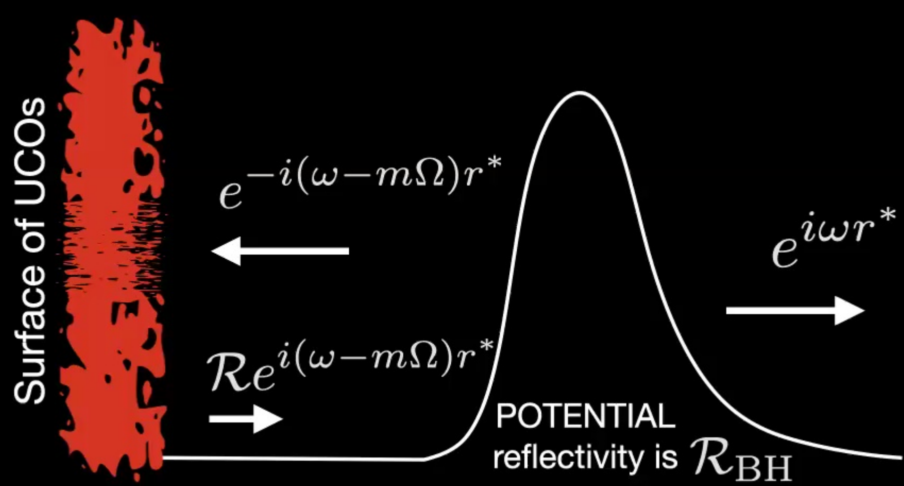
$$\psi_\omega = e^{-\pi\omega/(2\kappa)} A e^{i\omega r^*} + e^{\pi\omega/(2\kappa)} A^* e^{-i\omega r^*}$$



Boltzmann reflection rate!!

$$\mathcal{R} = e^{-\omega/T_H}$$

GW echoes -theoretical model-

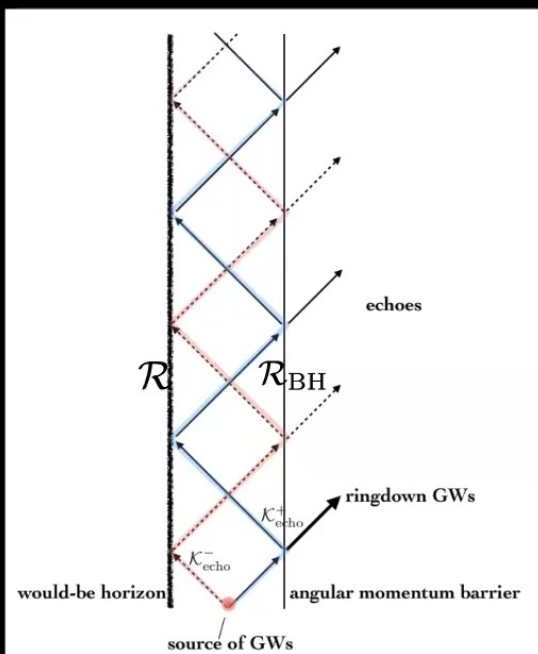


$$h^{(\rm echo)} = \frac{h^{(\rm GR)}}{1 - \mathcal{R}\mathcal{R}_{\rm BH} e^{-2i\tilde{\omega}x_0}}$$

$\mathcal{R}$ Reflectivity at would-be horizon

$\mathcal{R}_{\rm BH}$ Reflectivity of the angular momentum barrier

x_0 Position of the would-be horizon



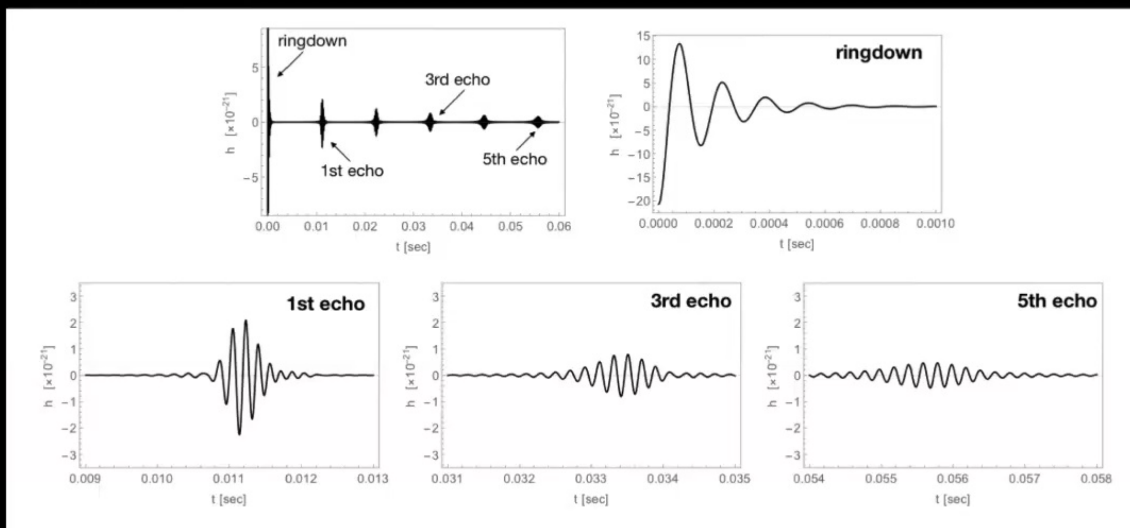


FIG. 8: A time domain function with $M = 2.7 M_\odot$, $\bar{a} = 0.7$, $D_o = 40$ Mpc, $\epsilon_{\rm rd} = 0.04$, $\theta = 90°$, and $\ell = m = 2$ in the Boltzmann reflectivity model with $\gamma = 1$, $T_{\rm H}/T_{\rm QH} = 0.6$, and $x_0 = x_{\rm m}$.

NO, Tsuna, Afshordi (2020) arXiv: 2001.11642

Ergoregion instability

superradiance + reflective boundary → ergoregion instability (rapid decay of rotation)

condition to avoid the instability

$$|\mathcal{R}\mathcal{R}_{\rm BH}| < 1$$

$$(\text{reflectivity in energy flux}) = |\mathcal{R}|^2$$

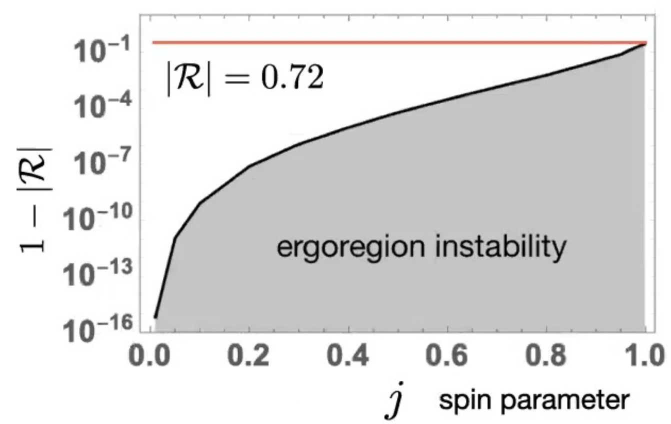
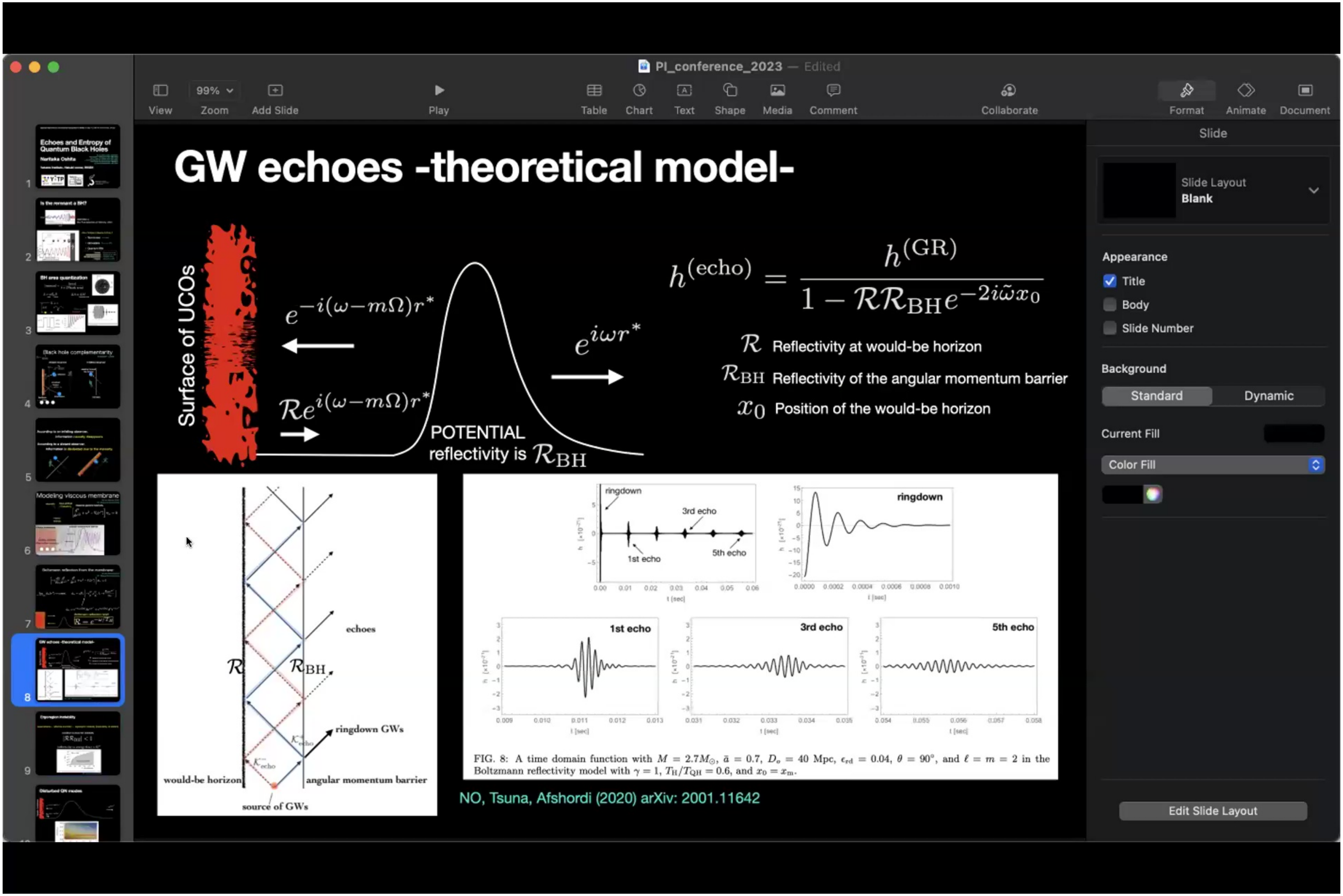


