Title: Relativistic superfluids, and the connection between finite density and spontaneous symmetry breaking for interacting scalar fields

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Series: Particle Physics

Date: May 02, 2023 - 11:00 AM

URL: https://pirsa.org/23050021

Abstract: We study the low-energy effective action for relativistic superfluids obtained by integrating out the heavy fields of a UV theory. A careful renormalization procedure is required if one is interested in deriving the EFT to all orders in the light fields (at a fixed order of derivatives per field). The result suggests a general relation between finite density and spontaneous symmetry breaking for QFTs of interacting scalars with an internal global symmetry. The ground state at finite chemical potential of these systems is usually associated with a superfluid phase, in which the global symmetry is spontaneously broken along with Lorentz boosts and time translations. We show that this expectation is always realized at one loop for complex scalar fields with arbitrary UV potential in d > 2 spacetime dimensions. The physically distinct phenomena of finite charge density and spontaneous symmetry breaking occur simultaneously. We quantify this result by deriving universal scaling relations for the symmetry breaking scale as a function of the charge density, at low and high density. Moreover, we show that the critical value of ? coincides with the pole mass. The same conclusions hold non-perturbatively for an O(N) theory with quartic interactions in d = 3, at leading order in the 1/N expansion. In order to do this we compute analytically the one-loop effective potential at finite ? and zero temperature. As an application we derive in closed form the one-loop EFT for superfluid phonons for arbitrary UV scalar potentials in d > 2. From this we obtain analytically the one-loop scaling dimension of the lightest charge n operator in the ϕ^6 conformal superfluid in d=3, at leading order in 1/n, reproducing a numerical result of Badel et al.

Zoom Link: https://pitp.zoom.us/j/97727675289?pwd=TWJyNzRucEhkM0JNM0NWeEdMM0xTQT09

Relativistic superfluids

and

The connection between finite density and SSB for interacting scalar fields

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Introduction

- The superfluid phase is characterized by the spontaneous breaking of time translation and an internal $\mathrm{U}(1)$ symmetry to the diagonal combination: $(H-\mu Q)|\Omega_\mu\rangle = 0$
- The system in the state $|\Omega_\mu\rangle$ has a finite density for the $\mathrm{U}(1)$ charge:
  - it defines a preferred reference system: rest frame
  - as a consequence boost invariance is spontaneously broken

  see e.g. Nicolis, Piazza- JHEP 06 (2012) 025

- So called "inverse Higgs constraints" ensure that a single Goldstone field can realise non-linearly the full set of broken symmetries

Ivanov, Ogievetsky '75 + Low, Manohar 2001 + [...] Nicolis, Penco, Piazza, Rosen 2013

Introduction

- Superfluids are interesting physical systems for many reasons
  - they describe a fascinating phase of condensed matter systems
  - Relativistic Superfluids can describe:
    1. cosmological fluids ("ghost condensate/inflation") or exotic DM phases.
    2. finite density phases of QCD (in the core of neutron stars and pulsars).
  - they are related to the large charge sector of CFTs (state/operator correspondence)

Hellerman et al. '15 + Monin et al. '17 + [...]

- We will consider **relativistic** superfluids at **zero temperature** in 3+1 D
  - the non-relativistic limit can be taken safely at the end by reintroducing units of c and taking the limit $c \to \infty$

The Superfluid EFT

- The most general low energy effective action is

$$S_{\text{eff}}[\theta] = \int d^4x \, P(X) + \text{higher derivative}\,, \qquad X = \partial_\mu \theta \, \partial^\mu \theta$$

D.T. Son '02

chemical potential ↗

- $\theta(t, x) = \mu t + \pi(t, x)$, where θ is a Poincaré scalar:

  it realizes non-linearly $\mathrm{U}(1)$ and time translations as $\pi \to \pi + \text{const}$

