Title: Do First-Class Constraints Generate Gauge Transformations? A Geometric Perspective.

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Abstract: The constrained Hamiltonian formalism is the basis for canonical quantization techniques. However, there are disagreements surrounding the notion of a gauge transformation in such a formalism. The standard definition of a gauge transformation in the constrained Hamiltonian formalism traces back to Dirac: a gauge transformation is a transformation generated by an arbitrary combination of first-class constraints. On the basis of this definition, Dirac argued that one should extend the form of the Hamiltonian in order to include all of the gauge freedom. However, Pitts (2014) argues that in some cases, a first-class constraint does not generate a gauge transformation, but rather "a bad physical change". Similarly, Pons (2005) argues that Dirac's analysis of gauge transformations is "incomplete" and does not provide an account of the symmetries between solutions. Both authors conclude that extending the Hamiltonian in the way suggested by Dirac is unmotivated. If correct, these arguments could have implications for other issues in the foundations of the constrained Hamiltonian formalism, including the Problem of Time. In this talk, I use a geometric formulation of the constrained Hamiltonian formalism to show that one can motivate the extension to the Hamiltonian independently from consideration of the gauge transformations, and I argue that this supports the standard definition of a gauge transformation without falling prey to the criticisms of Pitts (2014) and Pons (2005). Therefore, in order to maintain that first-class constraints do not generate gauge transformations, one must reject the claim that the constrained Hamiltonian formalism is fully described by the geometric picture; I suggest two avenues for doing so.

Zoom Link: https://pitp.zoom.us/j/98277900018?pwd=SW92OWYrRFpkWC9QOS9NeTlQWkY5dz09

Do First-Class Constraints Generate Gauge Transformations? A Geometric Perspective

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Background

- In several Lagrangian theories we find that there are (local) *symmetries* related (via Noether's second theorem) to dependencies among the Euler-Lagrange equations.

- In such cases, we seem to have a breakdown of determinism: there is arbitrariness in the solutions to the equations of motion.

Background

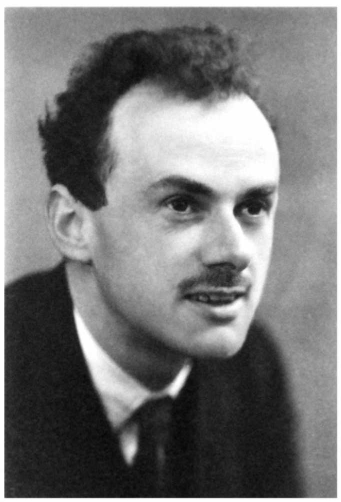
- In several Lagrangian theories we find that there are (local) *symmetries* related (via Noether's second theorem) to dependencies among the Euler-Lagrange equations.

- In such cases, we seem to have a breakdown of determinism: there is arbitrariness in the solutions to the equations of motion.

Gauge Transformation: A transformation that connects physically equivalent descriptions of the same *state* or *history* of a physical system.

How do we combat the threat of indeterminism?

The **Constrained Hamiltonian formalism** provides a framework for identifying the gauge-invariant variables (the 'observables') by linking constraints on what is a "physically allowed" state to gauge freedom.



Paul Dirac

Of particular importance are the *first-class* constraints: those that "commute" with each other.

Peter Bergmann

Dirac View of Gauge Transformations

Arbitrary combinations of first-class constraints generate a gauge transformation.

Therefore, one should extend the Hamiltonian to include all the gauge freedom.

Problems for the Dirac view

1. Counterexamples to the claim that first-class *secondary* constraints generate gauge transformations. (Henneaux and Teitelboim (1992))

2. Counterexamples to the claim that *each* first-class constraint generates a gauge transformation. (Pitts (2014a,b))

3. Arguments that Dirac's proof is faulty and limited in scope. (Pons (2005), Barbour and Foster (2008))

Alternative View of Gauge Transformations

A *specific combination* of first-class constraints generates a gauge transformation.

↓

Therefore, one *need not* extend the Hamiltonian (to the extent Dirac did).

