Title: Higher Weyl modules, coinvariants, and factorization homology

Speakers: Maciej Szczesny

Series: Mathematical Physics

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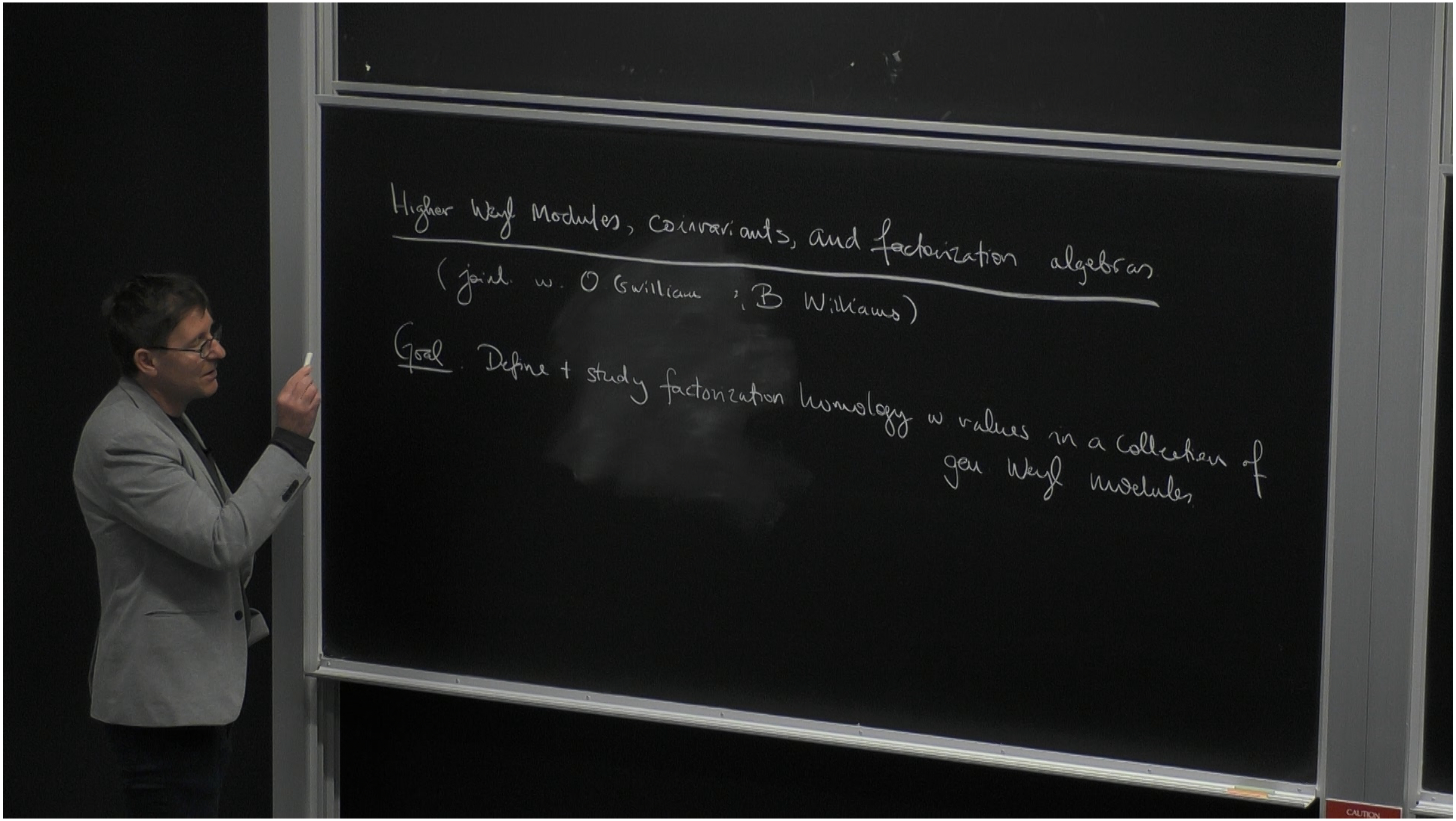
Abstract: This talk is based on joint work with Owen Gwilliam and Brian Williams. We define factorization homology of factorization envelopes valued in a collection of generalized Weyl modules supported on a cycle in a smooth complex projective variety X. When X is a smooth projective curve, and the cycle a collection of points, we recover the space of coinvariants studied in CFT.

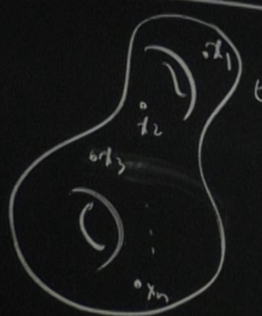
Zoom link: https://pitp.zoom.us/j/97473973419?pwd=QjRQUklac2lTRlduSVc3S0FKQW9IQT09


Higher Weyl Modules, coinvariants, and factorization algebras
(joint w. O Gwilliam & B Williams)
Goal. Define + study factorization homology w values in a collection of gen Weyl modules

(2) Factorization algebra story

(3) Examples / further directions

(1) $\underline{\dim_{\mathbb{C}} X = 1}$: (following F-BZ)



$(X, x_1, \ldots, x_n) \Leftarrow$ n-pointed sm proj $/\mathbb{C}$

• V - vertex alg $(Y : V \to \mathrm{End}\, V[[z, z^{-1}]])$

② Factorization algebra story

③ Examples / further directions

① $\dim_{\mathbb{C}} X = 1$: (following F-BZ).