- π is the superfluid phonon field

- The superfluid phonon EFT is obtained expanding around $X \simeq \mu^2$

- The phonon EFT expansion is controlled by $\dfrac{\partial}{m_{UV}}, \dfrac{\partial}{\mu}$, but μ can be large

- we would like to know $P(X)$ to all orders in X

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The Superfluid EFT

- Compare with the Lorentz invariant theory for a $U(1)$ Goldstone

$$\mathscr{L} = \frac{1}{2}(\partial\pi)^2 + \frac{c}{f_\pi^4}(\partial\pi)^4 + \dots$$

  - Taylor expansion in $Y = (\partial\pi)^2$ around $Y \simeq 0$
  - at low energy and with a finite experimental accuracy only a finite number of terms are needed

Matching in EFT

- Path integral approach: integrate out heavy fields

$h(x)$: heavy $\qquad \ell(x)$: light $\qquad e^{iS_{\rm eff}[\ell]} \equiv \int Dh\, e^{iS[h,\ell]}$

  - correlation functions of light fields computed with $S_{\rm eff}$ match those computed with S

$$\langle \ell(x_1)\ldots\ell(x_n)\rangle = \frac{\int D\ell Dh e^{iS[h,\ell]}\ell(x_1)\ldots\ell(x_n)}{\int D\ell Dh e^{iS[h,\ell]}} = \frac{\int D\ell\, e^{iS_{\rm eff}[\ell]}\ell(x_1)\ldots\ell(x_n)}{\int D\ell\, e^{iS_{\rm eff}[\ell]}}$$

- In general we don't know how to perform $\int Dh$ for arbitrary configurations

↓

derivative expansion / low energy EFT

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Matching in EFT

- The low energy expansion changes the structure of UV divergencies
  - heavy propagator:

$$\frac{i}{q^2 - M^2 + i\varepsilon} \to -\frac{i}{M^2}\left(1 + \frac{q^2}{M^2} + \frac{q^4}{M^4} + \ldots\right)$$

- How to "integrate out" *beyond tree level*?
  - usually one performs a matching computation

$$\langle \ell(x_1)\ldots\ell(x_n)\rangle_{\text{EFT}} = \langle \ell(x_1)\ldots\ell(x_n)\rangle_{\text{full}}$$

for as many correlates as needed to fix the free parameters at the desired order in the derivative expansion

All orders matching

- But we would like to compute $P(X)$ to all orders in X
  - possible strategy (first proposed by Georgi): choose S_{eff} such that

$$\Gamma_{\text{EFT}}[\ell] = \Gamma_{\text{full}}[\ell]$$

$$\left(\int D\delta\ell \, e^{iS_{\text{eff}}[\ell+\delta\ell]} \right)_{\text{1PI}} = \left(\int DhD\delta\ell \, e^{iS[h,\ell+\delta\ell]} \right)_{\text{1PI for } \delta\ell}$$

- We will carry out this computation at one-loop for a relativistic field theory

at first order in derivatives (one per field) but *all orders* in the light field

↓

derive the $P(X)$ effective lagrangian from an explicit UV completion

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An ultraviolet model

- Simplest weakly coupled UV completion of a superfluid:
  - complex scalar field with $\mathrm{U}(1)$ symmetry

$$\mathscr{L} = |\partial\Phi|^2 - m^2|\Phi|^2 - \lambda|\Phi|^4, \qquad \Phi = \frac{1}{\sqrt{2}}\rho(x)e^{i\theta(x)}$$

$$\theta(t,x) = \mu t + \pi(t,x)$$

  - if $m^2 > 0$: spontaneous symmetry breaking and finite density for $\mu > m$
  - if $m^2 < 0$: SSB always and finite density for $\mu > 0$

θ : light degree of freedom (Goldstone boson)

ρ : heavy degree of freedom (radial mode)