FIG. 1: Constraint on R_c from the ergoregion instability up to the Thorne limit $0 \leq \bar{a} \leq 0.998$.

NO, Tsuna, Afshordi (2020)


PI_conference_2023 — Edited
GW echoes -theoretical model-
Surface of UCOs
$e^{-i(\omega-m\Omega)r^*}$
$\mathcal{R}e^{i(\omega-m\Omega)r^*}$
$e^{i\omega r^*}$
POTENTIAL
reflectivity is $\mathcal{R}_{\rm BH}$
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$\mathcal{R}$ Reflectivity at would-be horizon
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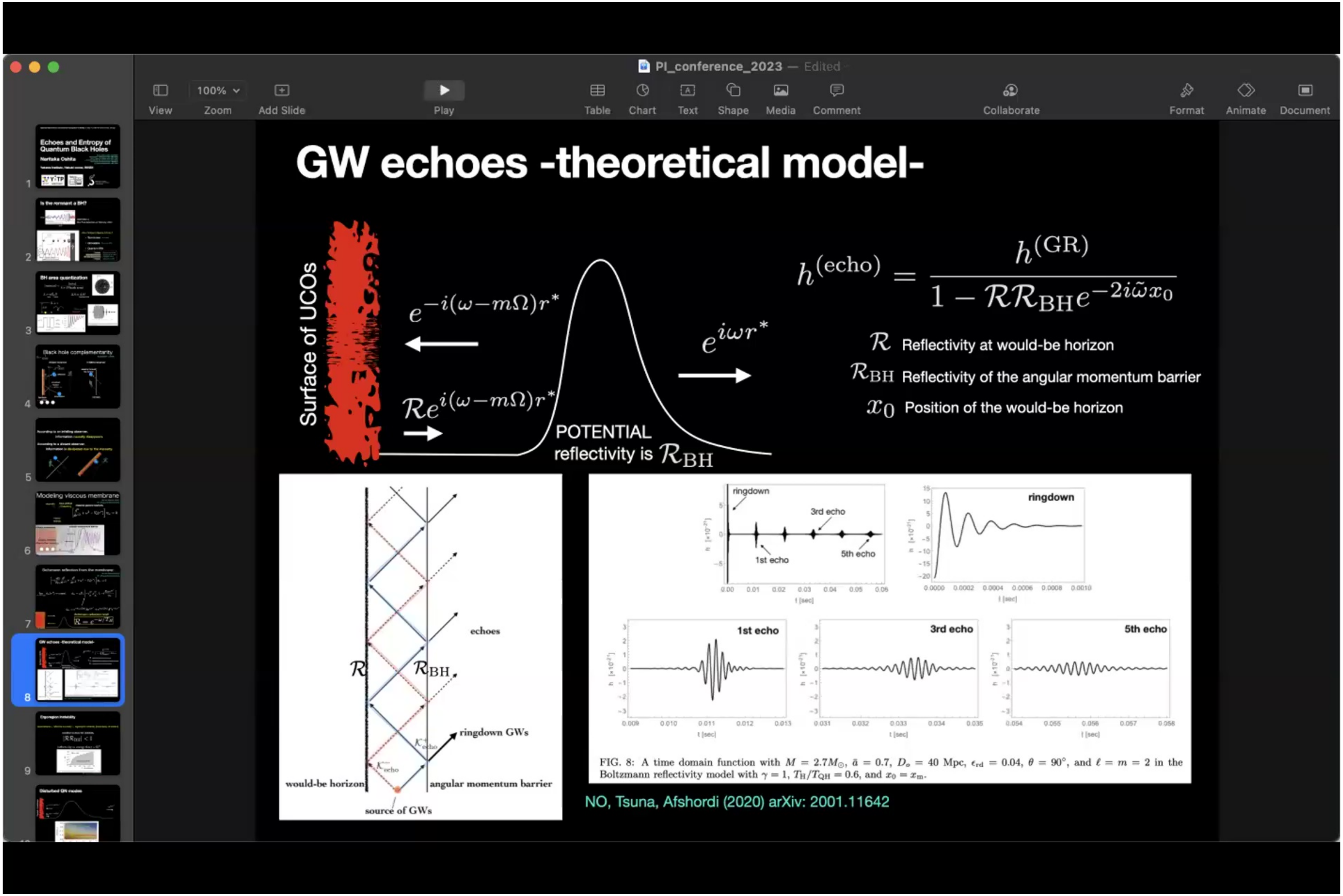
x_0 Position of the would-be horizon
echoes
$\mathcal{R}$
$\mathcal{R}_{\rm BH}$
ringdown GWs
would-be horizon
angular momentum barrier
source of GWs
ringdown
3rd echo
1st echo
5th echo
ringdown
1st echo
3rd echo
5th echo
FIG. 8: A time domain function with $M = 2.7M_\odot$, $\bar{a} = 0.7$, $D_o = 40$ Mpc, $\epsilon_{\rm rd} = 0.04$, $\theta = 90°$, and $\ell = m = 2$ in the Boltzmann reflectivity model with $\gamma = 1$, $T_{\rm H}/T_{\rm QH} = 0.6$, and $x_0 = x_m$.
NO, Tsuna, Afshordi (2020) arXiv: 2001.11642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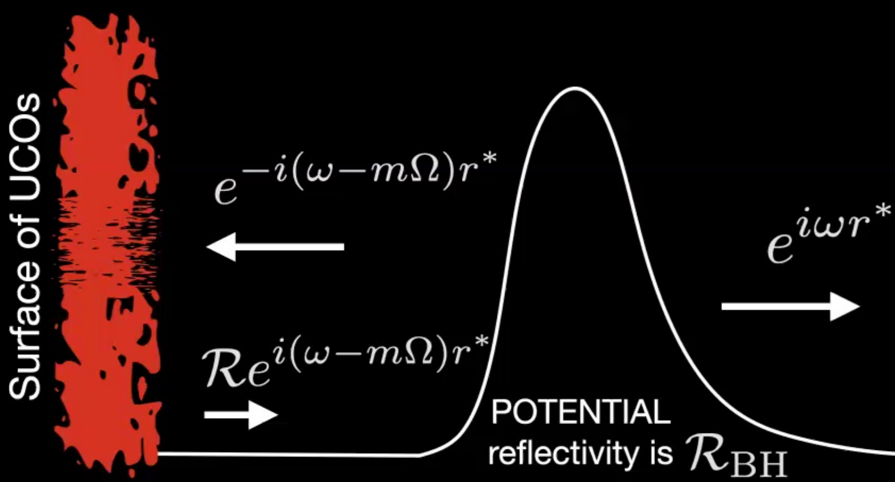