$(X, x_1, \dots, x_n)$ ← n-pointed sm proj / $\mathbb{C}$

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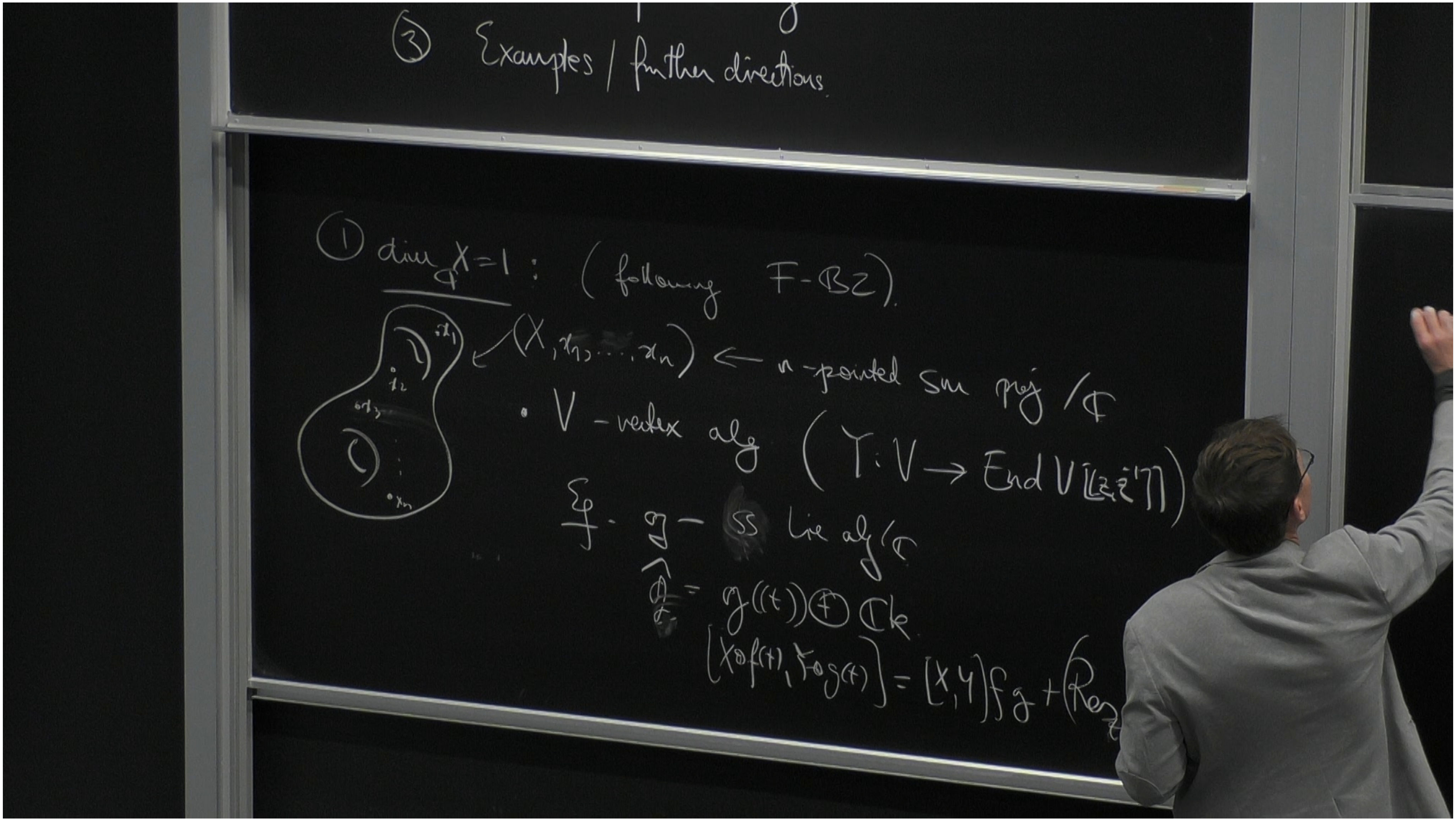
Eg.


③ Examples / further directions.
① dim_C X = 1 : (following F-BZ).
(X, x_1, ..., x_n) ← n-pointed sm proj / C
• V - vertex alg (Y: V → End V[[z, z^{-1}]])
Eg. g - ss Lie alg / C
ĝ = g((t)) ⊕ Ck
[X⊗f(t), Y⊗g(t)] = [X,Y] fg + (Res_z

$\mathfrak{g} = \mathrm{span}\{J^1, \dots, J^n\}$

- $V = V_k(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{g}[t]+\mathbb{C}k}^{\hat{\mathfrak{g}}} \mathbb{C} = \mathrm{Span}\{ J^{i_1}_{n_1} J^{i_2}_{n_2} \cdots J^{i_p}_{n_p} \mathbb{1} \}$, $\quad \mathbb{1} = |0\rangle$

$n_i \leq -1.$

$$Y(J^a_{-1}|0\rangle) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}$$

- $M_1, \dots, M_n$ V-modules.

