

Title: Asymptotically isometric codes

Speakers: Thomas Faulkner

Series: Quantum Fields and Strings

Date: May 23, 2023 - 2:00 PM

URL: <https://pirsa.org/23050022>

Abstract: I review a class of quantum error correcting codes that directly takes into account the large-N aspects of holographic theories. I will discuss some aspects of the vacuum sector of these codes and use them to show the equivalence between two different approaches to entanglement wedge reconstruction.

Zoom link: <https://pitp.zoom.us/j/96318197584?pwd=YXJEdkJrVktXVmI3SWVlRmlaK3A4Zz09>



Asymptotically Isometric Codes: 2211.12439
w/ Min Li

- Plan:
- summary of AIC
 - equivalence of QEC reconstruction & mod. flow "
 - Vacuum codes & the split [preliminary]

Math Pre-req's: \mathcal{X} , $L(\mathcal{X})$ \rightarrow type I_n von Neumann Alg.
 $n = \dim(\mathcal{X})$

* In finite dimensions we have norm



Aic



Lec16 x Lec17 x Review-2 x Aic x Lec18 x Lec19 x Lec20



Summary of AIC

- equivalence of QEC reconstruction & mod. flow
- Vacuum codes & the split [preliminary]

Math Pre-req's: \mathcal{X} , $L(\mathcal{X})$ → type I_n von Neumann Alg.

$$n = \dim(\mathcal{X})$$

* In finite dimensions we have norm

equivalence: $a \in B(\mathcal{X})$

$$\equiv \sqrt{\text{tr } a^* a}$$

$$\|a\| = \max \text{e-value } |a|$$



$$n = \dim(X)$$

✓ In finite dimensions we have norm

equivalence: $A \in B(X)$

$$\|A\| = \max \text{e-value } |a|$$

$$\equiv \sqrt{\lambda_{\max}}$$

$$\|A\|_1 = \text{tr } |A| \quad \|A\|_2 = \sqrt{\text{tr } |A|^2}$$

$$\|A\|_{\infty} = (\text{tr } |A|^p)^{1/p} \quad p \rightarrow \infty$$

$a_n \rightarrow 0$ in $\|A\|$ or $\|A\|_1$ or $\|A\|_2$

why? $\|A\|_{\infty} \leq \|A\|_2 \leq \|A\|_1 \leq (\dim)_X \|A\|_{\infty}$



$a_n \rightarrow 0$ in $\|a\|$ or $\|a\|_1$ or $\|a\|_2$

why?

$$\|a\|_\infty \leq \|a\|_2 \leq \|a\|_1 \leq (\dim x) \|a\|_\infty$$

* of course if we care about errors
as in how close two density matrices are

$$\|P_1 - P_2\|_1 \leq \epsilon$$

then we also often care how ϵ scales with

$(\dim x) \Rightarrow$ don't have norm equivalence

In ∞ dimensions we never have norm equiv.



$$L(\mathcal{H}) \rightarrow B(\mathcal{H}) \quad \text{where} \quad \|a\| < \infty$$

Diff Topol:

$$a_n \rightarrow a$$

$$\|a_n - a\| \rightarrow 0$$

$$\text{So } - a_n \rightarrow a$$

$$\|(a_n - a)\psi\| \rightarrow 0 \quad \forall \psi \in \mathcal{H}$$

$$\text{wo } - a_n \rightarrow a$$

$$\langle \psi | a_n - a | \psi \rangle \rightarrow 0 \quad \forall \psi, \psi \in \mathcal{H}$$

Norm \Rightarrow SO \Rightarrow WO But not other way around.

Von Neumann Algebra

$\mathcal{N} \subset B(\mathcal{H})$ subalgebra, closed

$$\text{under } \mathcal{N} = \overline{\mathcal{N}}^{\text{wo}} = \overline{\mathcal{N}}^{\text{so}}$$



Aic



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max

Kids day

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diff

$$SO - a_n \rightarrow a \quad \| (a_n - a) \psi \| \rightarrow 0 \quad \forall \psi \in \mathcal{H}$$

$$WO - a_n \rightarrow a \quad \langle \psi | a_n - a | \psi \rangle \rightarrow 0 \quad \forall \psi, \psi \in \mathcal{H}$$

Norm \Rightarrow SO \Rightarrow WO But not other way around.