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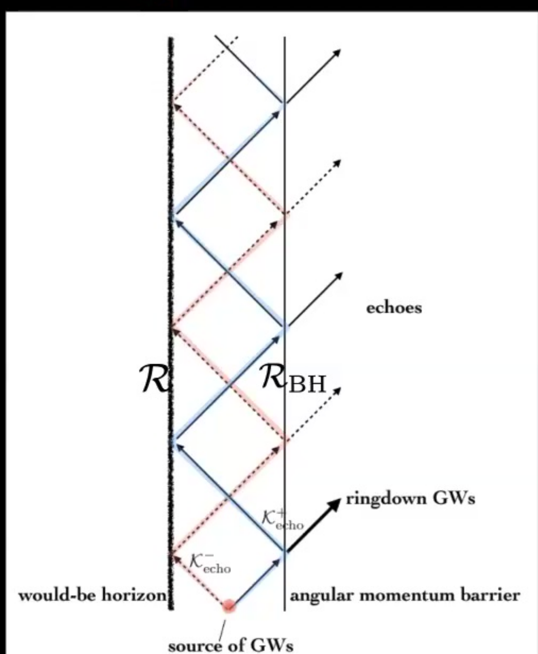

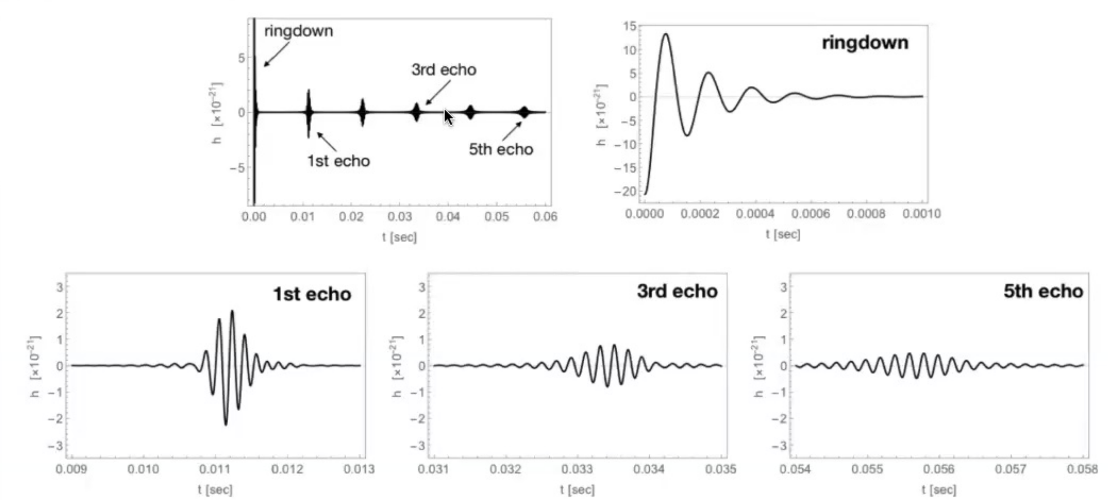

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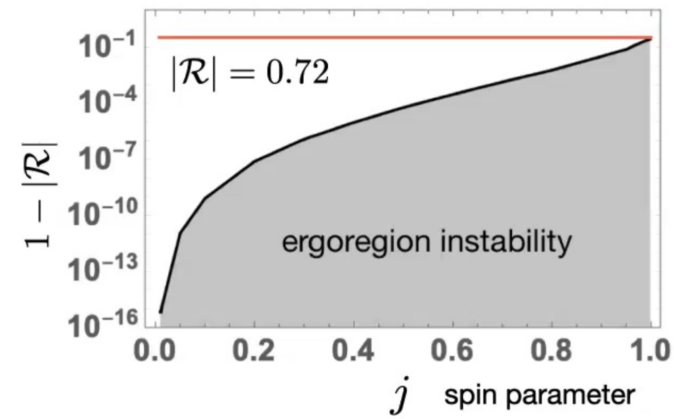

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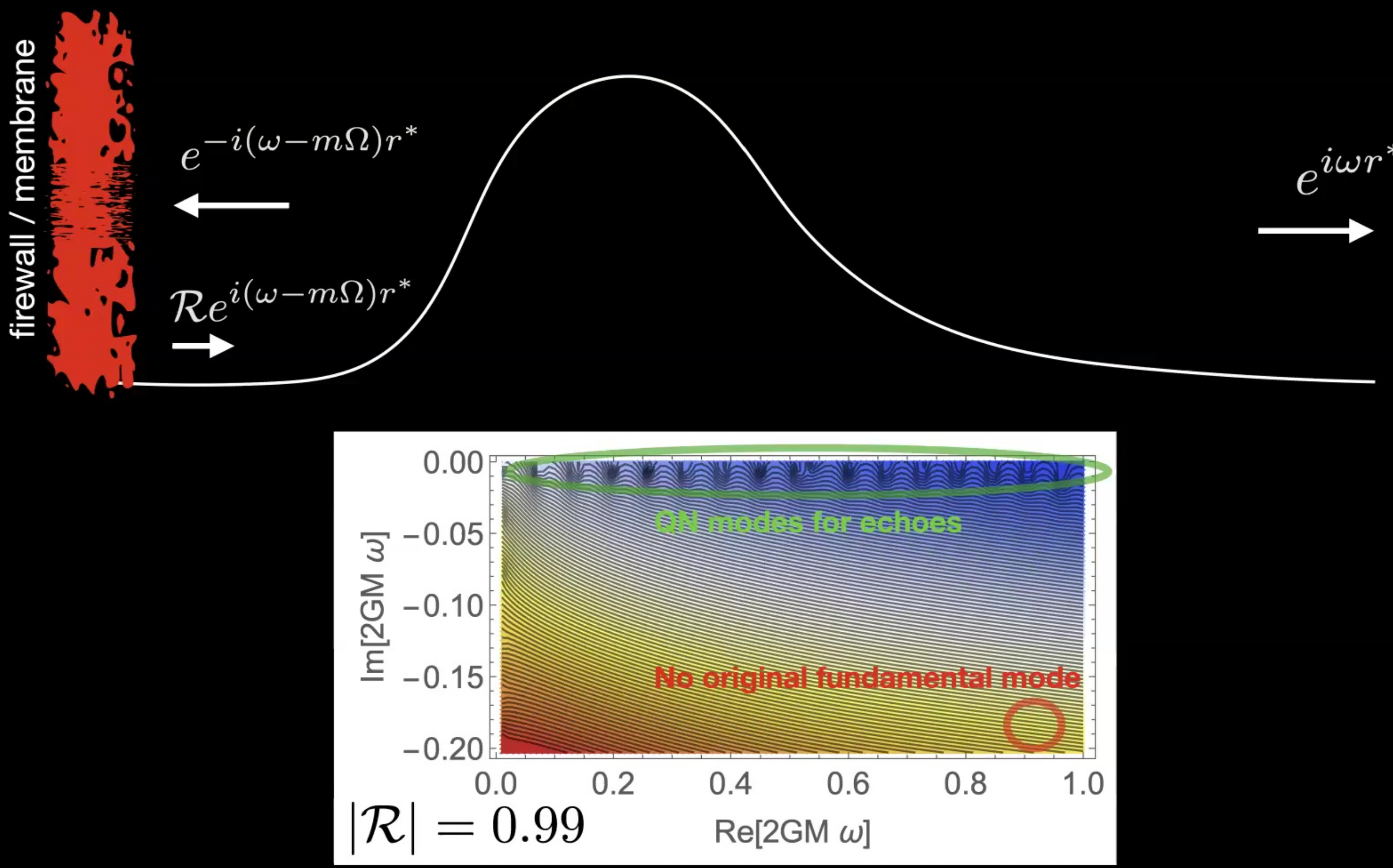
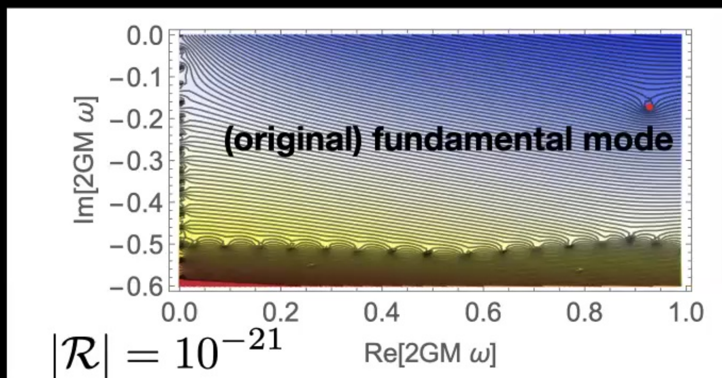
NO, Tsuna, Afshordi (2020)