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- $M_1, \dots, M_n$ V-modules.

① Weyl modules. $U \in \mathrm{Rep}(\mathfrak{g})$

$W(U)$ = "Weyl module induced from U"

$= \mathrm{Ind}_{\mathfrak{g}[t] + \mathbb{C}k}^{\hat{\mathfrak{g}}} U$

$$\cong \mathrm{Ind}_{\mathfrak{g}[t]+\mathbb{C}k}^{\hat{\mathfrak{g}}} U$$

② Another option. $\mathrm{Ind}_{t^N\mathfrak{g}[t]+\mathbb{C}k}^{\hat{\mathfrak{g}}} \mathbb{C}$

Given these ingredients.

$$\mathrm{Coinv}_{V_k(\mathfrak{g})}(X, \{x_i\}, \{M_i\}) = (M_1 \otimes \cdots \otimes M_n) \Big/ \mathfrak{g}[X \setminus \{x_i\}] \cdot (M_1 \otimes \cdots \otimes M_n)$$

$$\cong \mathrm{Ind}^{\hat{\mathfrak{g}}}_{\mathfrak{g}[t]+\mathbb{C}k} V$$

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mero. functions on $X \setminus \{x_1, \dots, x_n\}$

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Given these ingredients:

- $\mathrm{Coinv}_{V_k(\mathfrak{g})}(X, \{x_i\}, \{M_i\}) = (M_1 \otimes \cdots \otimes M_n) / \mathfrak{g}[X \setminus \{x_i\}] \cdot (M_1 \otimes \cdots \otimes M_n)$

- $\mathrm{ConfB}_{V_k(\mathfrak{g})}(X, \{x_i\}, \{M_i\}) = (\ldots)^*$ $\ni \Phi^{M}_{x_1 \ldots x_n}$, mero functions on $X \setminus \{x_1 \ldots x_n\}$

Conformal blocks are used to construct correlation functions

$M_1 \boxtimes \cdots \boxtimes M_n$

$\downarrow$

X^n

A conformal block $\phi^{M}_{x_1 \ldots x_n}$ is an element in the fiber of $(M_1 \boxtimes \cdots \boxtimes M_n)^*_{(x_1, \ldots, x_n)}$

fiber of $M_{i,x} \cong M_i$

$\Rightarrow$ the sheaf $M_1 \boxtimes \cdots \boxtimes M_n$ has a flat connection (KZ connection: $\nabla = d + L_{-1} \otimes dz$

Virasoro generators

$\otimes M_n)$

Conformal blocks are used to construct correlation functions

$$\begin{array}{c} \mathcal{M}_1 \boxtimes \cdots \boxtimes \mathcal{M}_n \\ \downarrow \\ X^n \supset \Delta_{ij} \end{array}$$

fiber of $\mathcal{M}_{i,x} \cong M_i$

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$\Rightarrow$ the sheaf $\mathcal{M}_1 \boxtimes \cdots \boxtimes \mathcal{M}_n$ has a flat connection (KZ connection: $\nabla = d + \underset{\uparrow}{L_{-1}} \otimes dz$, $L_{-1} = \frac{\partial}{\partial x}$