Matching at one-loop

- In polar coordinates the action takes the form

$$S[\rho,\theta] = \int \mathrm{d}^4x \left(\frac{1}{2}(\partial\rho)^2 + \frac{1}{2}\rho^2(\partial\theta)^2 - \frac{1}{2}m^2\rho^2 - \frac{1}{4}\lambda\rho^4 \right)$$

  - the action depends on θ only through $X = (\partial\theta)^2$
  - integrating out ρ we obtain an expansion in $X, \partial X, \ldots$
  - working at zeroth order in derivatives is analogous to a Coleman-Weinberg computation

- In dim-reg the path integral measure turns out to be invariant

$$D\Phi\, D\Phi^* = D\rho\, D\theta \exp\left(\cancel{\delta^{(4)}(0)} \int \mathrm{d}^4x \log(\rho) \right) = D\rho\, D\theta$$

Matching at one-loop

- One-θ irreducible effective action in the *full theory*

$$\theta(x) \to \theta(x) + \pi(x)$$

$$e^{i\Gamma_{\rm full}[\theta]} = \left(\int D\rho D\pi \, e^{iS[\rho,\theta+\pi]}\right)_{\rm 1PI\ for\ \pi}$$

- At tree level: $\Gamma^{\rm (tree)}_{\rm full}[\theta] = S[\rho_0(x), \theta], \qquad X = (\partial\theta)^2$

  - $\rho_0(x)$: solution to the classical equations of motion
  - $\left(\Box + m^2 - X\right)\rho_0 + \lambda\rho_0^3 = 0$
  - zeroth order in derivatives of X : $\cancel{\rho_0 = 0}, \qquad \rho_0^2 = \dfrac{X - m^2}{\lambda} \qquad \left(\text{for } X > m^2\right)$

$$\Gamma^{\rm (tree)}_{\rm full}[\theta] = \int {\rm d}^4x \frac{\left(X - m^2\right)^2}{4\lambda}$$

Babichev et al. '18

Creminelli et al. '19

Matching at one-loop

- At one-loop level: perturb about the saddle point and keep only quadratic terms

$$\rho = \rho_0 + h(x) \qquad\qquad \theta(x) \to \theta(x) + \pi(x)$$

$$S_2[h, \pi] = \int d^4x \left(\frac{1}{2}(\partial h)^2 + \frac{1}{2}\rho_0^2\,(\partial \pi)^2 + 2V^\mu\, h\partial_\mu \pi - \frac{1}{2}m_{\rm eff}^2 h^2 \right) ,$$

- $m_{\rm eff}^2 = 2(X - m^2), \qquad V_\mu = \rho_0 \partial_\mu \theta, \qquad \partial_\mu \theta = \text{const}$

- normalizing canonically and completing the square for h :

$$\int DhD\pi\, e^{iS_2} = \int Dh'D\pi'\ \exp \frac{i}{2}\left[-h'(\Box + m_{\rm eff}^2)h' - \pi'\left(\Box + 4\partial_\mu\theta\partial_\nu\theta \frac{\partial^\mu\partial^\nu}{\Box + m_{\rm eff}^2} \right)\pi' \right]$$

Matching at one-loop

- computing the determinant we obtain the effective action in the *full theory*

$$\Delta\Gamma^{(1\,\text{loop})}_{\text{full}}[\theta] = \int d^4x \Big[-\frac{1}{32\pi^2(d-4)} \left(m_{\text{eff}}^4 + 2m_{\text{eff}}^2 X + 2X^2\right) +$$

$$+\frac{1}{384\pi^2}\Big(9m_{\text{eff}}^4 + 18m_{\text{eff}}^2 X + 10X^2 - 6(m_{\text{eff}}^4 + 2m_{\text{eff}}^2 X + 2X^2)\log\frac{m_{\text{eff}}^2}{\bar{\mu}^2} +$$

$$-10\frac{X^3}{m_{\text{eff}}^2} f\left(-4X/m_{\text{eff}}^2\right)\Big)\Big]$$

where: $f(z) = {}_3F_2\left(1,1,\frac{7}{2};4,5;z\right)$

- the ordinary $\overline{\text{MS}}$ countqerterms $\delta m^2, \delta\lambda, \delta\Lambda_{\text{cc}}$ cancel all divergencies as expected.