Disturbed QN modes



NO's numerical result (unpublished)

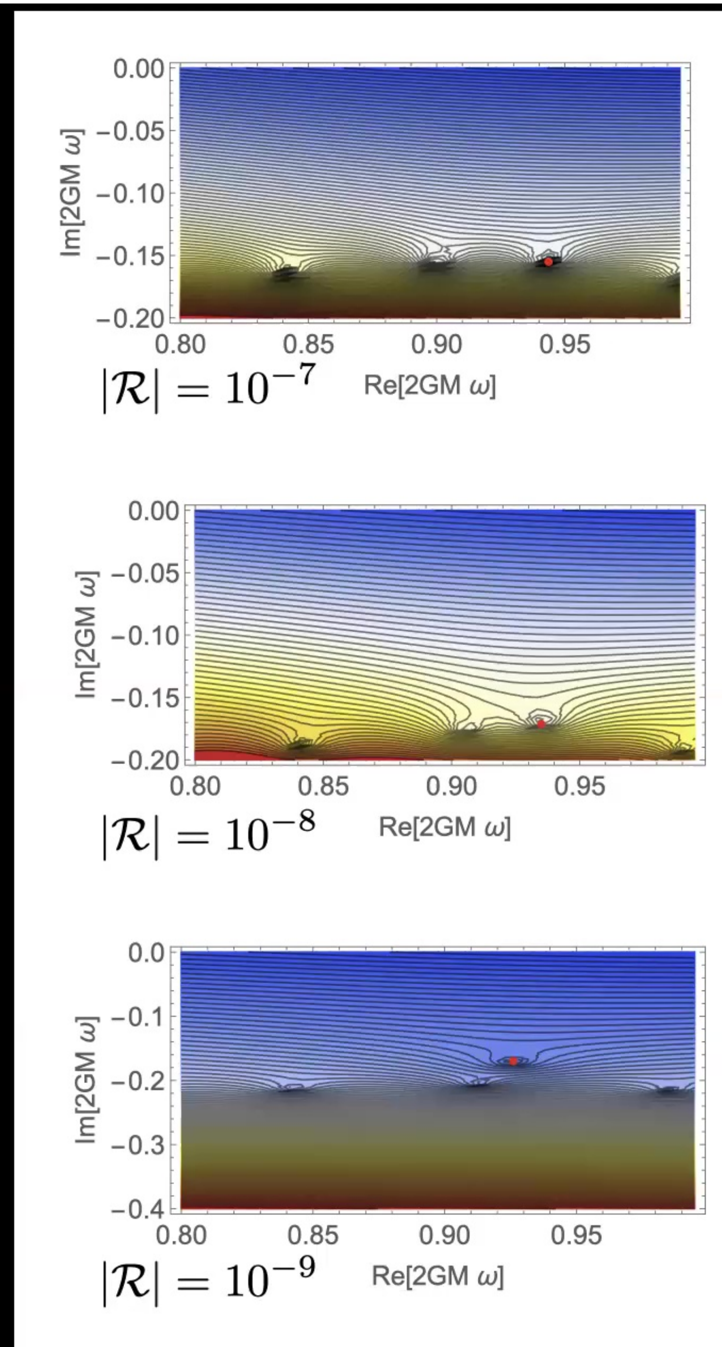
Disturbed QN modes



NO's numerical result (unpublished)

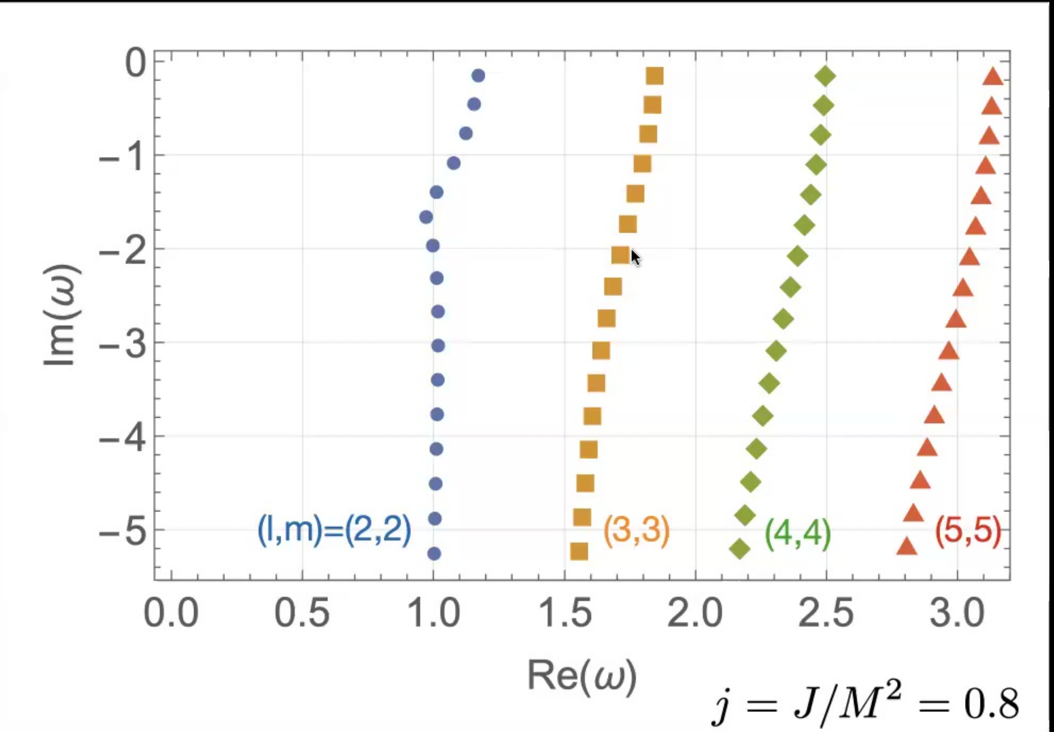
Structure of QNMs in GR is broken by the reflective boundary condition

QN modes are quite sensitive to reflection!!

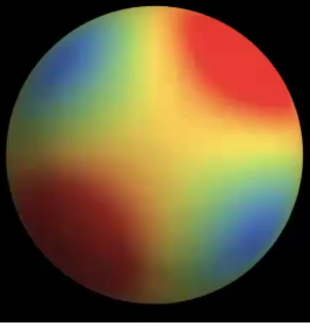


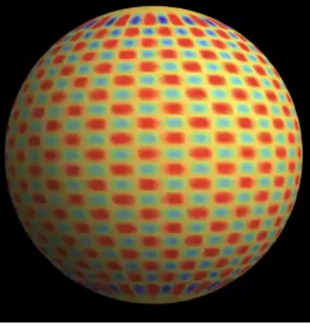
QNM counting and Entropy

free oscillation of a BH ~ ensemble of (damping) oscillators??



$j = J/M^2 = 0.8$





QNMs ~ damping oscillators

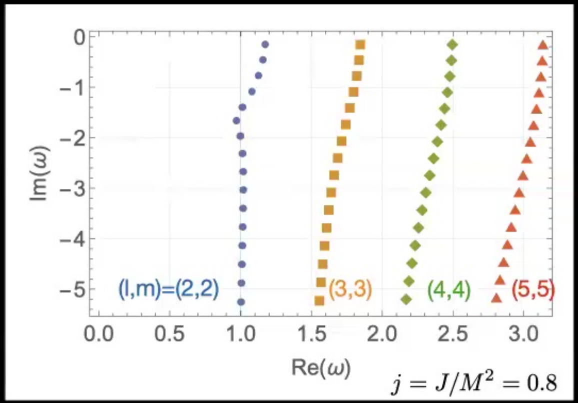
c.f. Maggiore's textbook (Gravitational Waves: Vol.2)

$$\ddot{\xi} + \gamma_0 \dot{\xi} + \omega_0^2 \xi = f(t)$$

friction — proper frequency

$$\xi(t) = -\int \frac{d\omega}{2\pi} \frac{\tilde{f}(\omega)}{(\omega - \omega_+)(\omega - \omega_-)} e^{-i\omega t}$$

$$\omega_\pm = \underbrace{\pm\sqrt{\omega_0^2 - (\gamma_0/2)^2}}_{\omega_\mathrm{R}} - i \underbrace{\frac{\gamma_0}{2}}_{\omega_\mathrm{I}} \rightarrow \omega_0 = \sqrt{\omega_\mathrm{R}^2 + \omega_\mathrm{I}^2}$$