Virasoro generator

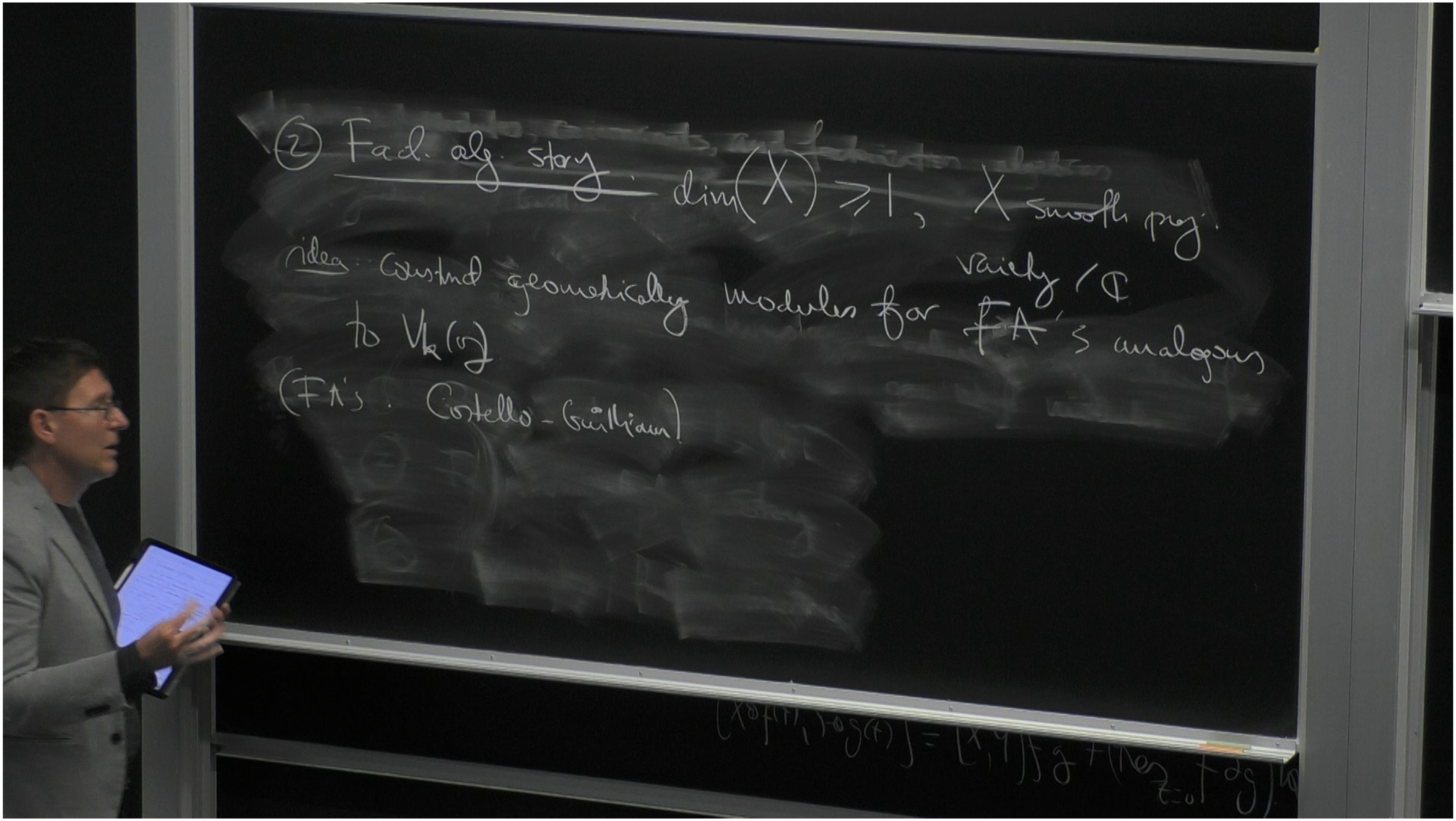
Simple but important result : Propagation of vacua

$$: C_{\mathrm{Coinv}_{V_k(\mathfrak{g})}}\big(X, \{x_i\} \cup \{y\}, \{M_i\} \cup \{V\}\big) \cong C_{\mathrm{Coinv}_{V_k(\mathfrak{g})}}\big(X, \{x_i\}, \{M_i\}\big).$$

② Fact. alg. story: $\dim(X) \geq 1$, X smooth proj. variety $/\mathbb{C}$

idea: Construct geometrically modules for FA's analogous to $V_k(\mathfrak{g})$

$[X \otimes f(t), Y \otimes g(t)] = [X, Y] fg + (\mathrm{Res}_{t=0} f\, dg)$


② Fact. alg. story : dim(X) ≥ 1, X smooth proj. variety /ℂ
idea: Construct geometrically modules for FA's analogous to V_k(g)
(FA's : Costello - Gwilliam)

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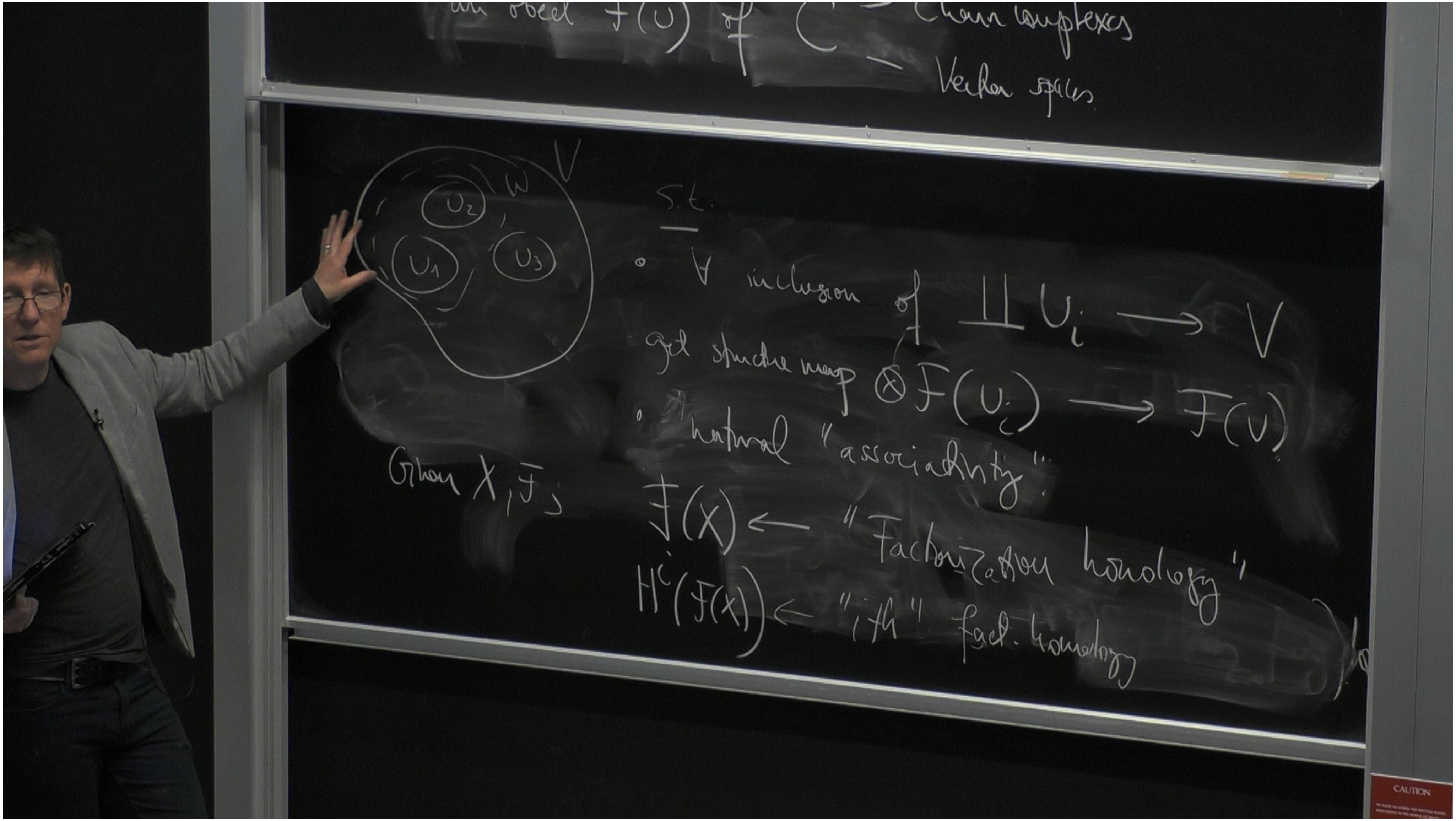
(FA's: Costello – Gwilliam)