Von Neumann Algebra

$\mathcal{N} \subset B(\mathcal{H})$ subalgebra, closed

$$\text{under } \mathcal{N} = \overline{\mathcal{N}}^{WO} = \overline{\mathcal{N}}^{SO}$$

$$\text{Also } \mathcal{N} = \mathcal{N}'' \quad \mathcal{N}' = \{ a \in B(\mathcal{H}) : [a, \mathcal{N}] = 0 \}$$

$$\mathcal{Z}(\mathcal{N}) = \mathcal{N} \cap \mathcal{N}' \text{ always } = \mathbb{C}I$$

Physics: • observables that can be approx



Aic



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$$w_0 - a_n \rightarrow a \quad \langle \phi | a_n - a | \psi \rangle \rightarrow 0 \quad \forall \psi, \phi \in \mathcal{H}$$

Norm \Rightarrow SO \Rightarrow W0 But not other way around.

Von Neumann Algebra

$\mathcal{N} \subset B(\mathcal{H})$ subalgebra, closed

$$\text{under } \mathcal{N} = \overline{\mathcal{N}}^{w_0} = \overline{\mathcal{N}}^{so}$$

$$\text{Also } \mathcal{N} = \mathcal{N}'' \quad \mathcal{N}' = \{ a \in B(\mathcal{H}) : \{a, \mathcal{N}\} = 0 \}$$

$$\mathcal{Z}(\mathcal{N}) = \mathcal{N} \cap \mathcal{N}' \text{ always } \mathcal{Z} = \mathbb{C}I$$

Physics: • observables that can be approx

in experiments $\mathcal{N} = \overline{\mathcal{N}}^{w_0}$, assuming idealized setting

• states: $\rho : \mathcal{N} \rightarrow \mathbb{C}$, $\rho(a^*a) \geq 0$



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$$H \leq 0 \quad N = N' \quad N = \sum_{\alpha \in \mathcal{B}(X)} \alpha \cdot \{ \alpha, N \} = 0$$

$$\mathbb{Z}(N) = N \cap N' \quad \text{always } \leq \mathbb{I}$$

Physics: • observables that can be approx

in experiments $N = \overline{N^{wo}}$, assuming idealized setting

• states: $\rho : N \rightarrow \mathbb{C}$, $\rho(\alpha) \geq 0$

$$\rho(1) = 1$$

$\rho \in N_*$ continuous on ultraweak topol.

$$\text{For } (\mathcal{B}(X))_* \rightarrow \text{tr } \rho(\cdot) \quad \|\rho\|_1 < \infty$$

$$\text{tr } \rho = 1, \quad \rho \geq 0$$

$$(M)_* \ni \rho|_M \quad (\text{like a partial trace})$$



For $(B(x))_k \rightarrow \text{tr } \rho(\cdot) \quad \|\rho\|_1 < \infty$
 $\text{tr } \rho = 1, \quad \rho \geq 0$

$(M)_k \ni \rho|_M$ (like a partial trace)

In quantum gravity there are reasons to expect experiments are limited at finite ℓ_P so we have to be especially careful about such seemingly technical things.

Also $\dim \mathcal{H} \sim e^{(bN)^2}$ so again careful!!



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have to be especially careful about such seemingly technical things.

Also $\dim \mathcal{L} \sim e^{(6N)^2}$ so again careful!!

Concretely study these issues in AdS/CFT
 \rightsquigarrow YM theory $SU(N)$ gauge thg.
 • CFT_N $6N \times 6N$ sphere S^5



On cylinder: $K \ni N \in \mathbb{N}$

$0 \rightarrow M_N(0) \subset B(K_N)$

AdS/CFT $O' = \{x \in K : x \perp 0\}$

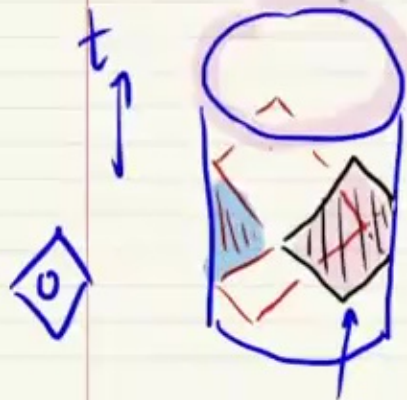
$M_N(0)$ satisfies reasonable



Concretely study these issues in AdS/CFT

~ YM theory $SU(N)$ gauge thg.