Matching at one-loop

- We need to match this with the quantum effective action of the *effective theory*

$$e^{i\Gamma_{\rm EFT}[\theta]} = \left(\int D\pi\, e^{iS_{\rm eff}[\theta+\pi]}\right)_{\rm 1PI} = e^{iS_{\rm eff}[\theta]}\int D\pi\, e^{iS_2[\pi]}$$

where $S_2[\pi] = \int \mathrm{d}^4x \frac{1}{2} Z^{\mu\nu}\partial_\mu\pi\partial_\nu\pi$ is the quadratic action for π

and $Z^{\mu\nu} = 2P'(X)\eta^{\mu\nu} + 4P''(X)\partial^\mu\theta\partial^\nu\theta$

- $Z^{\mu\nu}$ is constant for constant $\partial_\mu\theta$ and the determinant vanishes in dim-reg so that

$$\Gamma_{\rm EFT}[\theta] = S_{\rm eff}[\theta] = \int \mathrm{d}^4x\, P(X)$$

at first order in derivatives of θ and at one-loop in dimensional regularization

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The one-loop effective action

- We arrive at the result

$$m_{\text{eff}}^2 = 2(X - m^2)$$

$$P(X)\Big|_{\mathrm{U}(1)} = \frac{\left(X - m^2\right)^2}{4\lambda} +$$

$$+\frac{1}{384\pi^2}\Big[-8X^2 + (m_{\text{eff}}^4 + 2m_{\text{eff}}^2 X + 2X^2)\big(9 - 6\log(m_{\text{eff}}^2/\bar{\mu}^2)\big) - 10\frac{X^3}{m_{\text{eff}}^2}\, f\left(-\frac{4X}{m_{\text{eff}}^2}\right)\Big]$$

- $P(X)$ for $X = \mu^2$ is associated with the *equation of state* of the superfluid:

should be renormalization scale independent

↓

expressing the result in terms of on-shell couplings the $\bar{\mu}$ dependence cancels non-trivially

$O(N)$ model

- A different (more general) UV completion is the theory of
  - N real scalars $\Psi_a(x)$, $a = 1,\ldots,N$ with $O(N)$ symmetry

$$\mathscr{L} = \frac{1}{2}|\partial\vec{\Psi}|^2 - \frac{1}{2}m^2|\vec{\Psi}|^2 - \frac{1}{4}\lambda|\vec{\Psi}|^4$$

  - Introducing a chemical potential for *one* of the $SO(N)$ charges we get a superfluid state
  - massless Goldstone + radial mode + $(N-2)$ gapped Goldstones with fixed gap μ

Nicolis, Piazza '13

- Working in cylindrical coordinates after a similar computation we get

$$P(X)\Big|_{O(N)} = P(X)\Big|_{\mathrm{U}(1)} + \frac{1}{384\pi^2}(N-2)\,X^2\left(9 - 6\log\frac{X}{\bar{\mu}^2}\right)$$

Some physical consequences

- In the high density limit, $X \gg m^2_{\text{pole}}$, we can use the RG to resume the logs

$$P(X) \simeq \frac{X^2}{4\,\lambda(2X)}\,, \qquad \bar{\mu}\frac{\mathrm{d}\lambda}{\mathrm{d}\bar{\mu}} = \left(\frac{20+2(N-2)}{16\pi^2}\right)\lambda^2$$

- the sound speed is given by

$$c_s^2 = \frac{P'(X)}{P'(X)+2X\,P''(X)} = \frac{1}{3} + \frac{1}{9}\left(\frac{20+2(N-2)}{16\pi^2}\right)\lambda(2X)$$

- same result can be derived from the trace anomaly of the stress-energy tensor:

$T_{\mu\nu} = 2P'(X)\partial_\mu\theta\,\partial_\nu\theta - \eta_{\mu\nu}P(X)$, so that $\varrho = 2P'(X)X - P(X)$, $p = P(X)$

on the other hand, for our $P(X)$: $T^\mu{}_\mu = -\,\beta_1\lambda(2X)P(X)$. Differentiating wrt X it follows.