To Come…

I will argue that the reasoning in both the Dirac and alternative view of gauge transformations is flawed, but that correct reasoning validates the definition of a gauge transformation given by Dirac: arbitrary combinations of first-class constraints generate a gauge transformation.

Importance for Quantum Gravity

- Constrained Hamiltonian formalism provides the basis for canonical quantization methods.
- The constrained Hamiltonian formalism is the basis of the "Problem of Time": when the Hamiltonian is one of the first-class constraints, the Dirac view leads to the conclusion that time evolution is the unfolding of a gauge transformation.
  - If the Dirac view does not hold, then perhaps the Problem of Time can be avoided.

Wider Importance: Relationship between Lagrangian and Hamiltonian pictures

- Determining the gauge symmetries of the Hamiltonian and Lagrangian pictures is important for determining whether these two pictures can be said to be equivalent.
- Some advocates of the alternative view argue that it matches the Lagrangian gauge transformations, unlike the Dirac view.

Rest of the Talk

1. Dirac's recipe and argument for the Dirac view of gauge transformations
2. An argument against Dirac and in favor of alternative view
3. Geometric picture of the constrained Hamiltonian formalism
4. Resolving the debate
5. Upshots and possible response

Dirac's Recipe

Start with Lagrangian $L(q_i, \dot{q}_i)$ → Define canonical momentum as $p_i = \frac{dL}{d\dot{q}_i}$ → Derive primary constraints $\varphi_m(q_i, p_i) \approx 0$ → Define Total Hamiltonian as $H_T = p^i \dot{q}_i - L + u^m \varphi_m$

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4. Define Total Hamiltonian as $H_T = p^i \dot{q}_i - L + u^m \varphi_m$
5. Derive equations of motion as $\frac{df}{dt} = \{f, H_T\}$
6. Use internal consistency to get secondary constraints (if any) $\sigma_n(q_i, p_i) \approx 0$
7. Determine whether $\{\gamma_i, \gamma_j\} \approx 0$ for all constraints γ_n (first-class) or not (second-class)

Dirac Gauge Transformation

A gauge transformation relates any two states $(q_1(\delta t), p_1(\delta t))$ and $(q_2(\delta t), p_2(\delta t))$ that are possible evolutions from an initial $(q_0(t_0), p_0(t_0))$ under $H_T = H + u^m \varphi_m$ at some fixed (infinitesimal) interval δt.

By considering two different choices of u^m, Dirac shows:

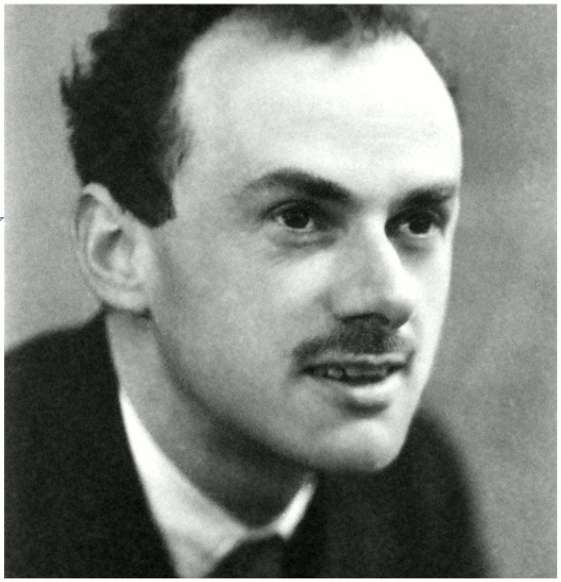
Each *primary* first-class constraint generates a gauge transformation.

Moreover, Dirac conjectures:

Each *secondary* first-class constraint generates a gauge transformation.

Extended Hamiltonian

"We were led to the idea that there are certain changes in the p's and the q's that do not correspond to a change of state, and which have as generators first-class secondary constraints. That suggests that… we should consider a more general equation of motion $\dot{g} = \{g, H_E\}$ with an ***extended Hamiltonian*** H_E, consisting of the previous Hamiltonian H_T , plus all those generators that do not change the state, with arbitrary coefficients."



Dirac's Final Picture

Gauge transformations are generated by each first-class constraint and arbitrary combinations of them.