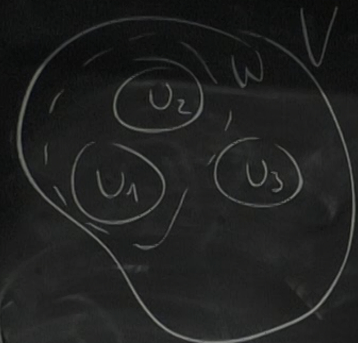
A fact. alg. $\mathcal{F}$ on X assigns to each open $U \subseteq X$ an object $\mathcal{F}(U)$ of $\mathcal{C}$ — chain complexes, vector spaces

① $\dim_{\mathbb{C}} X = 1$: (following F–BZ)


Chain complexes
Vector spaces
V
W
U₂
U₁
U₃
s.t.
∀ inclusion of ∐ Uᵢ ⟶ V
get structure map ⊗ F(Uᵢ) ⟶ F(V)
natural "associativity"
Given X, F,
F(X) ⟵ "Factorization homology"
Hⁱ(F(X)) ⟵ "i-th" fact. homology

an object $\mathcal{F}(U)$ of $\mathcal{C}$ — Chain complexes

— Vector spaces



s.t.

- $\forall$ inclusion of $\coprod U_i \longrightarrow V$

  get structure map $\bigotimes \mathcal{F}(U_i) \longrightarrow \mathcal{F}(V)$

- natural "associativity".

Given $X, \mathcal{F}$;

$\mathcal{F}(X) \leftarrow$ "Factorization homology"

$H^i(\mathcal{F}(X)) \leftarrow$ "i-th" fact. homology

One can construct such from a "nice" sheaf of Lie algebras $\mathfrak{g}$ - f.d. ss/$\mathbb{C}$

(1) $\mathfrak{g}_X^{dR} = (\mathfrak{g} \otimes \Omega^*_X, d_{dR})$

(2) $\mathfrak{g}_X^{Dol} = (\mathfrak{g} \otimes \Omega^{0,*}_X, \bar{\partial})$ + central ext.

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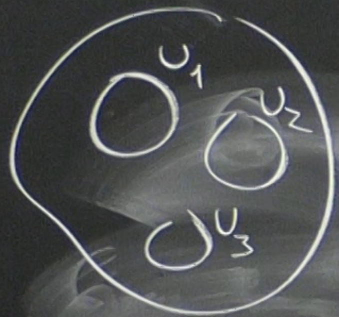
if $\mathcal{L}$ is (1) or (2), $\mathcal{L}_c \leftarrow$ compactly supported sections of $\mathcal{L}$

$$\mathcal{L}_c\left(\coprod U_i\right) = \bigoplus \mathcal{L}_c(U_i)$$

To get Fact alg, take Chevalley chains: $U \longrightarrow C_*(\mathcal{L}_c(U))$



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Relationship to VOA's: if $X = \mathbb{C}$

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If $X = \mathbb{C}$

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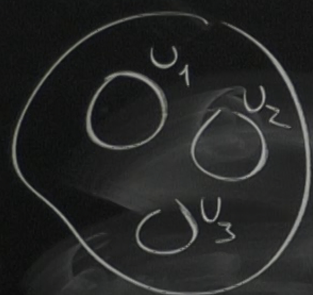

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Relationship to VOA's: if $X = \mathbb{C}$

then "nice" factorization algebras $\Longrightarrow$ Vertex algebras

$$V_{\mathcal{F}} = H^0(\mathcal{F}(\mathbb{C}))$$

In particular: $X = \mathbb{C}$, $\overset{\mathcal{F}}{\overset{\parallel}{C_X(\mathfrak{g}^{Dol}_X)}}$ $\rightsquigarrow$ $V_{\mathcal{F}} = V_k(\mathfrak{g})$

<u>Modules</u>: X – sm proj.