• CFT_N $g_N \times \mathbb{R}^4$ \neq S^1 \times \mathbb{R}^4 \neq S^1 \times \mathbb{R}^4



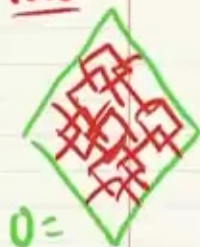
On cylinder: (K) $N \in \mathbb{N}$

$0 \rightarrow M_N(0) \subset B(K_N)$

AdFT $0' = \{x \in K : x \neq 0\}$

$M_N(0)$ satisfies reasonable

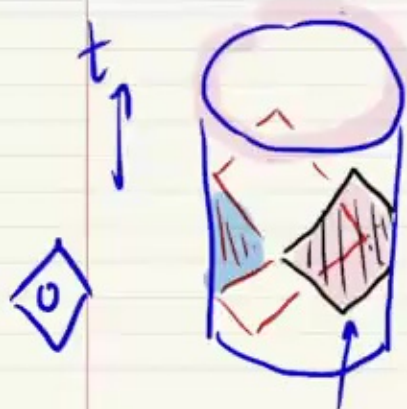
cover:



conditions. Complete: $M_N(0') = M_N(0)'$

(compare to: $\mathcal{X}_{Cyl} = \mathcal{X}_A \otimes \mathcal{X}_{\bar{A}}$)

Additive $M_N(0) = \bigvee M_N(\diamond)$ locally



On cylinder: $(K) \quad N \in N$

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A&FT $0' = \{x \in K : x \perp 0\}$

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cover:



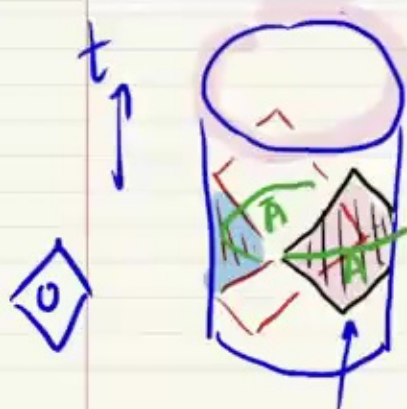
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(compare to: $\mathcal{X}_{Cyl} = \mathcal{X}_A \otimes \mathcal{X}_{\bar{A}}$)

Additive $M_N(0) = \bigvee M_N(\diamond)$ locally generated.
 \diamond cover of 0

type III, hyperfinite factor

- would like to model bulk in $N \rightarrow \infty$ limit



On cylinder: (K) $N \in N$

$0 \rightarrow M_N(0) \subset B(K_N)$

A&FT $0' = \{x \in K : x \perp 0\}$

$M_N(0)$ satisfies reasonable

cover:



conditions. Complete: $M_N(0') = M_N(0)'$

(compare to: $\mathcal{X}_{cyl} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$)

Additive $M_N(0) = \bigvee_{\diamond \text{ cover of } 0} M_N(\diamond)$ locally generated.

type III, hyperfinite factor

- would like to model bulk in $N \rightarrow \infty$ limit



A&FT $O' = \{x \in K : x \perp 0\}$
 $M_N(O)$ satisfies reasonable

cover:



conditions. Complete: $M_N(O') = M_N(O)'$
 (compare to: $\mathcal{X}_{(y)} = \mathcal{H}_A \otimes \mathcal{H}_{A'}$)

$O =$

Additive $M_N(O) = \bigvee M_N(\diamond)$ locally generated.
 \diamond cover of O

type III, hyperfinite factor

- would like to model bulk in $N \rightarrow \infty$ limit so semiclassical limit

$N = \infty$ theory is a theory of the finite



cover:



0 =

conditions. Complete: $M_N(o') = M_N(o)'$

(compare to: $X_{Cyl} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$)

Additive $M_N(o) = \bigvee_{\diamond \text{ cover of } o} M_N(\diamond)$ locally generated.

type III, hyperfinite factor

- would like to model bulk in $N \rightarrow \infty$ limit so semiclassical limit