- sound speed can been larger than the well-known relativistic 1/3

Some physical consequences

- For positive m^2, at the classical level, SSB occurs for $\mu > m$.

  What happens at 1-loop?

  - stability and subluminality of the configuration with $X = \mu^2$ implies

$$P'(X) > 0, \qquad P''(X) > 0, \qquad \Longrightarrow \; X > m^2 \left(1 - \frac{\lambda}{16\pi^2}(N+2)\right) = m^2_{\text{pole}}$$

- Low density limit $X \to m^2_{\text{pole}}$: the density is of order $(X - m^2_{\text{pole}})$

↓

It looks like a physical threshold. Relation between SSB and finite density?

Finite density vs Spontaneous Symmetry Breaking

- Consider a complex scalar Φ with U(1) internal symmetry.
- Consider the ground state of finite density for the scalar current $J_\mu = \Phi^\dagger \overleftrightarrow{\partial}_\mu \Phi$:

$$\langle \Phi^\dagger \overleftrightarrow{\partial}_0 \Phi \rangle \neq 0 \, .$$

Is it possible to have $\langle \Phi \rangle = 0$?

- For free bosons: the ground state at finite density always exhibits SSB (BEC)
- We are not aware of any proof that this has to be the case in the presence of interactions

  - in fact in 1+1 D Bose/Fermi duality gives counterexamples

  - in 2+1 D bosonic systems can have fermionic behavior if coupled to Chern-Simons terms

Minwalla et al. '20

Scalar fields at finite μ

- We consider a complex scalar in d > 2 space-time dimensions

$$\mathscr{L} = |\partial\Phi|^2 - V(\phi), \text{ where } \phi = |\Phi|$$

$$V(\phi) = m^2\phi^2 + V_{\text{int}}(\phi)$$

- We want now to compute one-loop effective action at finite μ

$$\mathscr{H}_\mu = \mathscr{H} - \mu J^0 \longrightarrow \text{project on ground state with } i\varepsilon \text{ term } \; -i\varepsilon\mathscr{H}_\mu$$

  - Going back to the lagrangian formalism we get

$$\mathscr{L}_\mu = \frac{1}{2}(\partial_\nu\varphi_1)^2 + \frac{1}{2}(\partial_\nu\varphi_2)^2 + \mu\left(\dot{\varphi}_1\varphi_2 - \dot{\varphi}_2\varphi_1\right) - V(\phi;\mu)$$

$$V(\phi;\mu) = (m^2 - \mu^2)\phi^2 + V_{\text{int}}(\phi)$$

$$i\varepsilon\mathscr{E}_\mu = \equiv \frac{i\varepsilon}{2}\left[\dot{\varphi}_i\dot{\varphi}_i + \vec{\nabla}\varphi_i \cdot \vec{\nabla}\varphi_i\right] + i\varepsilon\, V(\phi;\mu)$$

Scalar fields at finite μ - Functional approach

- In this field basis the $i\varepsilon\mathscr{E}_\mu$ term projects on time-independent configurations
- It is enough to compute the one-loop effective potential at finite μ

  - the radial mode is included (not integrated out): no assumption of SSB
  - we can check if at one-loop finite density is always tied to SSB:

$$Z[j_i;\mu] = \int D\varphi \exp\left[i\int \mathrm{d}^d x\left(\mathscr{L}_\mu + j_i(x)\varphi_i(x) + h.c. + i\varepsilon\,\mathscr{E}_\mu\right)\right]$$