Equation of motion given by $\dot{f} = \{f, H_E\}$ where $H_E = H + u^j \gamma_j$ where γ_j are the first-class constraints.

Response to Dirac

Dirac's proof is *incomplete;* it only gives an account of gauge transformations as a transformation on *states* that is distinct from the standard notion of a gauge transformation as a map between *solutions* to the equations of motion. (Pons (2005))

Pons' Definition of a Gauge Transformation

A gauge transformation relates any two states $(q_1(t), p_1(t))$ and $\big(q_2(t), p_2(t)\big)$ that are possible evolutions from an initial $(q_0(t_0), p_0(t_0))$ under $H_T = H + u^m \varphi_m$ for ***any* time *t***.

Under this account, Pons shows that the form of the gauge transformations is given by:

$$G(t) = \sum_{i=0}^{N} G_i \xi^{(i)}(t)$$

where each G_i is a first-class constraint (satisfying some conditions) and ξ is an arbitrary function of time.

At a particular time, this reduces to the Dirac gauge transformations.

Is there any tension?

Two notions of a gauge transformation:

1. A gauge transformation as a map from one state to a physically equivalent state (*Dirac view*)
2. A gauge transformation as a map from a solution to the equations of motion to another physically equivalent solution (*alternative view*)

Can't we keep these two things kept apart?

Where the tension lies

There is a conceptual issue with this division: it means that individual states along two curves can be gauge-equivalent without it being the case that if one curve is a solution, the other also is. But if gauge-equivalence means physical equivalence, how can this be so?

Debate assumes reasoning as follows:

Determine gauge transformations using total Hamiltonian.

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Use gauge transformations to say whether to extend the Hamiltonian.

Could we reason the other way round?

Let's consider things geometrically…

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- Phase space Γ $\{(q_i, p_i), i = 1, \ldots, N\}$ as cotangent bundle of configuration space T^*C

- Γ comes equipped with one-form given by $\theta = p_i dq^i$ with corresponding two-form $\omega = \boldsymbol{d}\theta$ which is *symplectic* (closed, non-degenerate)

- Given function f, can uniquely define smooth tangent vector field X_f through $\omega\left(X_f\,,\ \cdot\right) = df(\cdot)$

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- Equations of motion:

$$\{f, H\} = \omega\left(X_f, X_H\right) = df(X_H) = \frac{df}{dt}$$

Constraints, Geometrically

- (First-Class) constraints $\gamma_j(q,p) = 0$ for $j = 1, \dots, M$ give rise to $N - M$ dimensional submanifold, the *constraint surface, Σ.*

- Induced two-form $\widetilde{\omega}$ on the constraint surface is *degenerate:* possesses M null vectors associated with the first-class constraints:

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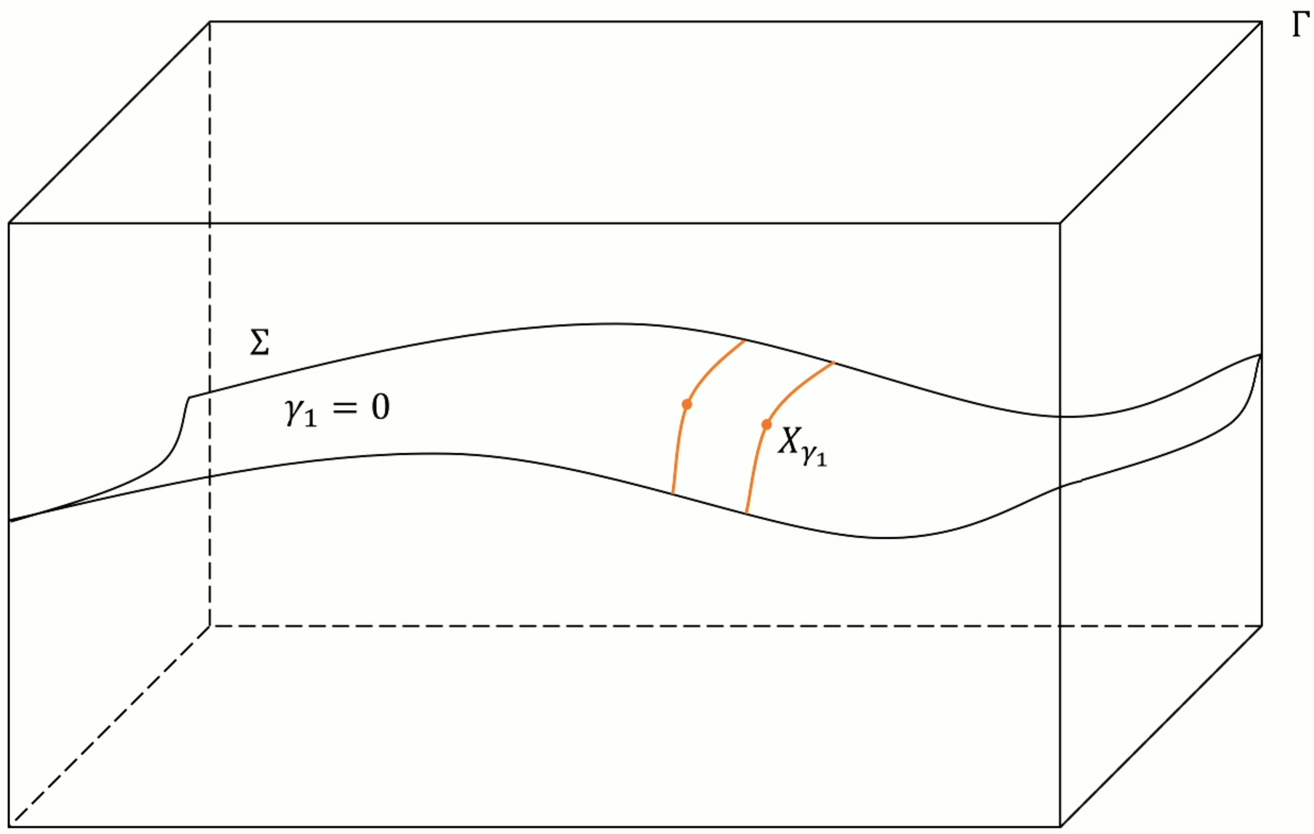
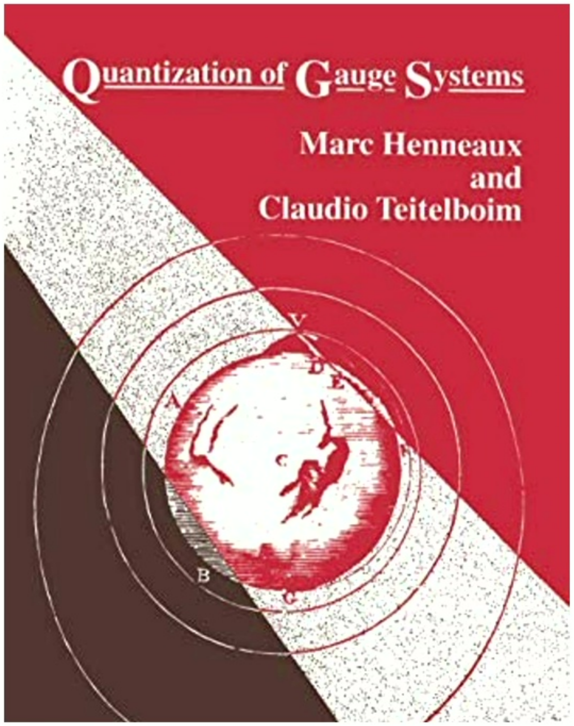
$$\widetilde{\omega}\left(X_{\gamma_j},\ \cdot\right) = d\gamma_j|_\Sigma = 0$$



Γ
Σ
$\gamma_1 = 0$
X_{γ_1}

Gauge Orbits

Henneaux and Teitelboim (1994): “The identification of the gauge orbits with the null surfaces of the induced two-form relies strongly on the postulate made throughout the book that all first-class constraints generate gauge transformations.”