$N = \infty$ theory is a theory of the single trace fields



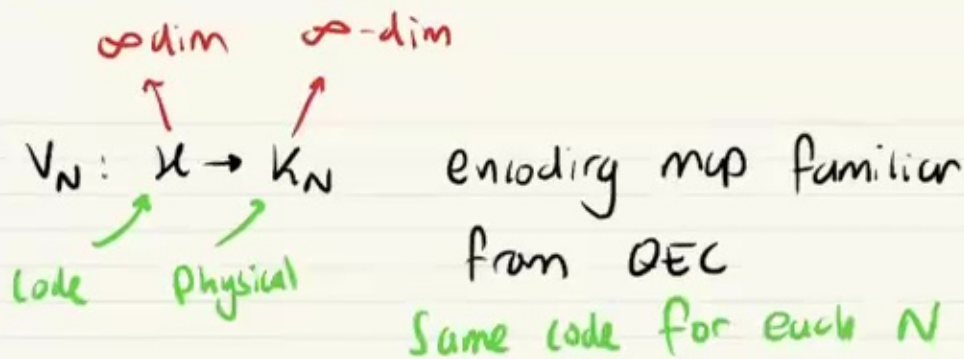
$N = \infty$ theory is a theory of the single trace fields

Say $+ \frac{1}{N}$ pert. expansion.

formal pert. expansion $\mathbb{C}[1/N]$ ring.

Here: understand the $N = \infty$ theory first!

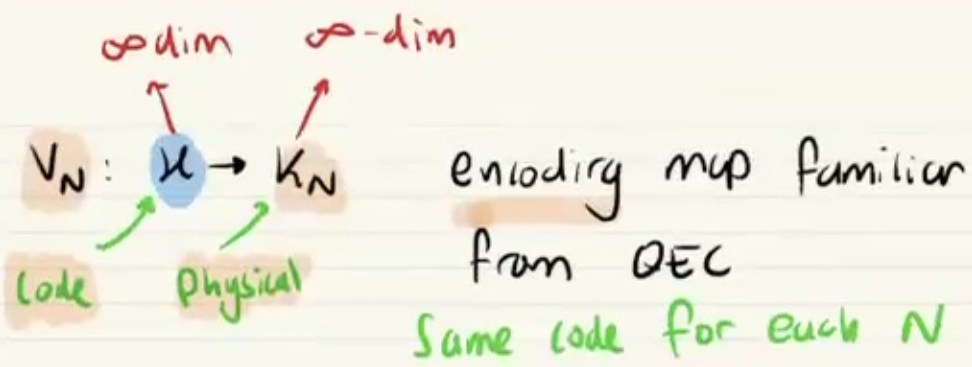
Use this to take the large- N limit.



$B(x)$ gen. by single trace fields



Use this to take the large-N limit.



$B(x)$ gen. by single trace fields.

$C(o)$ $C(B(x))$ gen additively by single trace fields in o .

Bdry Label

$\gamma_N: C(o) \rightarrow M_N(o)$ simple reconstruction map

~ Extrapolate Dictionary



AIC

Together these satisfy:

$$\lim_{N \rightarrow \infty} W_N \quad V_N^\dagger V_N = 1$$

$$\lim_{N \rightarrow \infty} \|\delta_N(c) V_N - V_N(c)\| \rightarrow 0 \quad \Rightarrow 0$$

$$\|V_N^\dagger V_N - 1\| \rightarrow 0$$

Physical motivation: fix obs. / experiments

& states:

$$W_\psi \equiv \langle \psi | \cdot | \psi \rangle \in \mathcal{B}(X)_*$$

then for sufficiently small ϵ_N expect code

to give good approx to this experiment.

(not allowed to change experiment with N)

Derived: from single trace fields

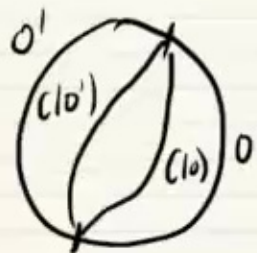


$$W_\psi \equiv \langle \psi | \cdot | \psi \rangle \in \text{BLN}^*$$

then for sufficiently small ϵ_N expect code
to give good approx to this experiment.
(not allowed to change experiment with N)

Derived: from single trace fields
following Liu, Leutheusser;
Witten

$C(0)$: non-trivial von Neumann algebra



Additive but not complete

$$\text{i.e. } C(0) \subset C(0)'$$

Haag dual violations.

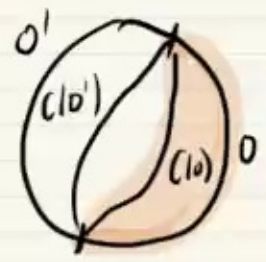


... for sufficiency small on expert case

to give good approx to this experiment.
(not allowed to change experiment with N)

Derived: from single trace fields
following Liu, Leutheusser;
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$\mathcal{C}(0)$: non-trivial von Neumann algebra



Additive but not complete

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Haag dual violations.