Free oscillation of a BH ~ ensemble of multiple damping oscillators

Canonical Ensemble of QNMs with the Hawking Temperature

for higher harmonics of $l \gg 1$

$$\omega_{\ell mn} = \omega_{R,\ell mn} - i\omega_{I,\ell mn},$$

$$\text{with } \omega_{R,\ell mn} \simeq \Omega\left(\ell + \frac{1}{2}\right),$$

$$\omega_{I,\ell mn} \simeq \Omega\left(n + \frac{1}{2}\right), \quad \Omega \equiv (3\sqrt{3}GM)^{-1}$$

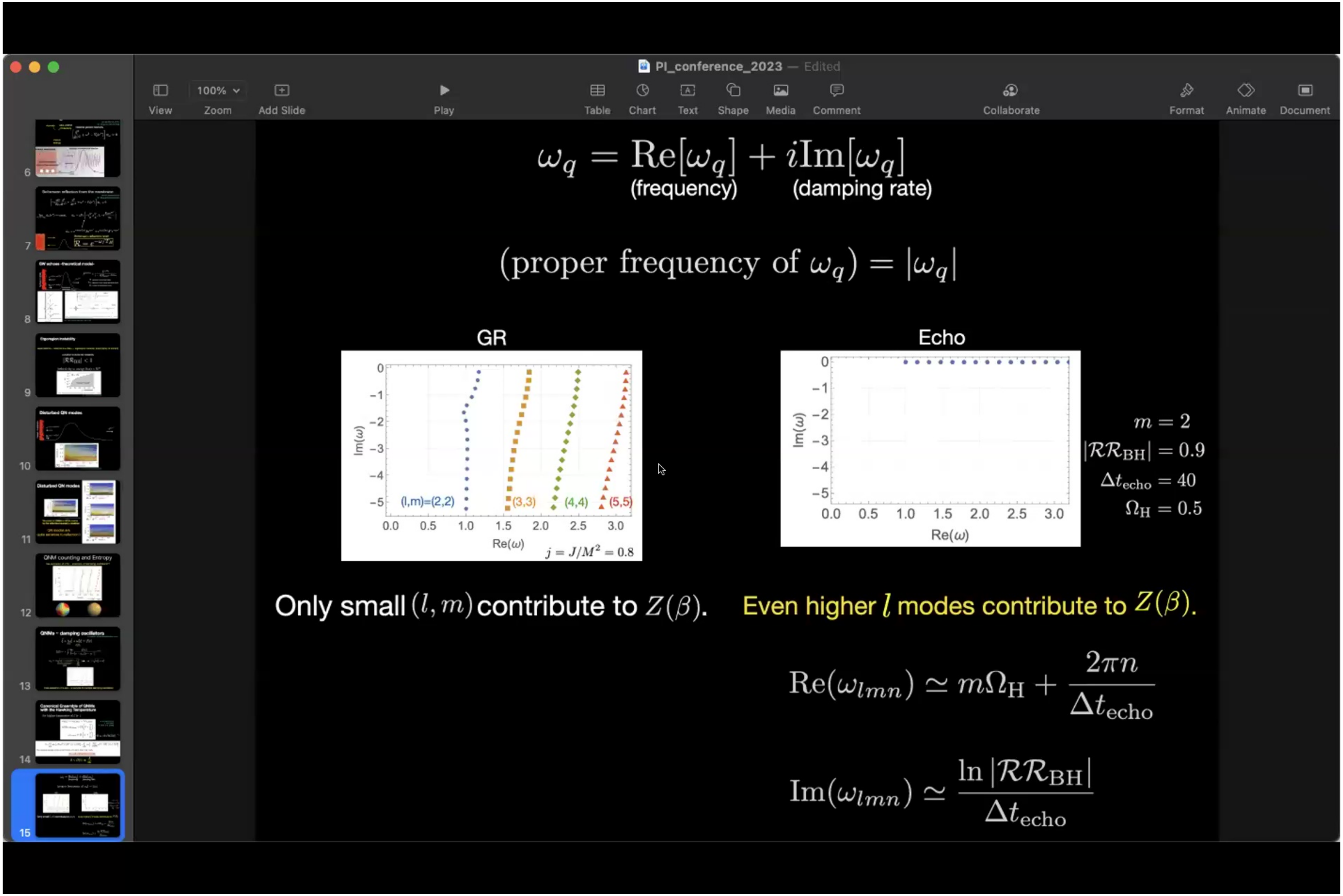
W. H. Press (1971)
S. Iter (1987)
E. Berti, et al. (2009)

$$Z \sim \sum_{\ell mn} \exp\left[-\beta\Omega\sqrt{(\ell+1/2)^2+(n+1/2)^2}\right] = \sum_{\ell mn} \exp\left[-\frac{8\pi\beta}{3\sqrt{3}\beta_{\rm H}}\sqrt{(\ell+1/2)^2+(n+1/2)^2}\right].$$

The canonical entropy of the excited modes of a static black hole reads

$$S = \ln Z - \beta\partial_\beta \ln Z|_{\beta=\beta_{\rm H}} \sim 0.32,$$

$$S \sim \mathcal{O}(1) \ll \frac{A}{4G}$$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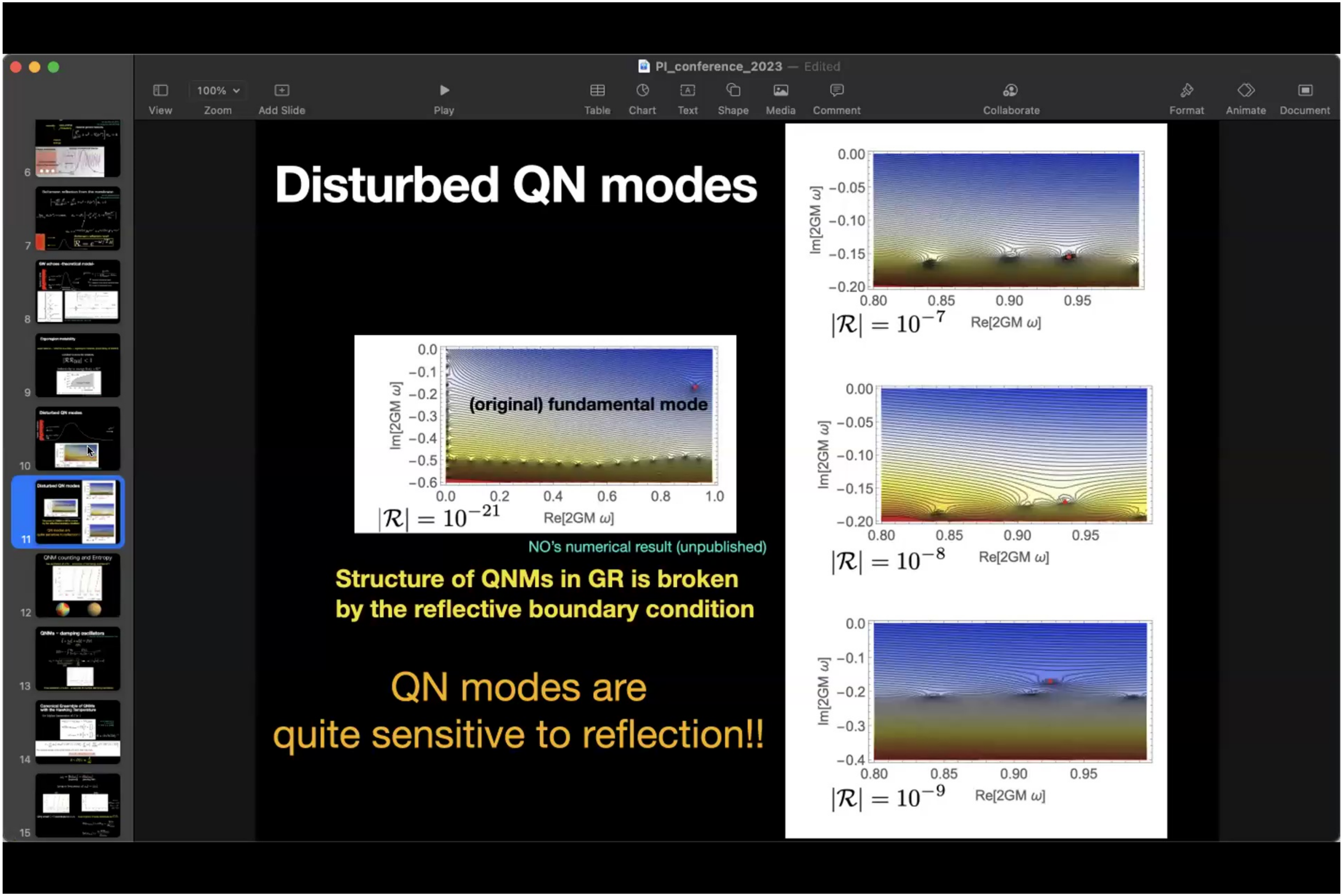
Collaborate
Format
Animate
Document
$\omega_q = \mathrm{Re}[\omega_q] + i\mathrm{Im}[\omega_q]$
(frequency)
(damping rate)
(proper frequency of ω_q) $= |\omega_q|$
GR
(l,m)=(2,2)
(3,3)
(4,4)
(5,5)
Re(ω)
Im(ω)
$j = J/M^2 = 0.8$
Echo
Re(ω)
Im(ω)
$m = 2$
$|\mathcal{R}\mathcal{R}_{\mathrm{BH}}| = 0.9$
$\Delta t_{\mathrm{echo}} = 40$
$\Omega_{\mathrm{H}} = 0.5$
Only small (l, m) contribute to $Z(\beta)$.
Even higher l modes contribute to $Z(\beta)$.
$\mathrm{Re}(\omega_{lmn}) \simeq m\Omega_{\mathrm{H}} + \frac{2\pi n}{\Delta t_{\mathrm{echo}}}$
$\mathrm{Im}(\omega_{lmn}) \simeq \frac{\ln|\mathcal{R}\mathcal{R}_{\mathrm{BH}}|}{\Delta t_{\mathrm{echo}}}$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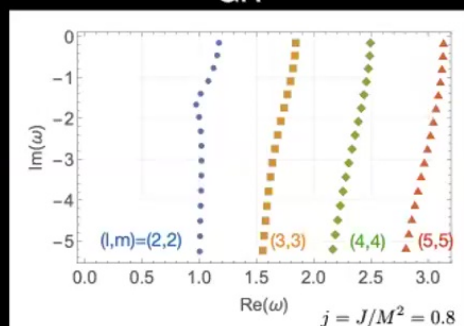
Disturbed QN modes
(original) fundamental mode
Im[2GM ω]
Re[2GM ω]
$|\mathcal{R}| = 10^{-21}$
NO's numerical result (unpublished)
Structure of QNMs in GR is broken
by the reflective boundary condition
QN modes are
quite sensitive to reflection!!
$|\mathcal{R}| = 10^{-7}$
$|\mathcal{R}| = 10^{-8}$
$|\mathcal{R}| = 10^{-9}$