↓

$$\frac{\delta\Gamma[\varphi_{cl}(x);\mu]}{\delta\varphi_{cl}(x)} = -j(x) \quad \Longrightarrow \quad \left.\frac{\mathrm{d}V_{\text{eff}}(\varphi_{cl};\mu)}{\mathrm{d}\varphi_{cl}}\right|_{\varphi_{\min}} = 0$$

$$Q(\mu) = \langle \hat{J}^0\rangle_\mu = -i\frac{\mathrm{d}}{\mathrm{d}\mu}\log Z[0;\mu] = -\frac{\mathrm{d}V_{\text{eff}}(\varphi_{\min};\mu)}{\mathrm{d}\mu}$$

Rephrasing the finite density - SSB connection

- The question that we are trying to address can be rephrased as follows:
  - assume that at $\mu = 0$ there is no SSB: $\varphi_{\min}(0) = 0, \qquad Q(0) = 0$
  - is it true that when $\mu > \mu_{\text{crit}}$: $Q(\mu) \neq 0 \implies \varphi_{\min}(\mu) \neq 0$?

$$Q(\mu) = -\frac{\mathrm{d}V_{\text{eff}}(\varphi_{\min};\mu)}{\mathrm{d}\mu} \qquad \longrightarrow \qquad -\frac{\mathrm{d}V_{\text{eff}}(0;\mu)}{\mathrm{d}\mu} = 0 \quad ?$$

- If yes:
  - what is the critical value of μ : $\mu_{\text{crit}} = m_{\text{pole}}$?
  - can we quantify the *"amount of SSB"* as a function of charge Q ?

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The quantum effective potential at finite μ

For arbitrary interaction potential, the one loop effective potential at finite μ is given by:

$$V_{\text{eff}}^{(1)}(\varphi;\mu) = \frac{1}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d} \log\left[(p^2+M^2)^2 - 4(p\cdot\xi)^2 - g^2\right],$$

where

$$\xi \equiv (i\mu, \vec{0}\,),$$

$$M^2 = M^2(\phi;\mu) \equiv \frac{1}{4\,\phi}\Big(V'(\phi;\mu) + \phi\, V''(\phi;\mu)\Big),$$

$$g^2 = g^2(\phi) \equiv \frac{1}{16\,\phi^2}\Big(V'(\phi;\mu) - \phi\, V''(\phi;\mu)\Big)^2.$$

We compute analytically the loop integral for arbitrary interaction potential.

We regularize using dim.reg. : the counterterms needed are exactly the same as those at $\mu = 0$.

The full expression in $d = 3$, $d = 4$

- In $d = 3$ we obtain

$$V_{\text{eff}}^{(1)}(\phi;\mu)\Big|_{d=3} = -\frac{1}{6\pi}(M^2+\mu^2)^{3/2} - \frac{g^2}{16\pi}\int_0^1 \mathrm{d}x\,\sqrt{y}\;{}_2F_1\left[\frac{1}{4},\,\frac{3}{4},\,2\,\middle|\,4g^2\,x(1-x)\,y^2\right],$$

$$y = \frac{1}{M^2+(1-2x)^2\mu^2}$$

- In $d = 4$ we obtain

$$V_{\text{eff}}^{(1)}(\phi;\mu)\Big|_{d=4} = -\frac{1}{64\pi^2}\left[4g^2+3(M^2+\mu^2)^2\right] + \frac{1}{32\pi^2}\left[g^2+(M^2+\mu^2)^2\right]\log\left(M^2+\mu^2\right)$$
$$+\frac{g^2}{16\pi^2}\frac{M}{\mu}\arctan\left(\frac{\mu}{M}\right) - \frac{g^2}{256\pi^2}\int_0^1 \mathrm{d}x\,y\;{}_3F_2\left[1,\,1,\,\frac{3}{2},\,2,\,3\,\middle|\,y\right],$$

$$y = \frac{4g^2\,x(1-x)}{(M^2+(1-2x)^2\mu^2)^2}$$

cfr. Kapusta '81, Benson et al. '91 , Brauner '06

Finite density and SSB at one loop

For arbitrary interaction potential, and in $d > 2$ we obtain:

- The value of the effective potential at $\phi = 0$ is μ-independent

$$-\frac{\mathrm{d}}{\mathrm{d}\mu} V_{\text{eff}}^{(1)}(\phi = 0; \mu) = 0 \quad \longrightarrow \quad Q(\mu) \neq 0 \implies \varphi_{\min}(\mu) \neq 0$$

- The critical value of μ coincides with the pole mass in the $\mu = 0$ theory: m_{pole}

$$Q > 0 \iff -\frac{\mathrm{d}}{\mathrm{d}\mu} V_{\text{eff}}(\phi_{\min}; \mu) > 0$$

[...] $\downarrow$

$$\mu^2 > V''_{\text{eff}}(0; 0) = m^2_{\text{pole}}$$

O(N) model in $d = 3$ at finite μ

- Consider an $O(N)$ vector multiplet of scalars $\vec{\Sigma}$ with quartic coupling:

  - introduce chemical potential in the plane 1-2 , with notation $\Sigma_I = (s_1, s_2, S_i)$

$$\mathscr{L}_\mu = \frac{1}{2}(\partial_\nu s_1)^2 + \frac{1}{2}(\partial_\nu s_2)^2 + \mu\left(\dot{s}_1 s_2 - \dot{s}_2 s_1\right) - \frac{1}{2}(m^2 - \mu^2)(s_1^2 + s_2^2) - \frac{\lambda_N}{4N}(s_1^2 + s_2^2)^2$$
$$+\frac{1}{2}(\partial_\nu S_i)^2 - \frac{1}{2}\left[m^2 + \lambda(s_1^2 + s_2^2)\right] S_i^2 - \frac{\lambda_N}{4N} S_i^4,$$

  - introduce Hubbard-Stratonovich auxiliary field: $\delta\mathscr{L} = \frac{N}{4\lambda_N}\left(\chi - m^2 - \frac{\lambda_N}{N}\Sigma_I^2\right)^2,$

  - at leading order in $1/N$, and to all orders in λ_N:

$$V_{\text{eff}}(s, S, \chi; \mu) = \frac{1}{2}(\chi - \mu^2)(s_1^2 + s_2^2) + \frac{1}{2}\chi S_i^2 - \frac{N}{4\lambda_N}\left(\chi - m^2\right)^2 - \frac{N}{12\pi}\chi^{3/2}.$$

→ $Q(\mu) \neq 0 \implies \varphi_{\min}(\mu) \neq 0\,, \qquad \mu_{\text{crit}}^2 = m_{\text{pole}}^2$

Quantifying SSB : scaling relations

- Having found a relation between finite density and SSB we would like to quantify it.

  - physical definition of symmetry breaking scale?

  Coefficient of the term $-(\vec{\nabla}\pi)^2/2$ in the superfluid phonon EFT:

  - from the shift-symmetry current of the EFT we also have:

$$f_\pi^{d-2} = 2P'(\mu^2)$$

$$Q = 2P'(\mu^2)\,\mu$$

  - using these relations, the sound speed and that $\mu_{\rm crit} = m_{\rm pole}$ we obtain:

$$f_\pi^{d-2} = \frac{Q}{m_{\rm pole}} - \frac{1}{4m_{\rm pole}^4 P''(m_{\rm pole}^2)} Q^2 + \dots \qquad (Q \to 0),$$

  - positivity of c_s^2 implies the general upper bound $f_\pi^{d-2} \leq \frac{Q}{m_{\rm pole}}$ for arbitrary Q

  - if the superfluid EFT flows to a conformal superfluid in the UV:

$$f_\pi \simeq e^{-\frac{K_\star}{d-2}} \left(\frac{Q}{\sqrt{\lambda}}\right)^{\frac{1}{d-1}} \qquad (Q \to \infty)$$