\mathcal{C}_I

fixed
Background
 $b_N = \infty$

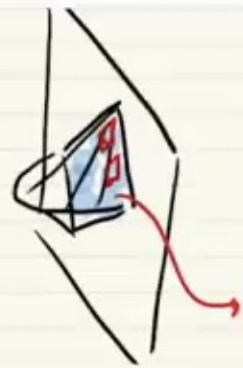


$\mathcal{C}(0)$: causally
complete bulk region

Additivity (also: constrained)



fixed
Background
 $b_N = \infty$



$(I_0) :$ causally complete bulk region
Additivity gen: constrained by bulk causality

$(I_0)' = (I_0')$

$(I_0) =$ causal completion of causal wedge

$J_+(I_0) \cap J_-(I_0)$

\sim HKLL, Modern: Time like Tube Theorem.

Entanglement Wedges:





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Kids play

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$I(\mathcal{O}) =$ causal completion of causal wedge

$$J_+(\mathcal{O}) \cap J_-(\mathcal{O})$$

\sim HKLL, Modern: Time like Tube Theorem.

Entanglement Wedges:



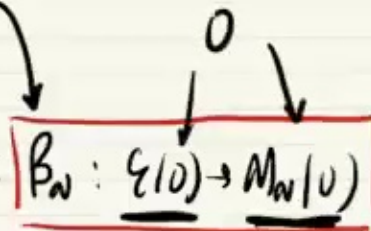
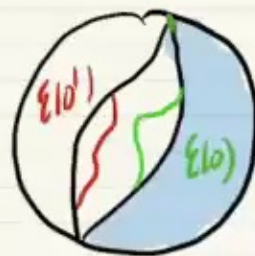
Extension of the causal wedge: $\beta_N : \mathcal{E}(\mathcal{O}) \rightarrow M_N(\mathcal{U})$

$$s \cdot \beta_N(n) \cdot v_N - v_N \cdot n = 0 \quad \forall n \in \mathcal{E}(\mathcal{O})$$

β_N : quantum channel, entanglement

~ HKLL, Modern: Time like Tube Theorem.

Entanglement Wedges:



Extension of the causal wedge:

$$B_N : E(10) \rightarrow M_N(U)$$

$$s.t. \quad s_0 - B_N(n) V_N - V_N n = 0 \quad \forall n \in E(10)$$

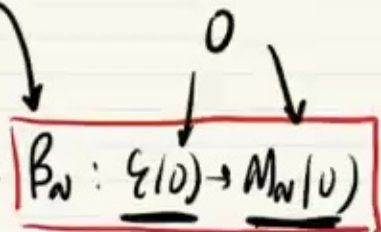
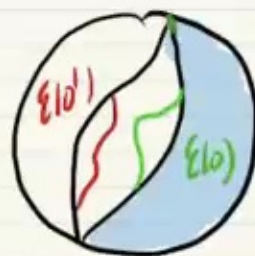
B_N : quantum channel, entanglement wedge reconstruction.

Theorem 1: ^{Bulk} maintains bdy causality





Entanglement Wedges:



Extension of the causal wedge:

st $s_0 - P_N(n) V_N - V_N n = 0 \quad \forall n \in E(0)$

P_N : quantum channel, entanglement wedge reconstruction.

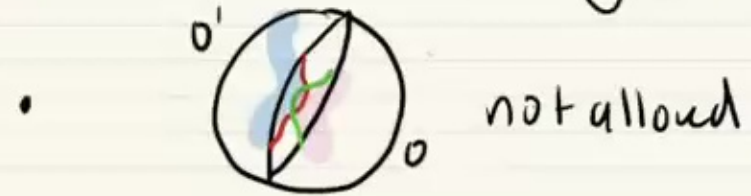
Theorem 1: ^{Bulk} maintains bdy causality



not allowed



Theorem 1: ^{351K} maintains bdy causality



• i.e. $\mathcal{E}(O) \subset \mathcal{E}(O')$ Einstein causality.

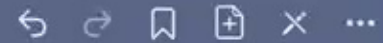
Theorem 2: Maximal extension (JLMS)

Equivalent conditions: $(1 \iff 2)$

1) $\mathcal{E}(O) = \mathcal{E}(O')$ Entanglement wedge

2) "V" $\mathcal{D}, \mathcal{E} \in B(X)$

$\lim_{\text{Sur}} (\rho_{\mathcal{D} \cup \text{Adv.}} | \mathcal{E} \cup \text{Adv.} : M_{\mathcal{D}}(O))$



- i.e. $\mathcal{E}(O) \subset \mathcal{E}(O)'$ Einstein causality.