$Z \subseteq X$ closed subscheme

$M \in \mathrm{Rep}(\mathfrak{g})$

We construct a "generalized Weyl module" as follows:

In particular: $X = \mathbb{C}$, $\underset{\textstyle C_x(\mathfrak{g}^{Dol}_x)}{\overset{\mathcal{F}}{\|}} \rightsquigarrow V_{\mathcal{F}} = V_k(\mathfrak{g})$

Modules: X — sm proj.

$Z \subseteq X$ closed subscheme

$M \in \mathrm{Rep}(\mathfrak{g})$

We construct a "generalized Weyl module" as follows:

$W(M, Z_p)$: $\mathcal{E}^{\bullet} \longrightarrow \mathcal{O}_Z$ locally free resolution of $\mathcal{O}_Z$.

$\mapsto \mathcal{E}^{\bullet} \otimes \Omega^{p*}_X \otimes M.$

$\hookrightarrow \mathcal{E}\otimes\Omega^{p^*}_X\otimes M.$ of $\mathcal{O}_Z$.

$W(M,Z,p)$ is acted on by $\mathfrak{g}^{Dol}_X$.

We can form a "square zero extension" Lie alg.

$$\mathfrak{g}^M_X := \mathfrak{g}^{Dol}_X \ltimes \left(\bigoplus W(M_i, Z_i, p_i)[-\operatorname{codim} Z_i]\right)$$

We get a new factorization alg $C_*(\mathfrak{g}^M_{X,c}) = C_*(\mathfrak{g}^{Dol}_X, \mathrm{Sym}(W(M,Z)))$

$\mathcal{F}^M$ is multigraded. Each factor of $W(M_i, \mathbb{Z}_{(i)})$ is assigned deg $\mathbb{N}^n$ $(0, \ldots, \underset{\uparrow}{1}, \ldots, 0)$

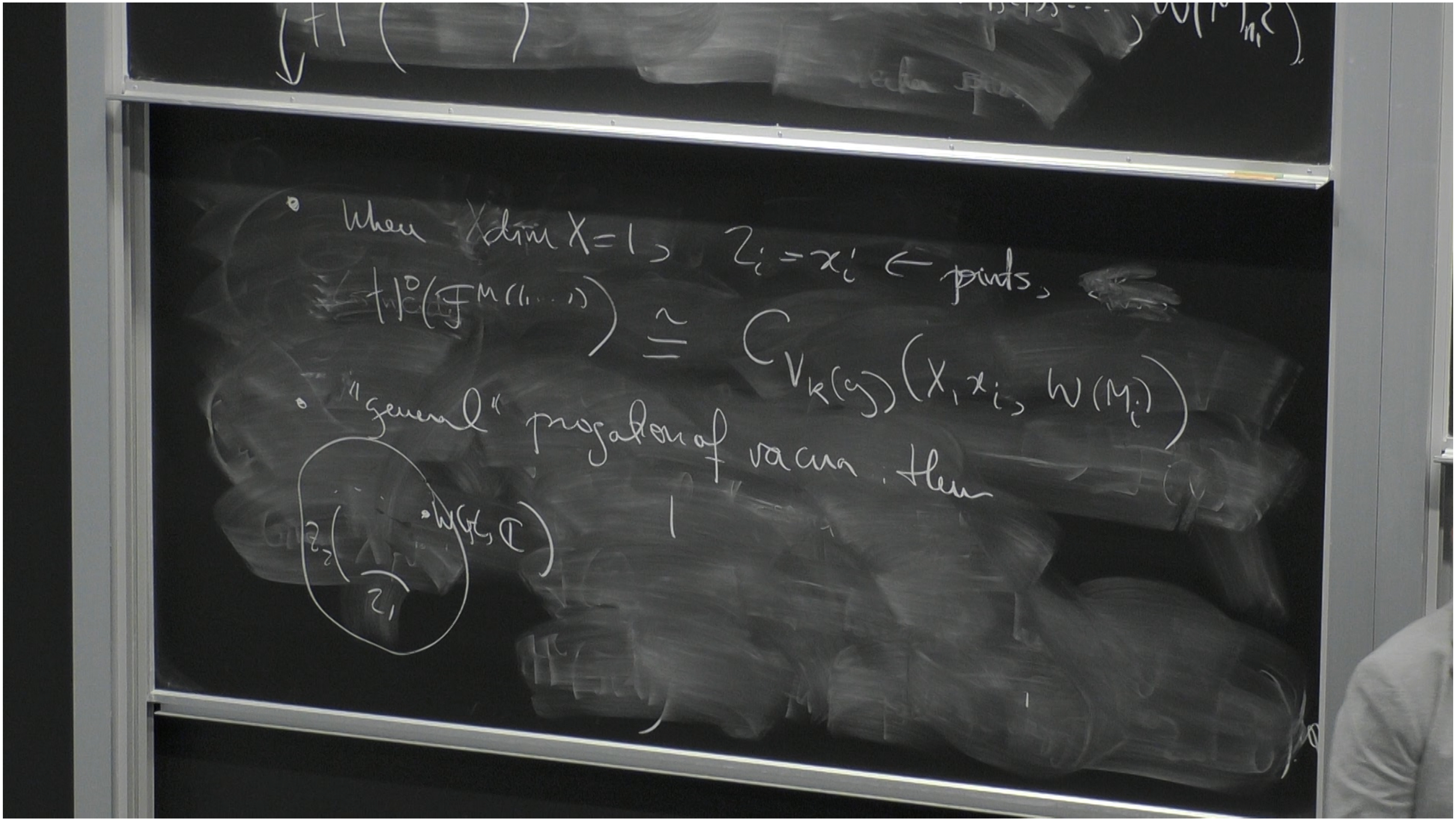
$\qquad\qquad\qquad\qquad\qquad\qquad$ ith spot