$$\omega_q = \mathrm{Re}[\omega_q] + i\mathrm{Im}[\omega_q]$$

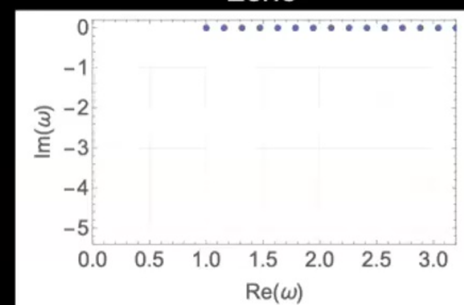
(frequency) (damping rate)

$$(\text{proper frequency of } \omega_q) = |\omega_q|$$

GR



Echo



$m = 2$

$|\mathcal{R}\mathcal{R}_{\mathrm{BH}}| = 0.9$

$\Delta t_{\mathrm{echo}} = 40$

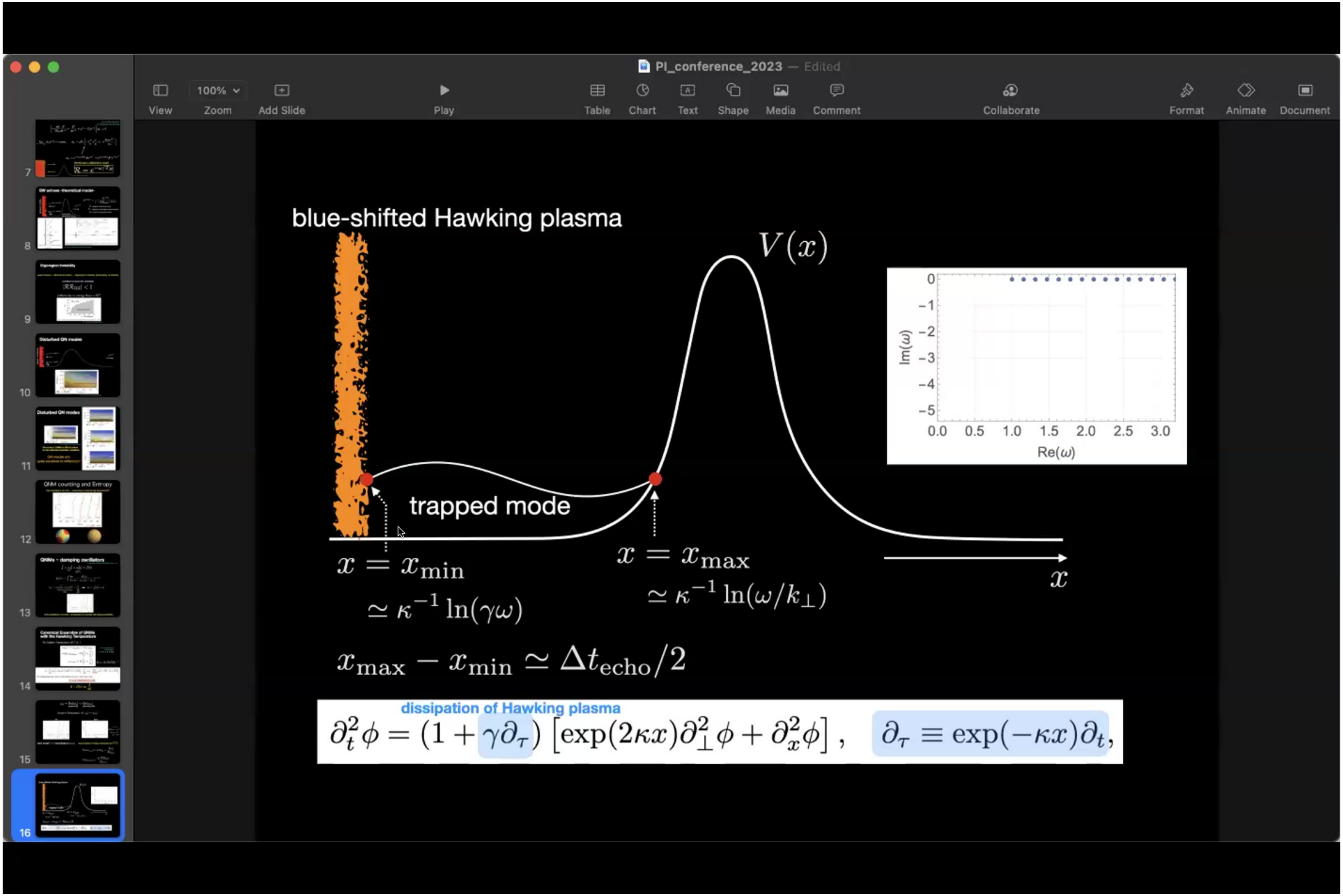
$\Omega_{\mathrm{H}} = 0.5$

Only small (l, m) contribute to $Z(\beta)$.

Even higher l modes contribute to $Z(\beta)$.

$$\mathrm{Re}(\omega_{lmn}) \simeq m\Omega_{\mathrm{H}} + \frac{2\pi n}{\Delta t_{\mathrm{echo}}}$$

$$\mathrm{Im}(\omega_{lmn}) \simeq \frac{\ln|\mathcal{R}\mathcal{R}_{\mathrm{BH}}|}{\Delta t_{\mathrm{echo}}}$$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blue-shifted Hawking plasma
V(x)
trapped mode
x = x_min ≃ κ^{-1} ln(γω)
x = x_max ≃ κ^{-1} ln(ω/k_⊥)
x
Im(ω)
Re(ω)
x_max − x_min ≃ Δt_echo/2
dissipation of Hawking plasma
∂_t^2 φ = (1 + γ∂_τ) [exp(2κx)∂_⊥^2 φ + ∂_x^2 φ] , ∂_τ ≡ exp(−κx)∂_t,

Entropy of an echoing BH

$$S_{\mathrm{Q}} = 2 \times \mathrm{Area} \times \int \frac{d^2k_\perp}{(2\pi)^2} \sum_n \left\{ \frac{\omega_n/T_H}{\exp(\omega_n/T_H)-1} - \ln\left[1-\exp(-\omega_n/T_H)\right] \right\}$$

graviton's polarization

$\sim \sum_{lm} \sim l^2$

summing up overtones

entropy of a thermal harmonic oscillator

$$\simeq \frac{\mathrm{Area}}{55166 \times \gamma^2}$$

$$\omega_n = -\frac{n\pi\kappa}{\ln(\gamma k_\perp)}$$

$$\gamma \simeq 8.52 \times 10^{-3} \times (\text{Planck time})$$

dissipation scale

$$S_{\mathrm{Q}} \simeq \frac{\mathrm{Area}}{4G}$$

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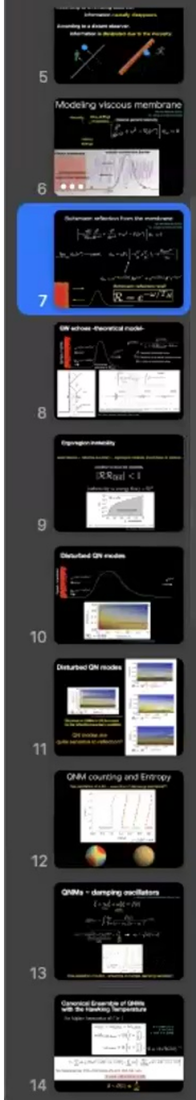