Theorem 2: Maximal extension (JLMS)

Equiv. conditions: $(1 \Leftrightarrow 2)$

1) $\mathcal{E}(O) = \mathcal{E}(O)'$ Entanglement wedge

2) " \forall " $\rho, \sigma \in \mathcal{B}(X)$

$$\lim_N S_{rel}(\rho_{O \cap A_N} | \sigma_{O \cap A_N}; M_N(O)) \\ = S_{rel}(\rho | \sigma; \mathcal{E}(O))$$

Equiv. b/w QEC & Modular Flow Reconstruction

Equiv. b/w QEC & Modular Flow Reconstruction

* New result: Assume $\exists \eta \in \mathcal{H}$ s.t

jointly cyclic & sep for (ρ) & (ρ)

$$\overline{(\rho)|\eta\rangle} = \mathcal{X} \quad \begin{matrix} \rightarrow \\ \downarrow \end{matrix} \quad \begin{matrix} \langle \rho | \eta \rangle = 0 \\ \rho = 0 \end{matrix}$$

then we can add a 3rd condition
to thm 2

$$3) \quad \text{so-lim}_N \quad \sigma_N^S(\gamma_N(\rho)) V_N - V_N \sigma^S(\rho) = 0$$

Modular
Flow!

for $M_N(\rho); V_N \eta$

for $\mathcal{E}(\rho); \eta$



$C=0$

then we can add a 3rd condition to thm 2

3) $\text{so-lim}_N \sigma_N^S(\gamma_N(c)) V_N - V_N \sigma^S(c) = 0$

Modular Flow!

for $M_N(c); V_N \eta$

for $\mathcal{E}(c); \eta$

$\forall c \in C(c)$

Assumed in LM

$\sigma_N^{S(c)} = P_N^{i,j}(\cdot) P_N^{-i,j}$
Finite dimensions.

How? uses $\forall \sigma^S(c(c)) = \mathcal{E}(c)$
 $S \in \mathbb{R}$

Follows from joint cyclic & sep condition



Modular Flow!

for $M_N(t); V_N \eta$

for $\mathcal{E}(t); \eta$

$$\forall C \in C(t)$$

Assumed in LM

$$\sigma_N^{S(\cdot)} = \left(P_N^{i,j}(\cdot) \right) \left(P_N^{-i,j} \right)$$

Finite dimensions.

How? uses $\forall_{S \in R} \sigma_N^S(C(t)) = \mathcal{E}(t)$

Follows from joint cyclic & sep. condition

Requires ∞ dims. Nat in QFT.



A kind of operator ergodicity

Also a simple time like tube thm:



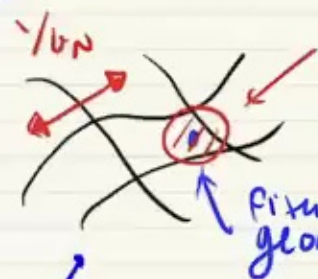
CFT mod flow is geometric



Vacuum Codes: (still preliminary)

Interactions:

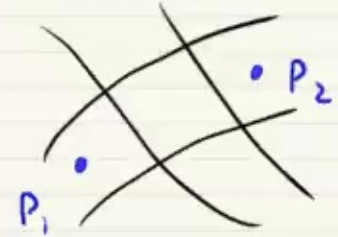
(classical) soln to gravity in ADS b/c



Quadratic: QFT fluctuations to fixed. $\omega \rightarrow 0$

fixed geometry

Phase space



multiple codes $\gamma_w(p_i)$

Open: how to patch together ($1/\omega$)

Now: Study one particular code. A: Vacuum ADS



Aic



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Lec17

Review-2

Aic

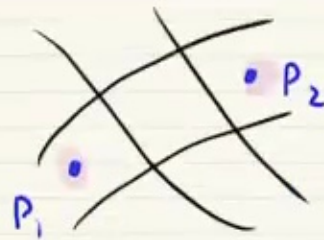
Lec18

max

Kids-pity

Lec19

Lec20



multiple codes

 $V_N(p_i)$ Open: how to patch together ($1/N$)Now: Study one particular code P : vacuum Ads.i.e. $\Omega_N \in K_N$ s.t.