Superfluid phonons at one loop for arbitrary UV potential

- We can obtain the $P(X)$ for the phonons from the value of the effective potential:

$$P(X) = \mathscr{L}_{\text{eff}}\left[\left((D_\nu\pi)(D^\nu\pi)\right)^{1/2}\right] \xrightarrow[\pi=\text{const}]{} -V_{\text{eff}}(\Phi_{\text{min}};\mu) \qquad \left(D_\nu\pi = \partial_\nu\pi - \mu\delta_\nu^0\right)$$

- we obtain the superfluid EFT for arbitrary UV potential in closed form

$$g_{\text{min}} = \frac{V''(\phi_0)}{4}\,.$$

$$P(X)\Big|_{d\,\text{odd}} = -V(\phi_0;\mu = X^{1/2}) + (-1)^{(d+1)/2}\frac{\pi}{2\,\Gamma\left(d/2+1\right)}\left(\frac{g_{\text{min}}}{2\pi}\right)^{d/2} {}_2F_1\left[\frac{1}{2},\,-\frac{d}{2},\,\frac{d}{2}\,\middle|\,-\frac{2X}{g_{\text{min}}}\right],$$

$$P(X)\Big|_{d=4} = -V(\phi_0;\mu = X^{1/2}) + \frac{1}{192\pi^2}\Bigg(-4X^2 + \left(X^2 + 2g_{\text{min}}X + 2g_{\text{min}}^2\right)\left(9 - 6\log(2g_{\text{min}})\right) - \frac{5X^3}{2g_{\text{min}}}\,{}_3F_2\left[1,\,1,\,\frac{7}{2},\,4,\,5\,\middle|\,-\frac{2X}{g_{\text{min}}}\right]\Bigg).$$

Conformal superfluid and large charge operators

- Considering the $\lambda\phi^6$ model in $d = 3$ $\lambda = \hat{\lambda}^2/36$

  - the model is conformal at one loop. Finite μ : conformal superfluid

$$P(X)\Big|_{d=3,\,\lambda\phi^6} = \left(\frac{2}{3^{3/2}\sqrt{\lambda}} + \frac{7\sqrt{2} + 3\,\mathrm{arcsinh}(1)}{12\pi}\right) X^{3/2},$$

- The scaling dimension of the lowest dimensional charge n operator, for large n :

$$\Delta_{\phi^n} = \left(\frac{\hat{\lambda} n}{\sqrt{3}\pi}\right)^{3/2} \left[c_{3/2} + \mathcal{O}\left(\frac{\sqrt{3}\pi}{\hat{\lambda} n}\right)\right],$$

Hellerman et al. '15
Monin et al. '17,

  - $c_{3/2}$ can be related to the coefficient of the $P(X)$:

$$c_{3/2} = \frac{\sqrt{3}\pi}{6\hat{\lambda}} - \frac{7\sqrt{2} + 3\,\mathrm{arcsinh}(1)}{192} = \frac{\sqrt{3}\pi}{6\hat{\lambda}} - 0.0653313\ldots$$

Matches (to all digits) the numerical result of: Badel et al. '20

Summary

- We computed the low energy effective theory $P(X)$ at one-loop for relativistic superfluids from a UV theory with arbitrary potential
  - two different and independent computations, they agree non-trivially
  - reproduces large charge scaling dimensions in $\lambda\phi^6, \quad d = 3$
- This gives the equation of state for a zero-temperature superfluid
  - it can be fully expressed in terms of on-shell quantities, RG independent

phonon

$\pi(t, x)$

- We computed the full one-loop effective potential at finite μ
- We checked explicitly that finite density $\Longrightarrow$ SSB at one-loop
  - What about higher orders? Is it possible to give a non-perturbative proof?
- We quantified the amount of SSB by finding (universal) scaling relations

Complex

$\Phi(t, x)$