Boltzmann reflection from the membrane

NO and Afshordi (2019)

NO, Wang and Afshordi (2020)

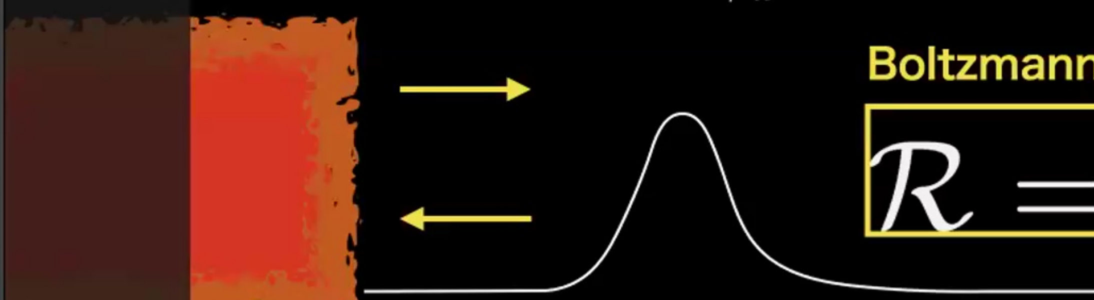
$$\left[-i\frac{\gamma\Omega}{E_{\rm Pl}}\frac{d^2}{dr^{*2}}+\frac{d^2}{dr^{*2}}+\omega^2-V_\ell(r^*)\right]\psi_\omega=0$$

$$\lim_{r^*\to-\infty}\psi_\omega(r^*)=\text{const.}\qquad \psi_\omega={}_2F_1\left[-i\frac{\omega}{\kappa},i\frac{\omega}{\kappa},1,-i\frac{E_{\rm Pl}e^{\kappa r^*}}{\gamma\omega}\right]$$

$$\psi_\omega=e^{-\pi\omega/(2\kappa)}Ae^{i\omega r^*}+e^{\pi\omega/(2\kappa)}A^*e^{-i\omega r^*}$$

Boltzmann reflection rate!!

$$\mathcal{R}=e^{-\omega/T_H}$$



Tentative detection of GW echoes

GW170817

Figure 4: A 3d rendition of Fig. (3) within our echo search frequency range $f = 63 - 92$ Hz, showing that our tentative detection of echoes at $f_{\rm peak} = 72\ (\pm 0.5)$ Hz and $t - t_{\rm merger} \simeq 1.0$ sec clearly stands above noise.

Abedi and Afshordi (2020) arXiv: 1803.10454

The significance depends on the methodology of data analysis.
There are positive and negative results for the detections.

Positive	Negative
Abedi, et al. (2017)	Uchikata, et al. (2019)
Conklin, et al. (2018)	Abbott, et al. (LIGO) (2020)
Holdom (2020)	Wang, et al. (2020)
Uchikata et al. (2019)	Westerweck, et al. (2021)

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Echoes and the third generation of GW detectors

Abedi, Afshordi, NO, Wang (2020) arXiv: 2001.09553

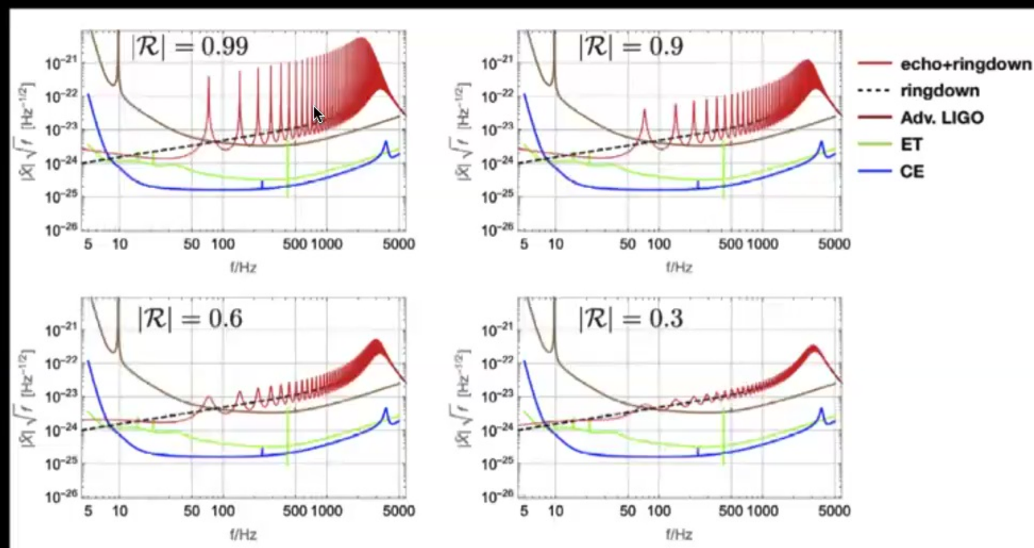
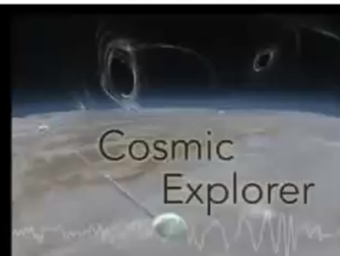
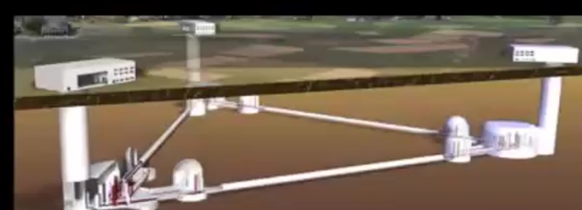


Figure 39. Spectra of ringdown and echo phases with the reflectivity of $|\mathcal{R}| = 0.99$, 0.9, 0.6, and 0.3. We set $D_o = 40$ Mpc, $\bar{a} = 0.1$, $\ell = m = 2$, $M = 4M_\odot$, $\theta = 20°$, and $\epsilon_{\rm rd} = 0.1\%$.



Cosmic Explorer website



Einstein Telescope website

Evaporation of echoing black holes?

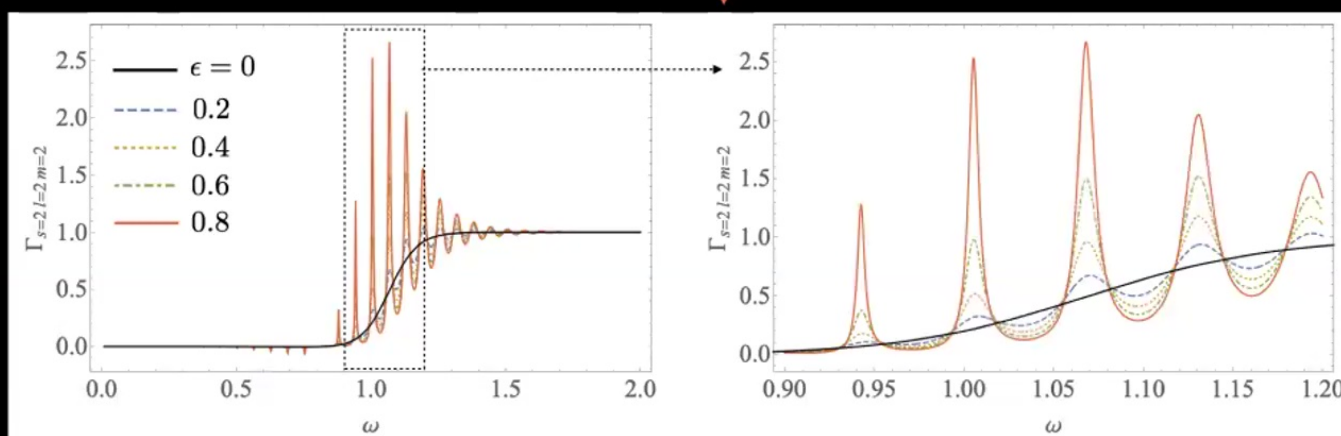
Time development of the mass and angular momentum

$$\frac{dM}{dt} = -\sum_{slm} \frac{1}{2\pi} \int_0^\infty d\omega \frac{\omega \Gamma_{slm}(\omega, \epsilon, r_{\mathrm{w}}^*, \Theta)}{e^{k_{\mathrm{H}}/T_{\mathrm{H}}} - (-1)^{2s}} = \sum_{slm} \left(\frac{dM}{dt}\right)_{slm}$$

$$\frac{dJ}{dt} = -\sum_{slm} \frac{1}{2\pi} \int_0^\infty d\omega \frac{m \Gamma_{slm}(\omega, \epsilon, r_{\mathrm{w}}^*, \Theta)}{e^{k_{\mathrm{H}}/T_{\mathrm{H}}} - (-1)^{2s}} = \sum_{slm} \left(\frac{dJ}{dt}\right)_{slm}$$