- $\mathcal{F}^M \leq (1, 1, \ldots, 1)$ is a fact. algebra

$\mathcal{F}^M$ is multigraded: each factor of $W(M_i; \mathcal{Z}_i, \gamma)$ is assigned deg $\mathbb{N}^n$ $(0, \dots, \underset{\substack{\uparrow \\ i\text{th spot}}}{1}, \dots, 0)$

- $\mathcal{F}^M_{\leq (1,1,\dots,1)}$ is a fact. algebra

- $(\mathcal{F}^M)_{(1,1,\dots,1)} \longleftarrow$ "factorization homology w. values in $W(M_1; \mathcal{Z}_1), \dots, W(M_n; \mathcal{Z})$"



When dim X = 1, z_i = x_i ← pints,
H^0(F^{M(1,...,1)}) ≅ C_{V_k(g)}(X, x_i, W(M_i))
"general" propagation of vacua: then

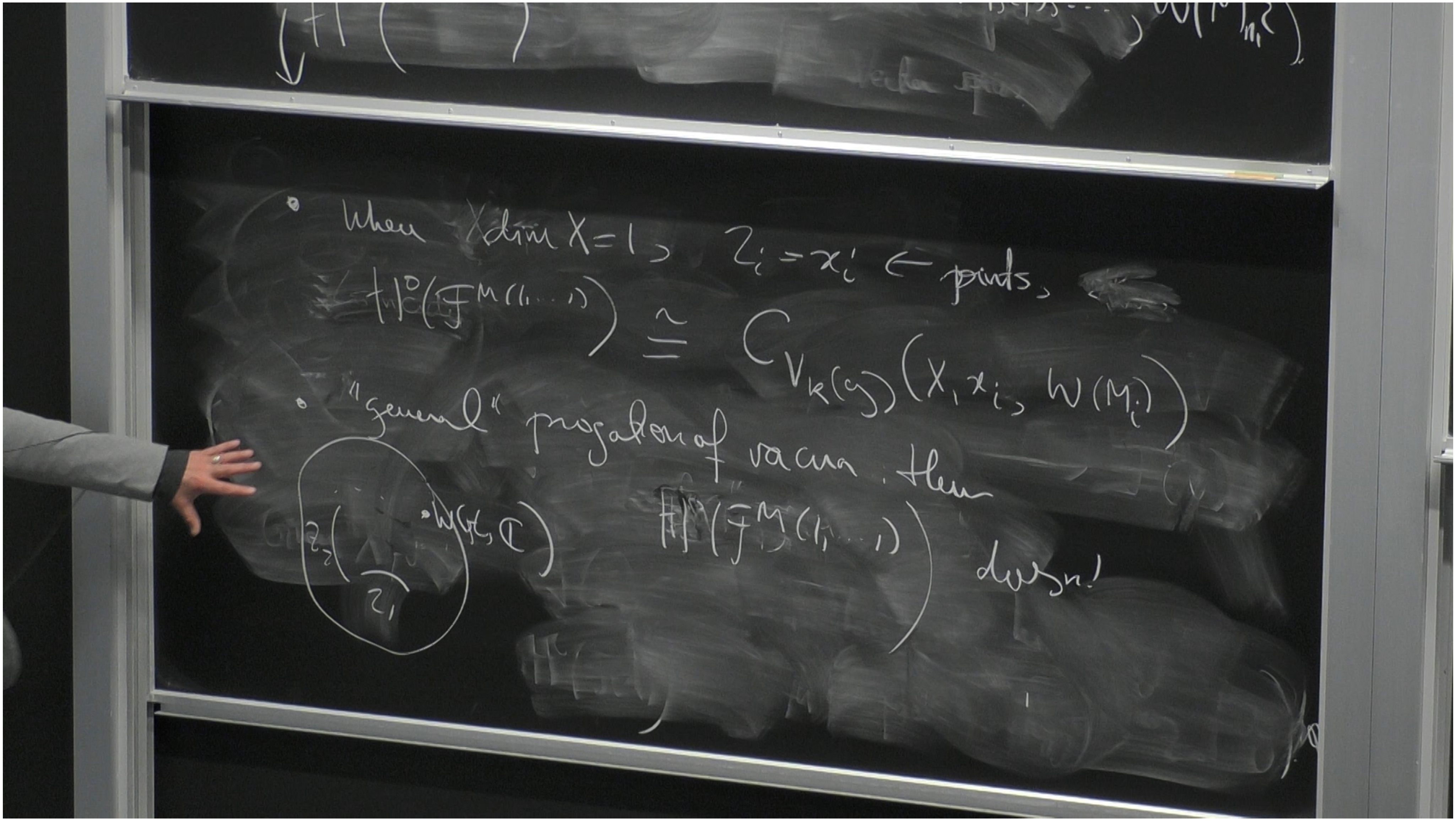
- When dim $X = 1$, $z_i = x_i \leftarrow$ points,