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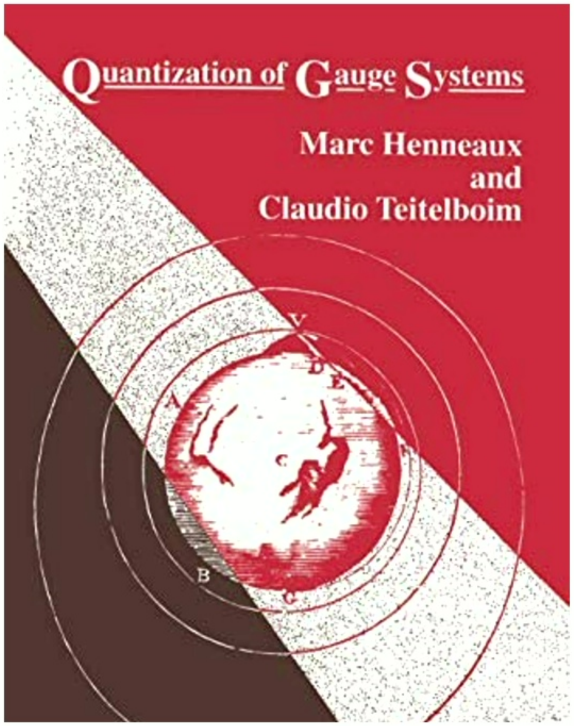

Does this mean that the geometric view is subject to the criticisms of Pons?

No: The geometric picture can be used to show that the reasoning in the debate is wrong.

Reasoning in Debate

Determine gauge transformations using total Hamiltonian.

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Use gauge transformations to say whether to extend the Hamiltonian.

Geometric Reasoning

Motivate Extended Hamiltonian.

Determine gauge transformation of states.

Use extended Hamiltonian and gauge freedom in states to determine gauge transformations on solutions.

Motivating Extended Hamiltonian without assumptions about gauge transformations

Consider $X_H \to X_H + a^j X_{\gamma_j}$

Since $\widetilde{\omega}\left(a^j X_{\gamma_j}, \quad \cdot\right) = 0$,

$$\widetilde{\omega}\left(X_H + a^j X_{\gamma_j}\,\cdot\right) = \widetilde{\omega}(X_H\,,\ \cdot\,) + \widetilde{\omega}\left(a^j X_{\gamma_j}\,,\ \cdot\,\right) = \widetilde{\omega}(X_H,\cdot) = dH|_\Sigma$$

The evolution generated by X_H and $X_H + a^j X_{\gamma_j}$ is not distinguished on Σ.

Motivating gauge transformations on states without dynamics

- On the constraint surface, $\tilde{\omega}$ only acts on vector fields that are tangent to the constraint surface i.e. the vector fields that are constant along the gauge orbits: $\omega\left(X_f, X_{\gamma_a}\right) = 0$ on the constraint surface.
- This means that for functions that vary along the gauge orbits, the induced two-form cannot 'see' this change.

Each first-class constraint generates a gauge transformation on states.

Motivating gauge transformations on solutions from extended Hamiltonian and gauge freedom in states

- $X_H \to X_H + a^j X_{\gamma_j}$ generates a curve that differs only with regards to where on the gauge orbit it lies at each point in time.
- Moreover, each point along a gauge orbit form an equivalence class of states.

Each first-class constraint generates a gauge transformation from a solution to the equations of motion to another solution.

Upshot

- Dirac was right about the definition of the gauge transformations and the role of the extended Hamiltonian, but his reasoning was flawed.
- Restricting the equivalence class of Hamiltonians to those that only include an arbitrary combination of *primary* first-class constraints is unnatural since:
  1. It distinguishes different null vectors that are structurally equivalent.
  2. It would be to say that the dynamics can distinguish states along a gauge orbit, even though the structure of the constraint surface cannot distinguish these states.

Possible Response:

"Only the total Hamiltonian produces the *correct, Lagrangian equivalent* gauge transformations."

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- This is to deny that the geometry of the constraint surface captures the full picture.

Option 1: Argue that the interpretation of primary and secondary constraints is distinct within the Hamiltonian formalism.

Option 2: Argue that the Lagrangian formalism provides the basis for the distinction between primary and secondary constraints.

Thank you!

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