$$|\Omega_N\rangle - V_N|\eta\rangle \rightarrow 0 \quad \text{some } \eta$$

bulk vacuum
state

\mathcal{K}_N furnishes a unitary rep. of $SO(d,2)$
 $V_N|\eta\rangle$ acting wr. on $M_N(0)$ & spectrum



\mathcal{K}_N furnishes a unitary rep. of $SO(d, 2)$
 $U_N(g)$ acting cov. on $M_N(0)$ & spectrum condition:

Theorem (Sketch):

Equivalent: $(1 \leftrightarrow 2)$

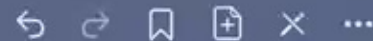
$$H_{sd-1} \geq 0$$

1) $\mathcal{E}(\diamond) = \mathcal{C}(\diamond)$ for diamonds

2) \mathcal{K} furnishes a unitary rep of $SO(d, 2)$
 $U(g)$ acting cov. on $\mathcal{C}(0)$ w/ spectrum condition:

$$\text{s.t.} \quad \mathfrak{so} - U_N(g) V_N - V_N U(g) = 0$$

$$U|\eta\rangle = |\eta\rangle$$



(2) \mathcal{U} finishes a unitary rep of $SO(d, 2)$
 $U(g)$ acting cov. on (h) w/ spectrum condition:

$$\text{S.t.} \quad SO - U_N(g) V_N - V_N U(g) = 0$$

$$U|\eta\rangle = |\eta\rangle$$

Why? Since mod. theory for vacuum gives:



(conformal boost.



combine to null translations

etc.

this is a simple form of bulk reconstruction:



Isometries of AdS background.

How about a non-trivial entanglement wedge?

Theorem (sketch):

Assuming: $\lim_N \text{Tr}_{k_N} e^{-\beta H_{\text{SH}}^{(N)}} = \text{Tr}_{\mathcal{H}} e^{-\beta H_{\text{SH}}}$

$T < T_{\text{HP}}$ (Hawking Page)

then:

i) $\mathcal{E}(\diamond_1, \vee \diamond_2) = \mathcal{C}(\diamond_1) \vee \mathcal{C}(\diamond_2)$

$\diamond_1 \xrightarrow{d} \diamond_2$ for $d > \# / T_{\text{HP}}$



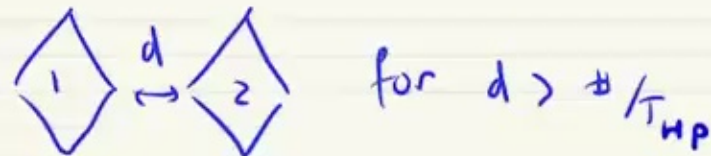
Assuming:

$$\lim_N \text{Tr}_{k_N} e^{-\beta H_{S_{N+1}}^{(N)}} = \text{Tr}_{\mathcal{H}} e^{-\beta H_{S_{N+1}}}$$

$$T < T_{HP} \text{ (Hawking Page)}$$

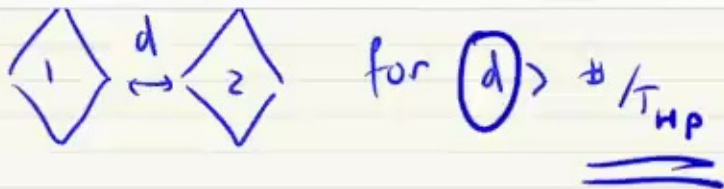
then:

$$i) \quad \mathcal{E}(\diamond_1 \vee \diamond_2) = \mathcal{L}(\diamond_1) \vee \mathcal{L}(\diamond_2)$$



ii) Bulk satisfies split property

$$\text{i.e. } \exists \text{ set } S \text{ s.t. } \omega_S(c_1, c_2) = \omega_S(c_1) \omega_S(c_2)$$



ii) Bulk satisfies split property

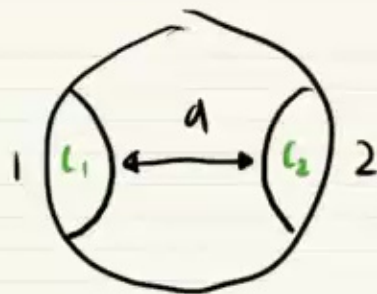


i.e. $\exists S \in \mathcal{K}$ s.t. $\underline{w_S(c_1, c_2)} = \frac{w_\eta(c_1)}{w_\eta(c_2)}$