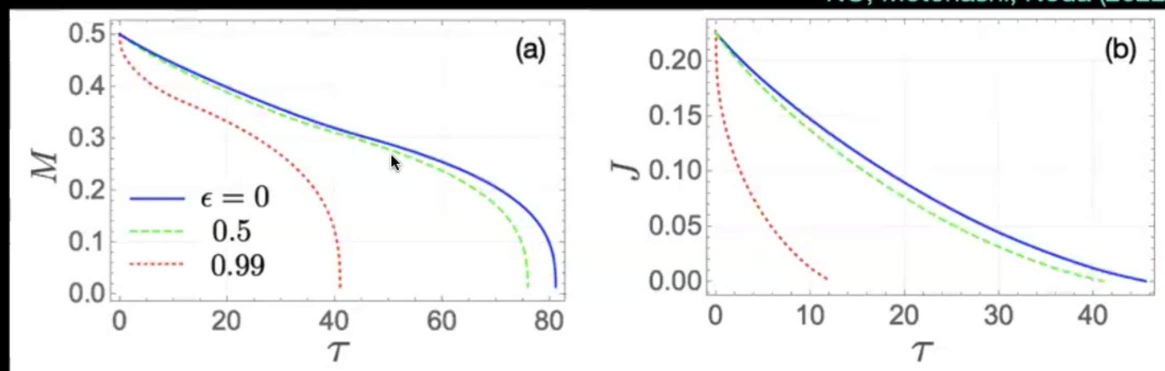
Graybody factor

NO, Motohashi, Noda (2022)



Lifetime of an echoing BH

NO, Motohashi, Noda (2022)



BH's lifetime is shortened only by $\mathcal{O}(1)$

Cosmological constraint on the mass of PBHs is NOT severely affected by the reflection, provided that $|\mathcal{R}\mathcal{R}_{\mathrm{BH}}| < 1$.

Summary

There is a possibility that the remnant of a compact binary merger is NOT a BH.

Possible candidates

- Wormholes
- Gravastars
- Quantum BHs
- and more?

Many ultra compact objects are predicted to have a reflective surface.

It leads to a GW echoes after the ringdown phase.

Perfect reflection leads to the ergoregion instability.
(partial reflection can evade the instability)

Entropy of a quantum BH was estimated by counting QN modes.

Echo model would be reasonable from the entropic point of view.

Echo model shorten the life time of evaporating BHs (by a few factor).

Evaporation of echoing black holes?

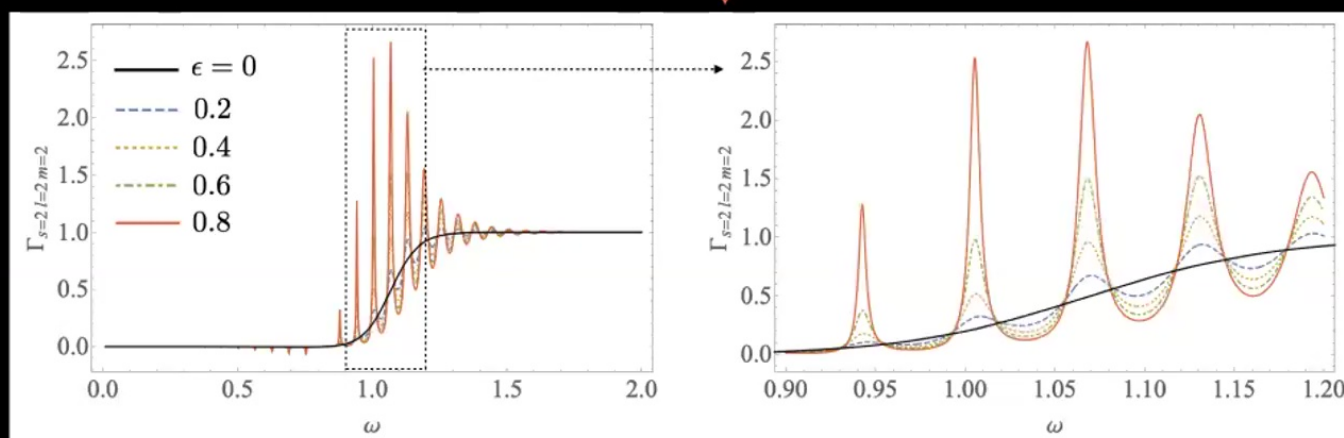
Time development of the mass and angular momentum

$$\frac{dM}{dt} = -\sum_{slm} \frac{1}{2\pi} \int_0^\infty d\omega \frac{\omega \Gamma_{slm}(\omega, \epsilon, r_{\rm w}^*, \Theta)}{e^{k_{\rm H}/T_{\rm H}} - (-1)^{2s}} = \sum_{slm} \left(\frac{dM}{dt}\right)_{slm}$$

$$\frac{dJ}{dt} = -\sum_{slm} \frac{1}{2\pi} \int_0^\infty d\omega \frac{m \Gamma_{slm}(\omega, \epsilon, r_{\rm w}^*, \Theta)}{e^{k_{\rm H}/T_{\rm H}} - (-1)^{2s}} = \sum_{slm} \left(\frac{dJ}{dt}\right)_{slm}$$

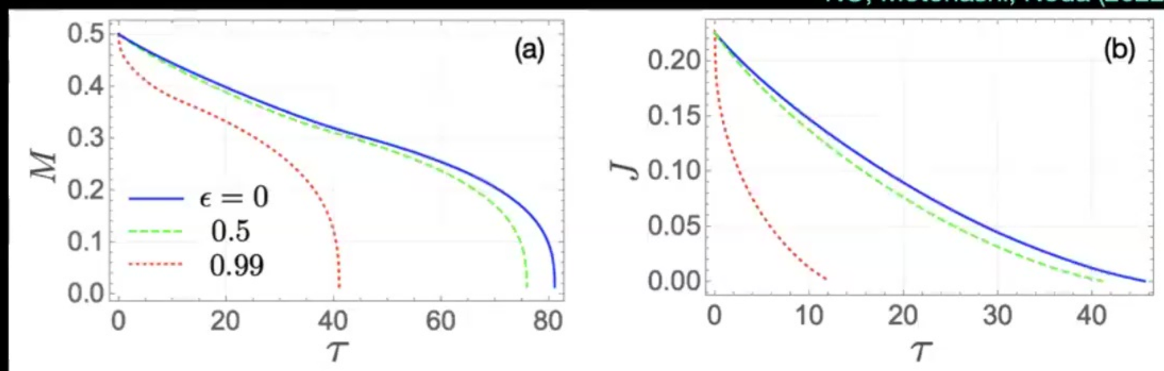
Graybody factor

NO, Motohashi, Noda (2022)



Lifetime of an echoing BH

NO, Motohashi, Noda (2022)



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Cosmological constraint on the mass of PBHs is NOT severely affected by the reflection, provided that $|\mathcal{R}\mathcal{R}_{\rm BH}| < 1$.

Evaporation of echoing black holes?

Time development of the mass and angular momentum

$$\frac{dM}{dt} = -\sum_{slm} \frac{1}{2\pi} \int_0^\infty d\omega \frac{\omega \Gamma_{slm}(\omega, \epsilon, r_{\rm w}^*, \Theta)}{e^{k_{\rm H}/T_{\rm H}} - (-1)^{2s}} = \sum_{slm} \left(\frac{dM}{dt}\right)_{slm}$$

$$\frac{dJ}{dt} = -\sum_{slm} \frac{1}{2\pi} \int_0^\infty d\omega \frac{m \Gamma_{slm}(\omega, \epsilon, r_{\rm w}^*, \Theta)}{e^{k_{\rm H}/T_{\rm H}} - (-1)^{2s}} = \sum_{slm} \left(\frac{dJ}{dt}\right)_{slm}$$

Graybody factor

NO, Motohashi, Noda (2022)