$$H^0(\mathcal{F}^{M}(1,\dots,1)) \cong C_{V_k(\mathfrak{g})}(X, x_i, W(M_i))$$

- "general" propagation of vacua: then

$z_2(\ \cdot W(\mathfrak{g}, \mathbb{C}))$ z_1

$H^0(\mathcal{F}^{M}(1,\dots,1))$ doesn't!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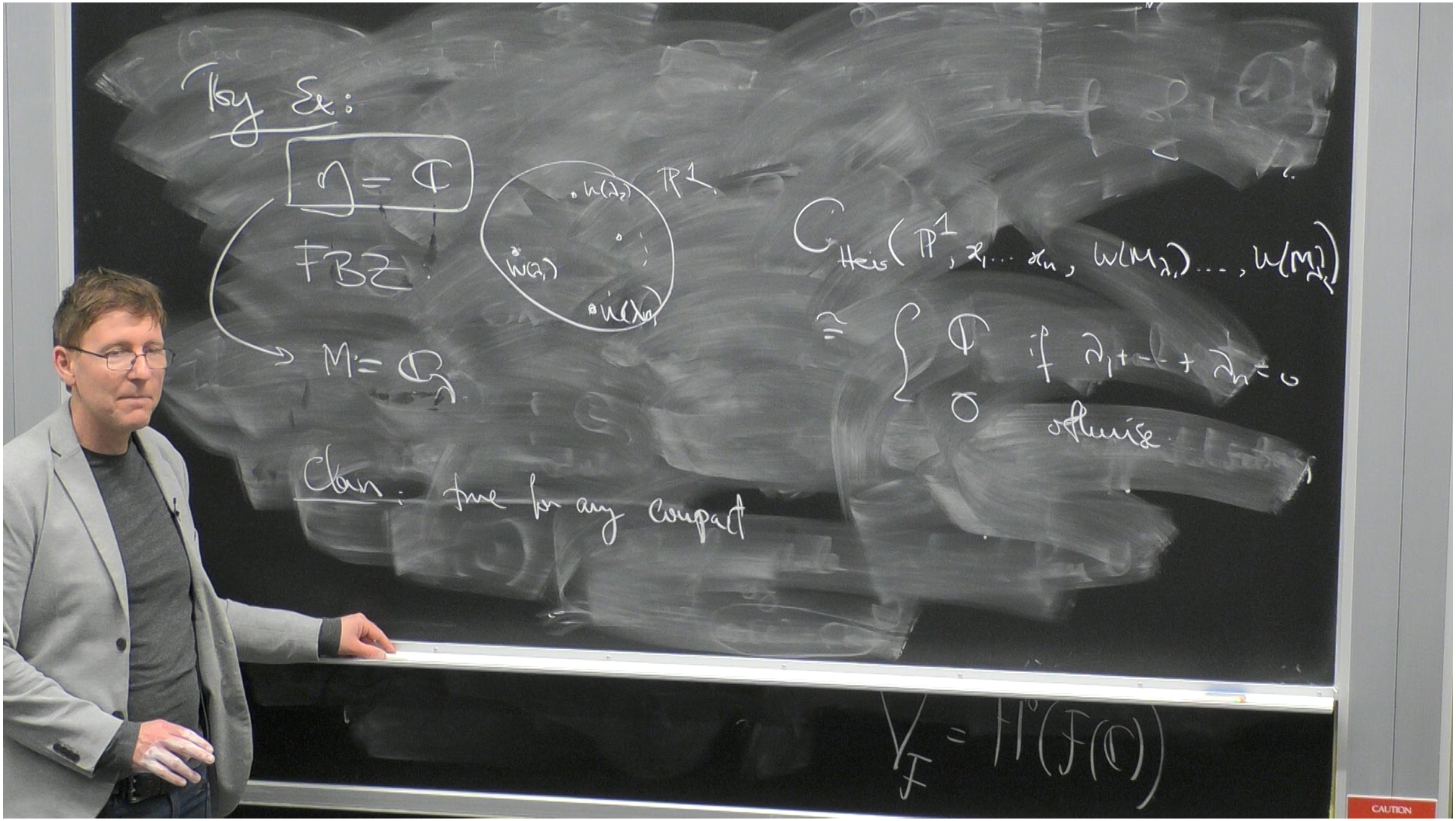
"general" propagation of vacua: then
W(g, C)
z_2
z_1
H^0(F^M(1,...,1)) dvgn!


Toy Ex:
FBZ
M = C
P1
Clm: true for any compact
otherwise