\downarrow s.t. $\lim_N |S_N\rangle - V_N |S\rangle = 0$

iii) $\lim_N \mathbb{I}(0_1; 0_2)_{V_N \Psi} = \mathbb{I}(0_1; 0_2)_{\Psi}$

Expectations:

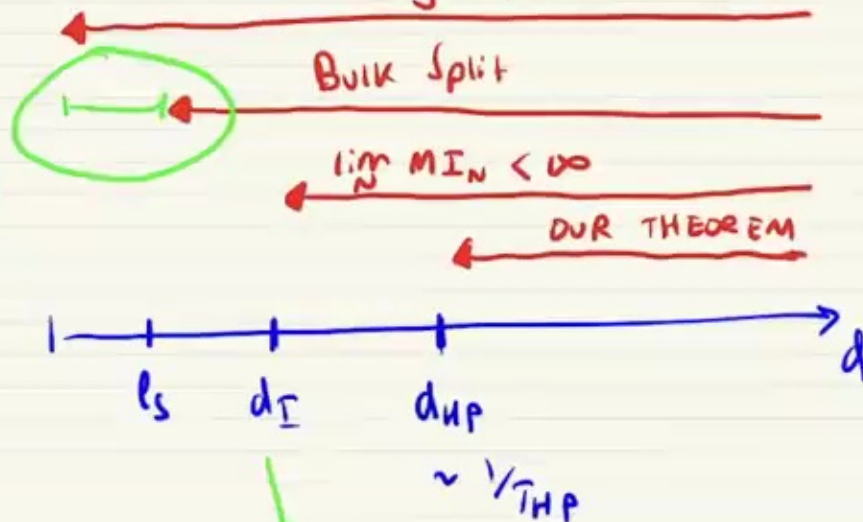


Bary Split $\forall d$

Bulk Split

$\liminf_N M_{IN} < \infty$

DVR THEOREM



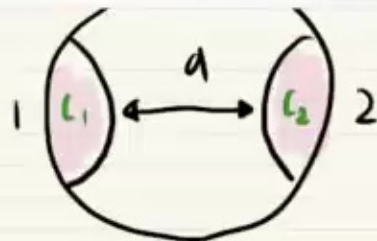
Hage Aorn

Diverge in
bulk. Condition
for bulk

Mutual Information
Transition



Expectations:

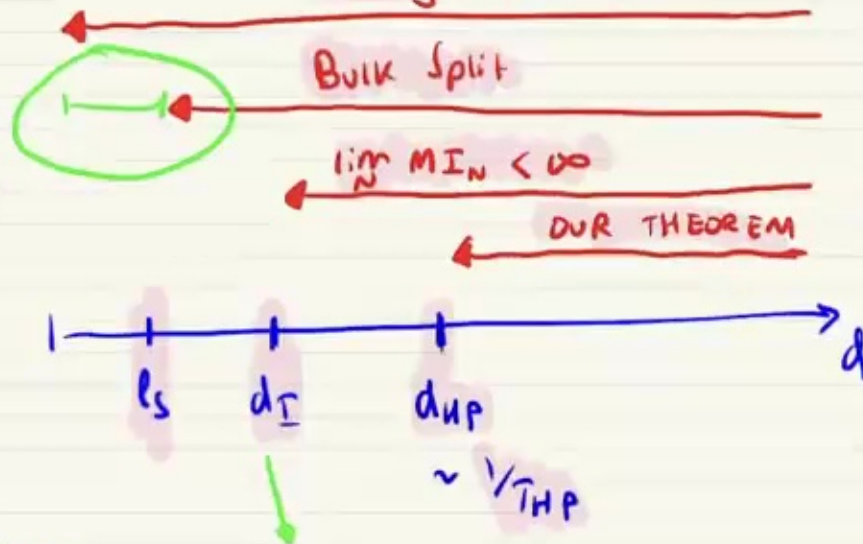


Bary Split $\forall d$

Bulk Split

$\lim_N M_{IN} < \infty$

DUR THEOREM



Hageorn

Divergence in
bulk. condition

for bulk
locality

Mutual Information

Transition





Huye Aorn

Divergence in
bulk. Condition
for bulk
locality

$l_s \rightarrow 0$

Mutual Information

Transition



$E_{12} = E_1 \vee E_2$

Thanks!