Title: Quantum Gravity Lecture (230502)

Speakers: Aldo Riello

Collection: Quantum Gravity (2022/2023)

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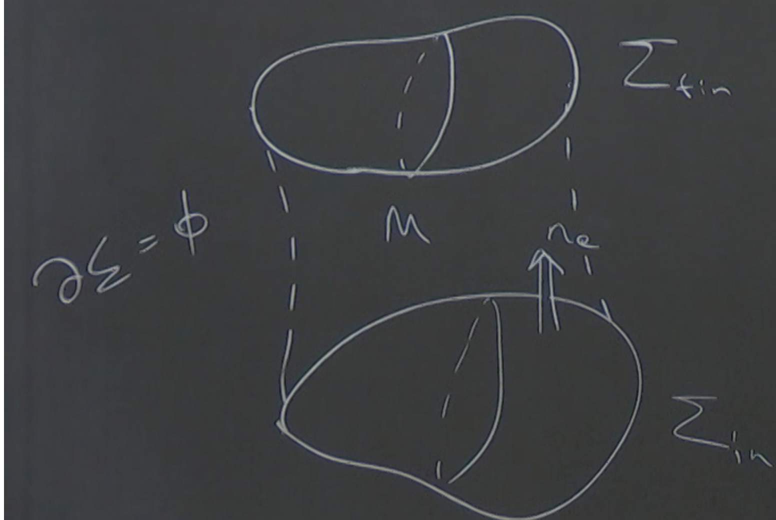
FROM COVARIANT TO CANONICAL PH. SPACE



$$M : (\mathcal{F}, \Omega) \xleftrightarrow{E=0} (\mathcal{F}, \iota^*_{EL}\Omega)$$

off shell

on-shell
(Euler-Lagrange locus

?

?

$\Sigma_{(in)}$:

$(\mathcal{P}, \omega)$

canonical ph. sp.

$\to \left(\underline{\mathcal{F}}, \iota^*_{EL}\,\underline{\Omega}\right)$

on-shell
(Euler-Lagrange locus

?

)

ph. sp.

Exemple 1

Particle (0+1 d field theory)

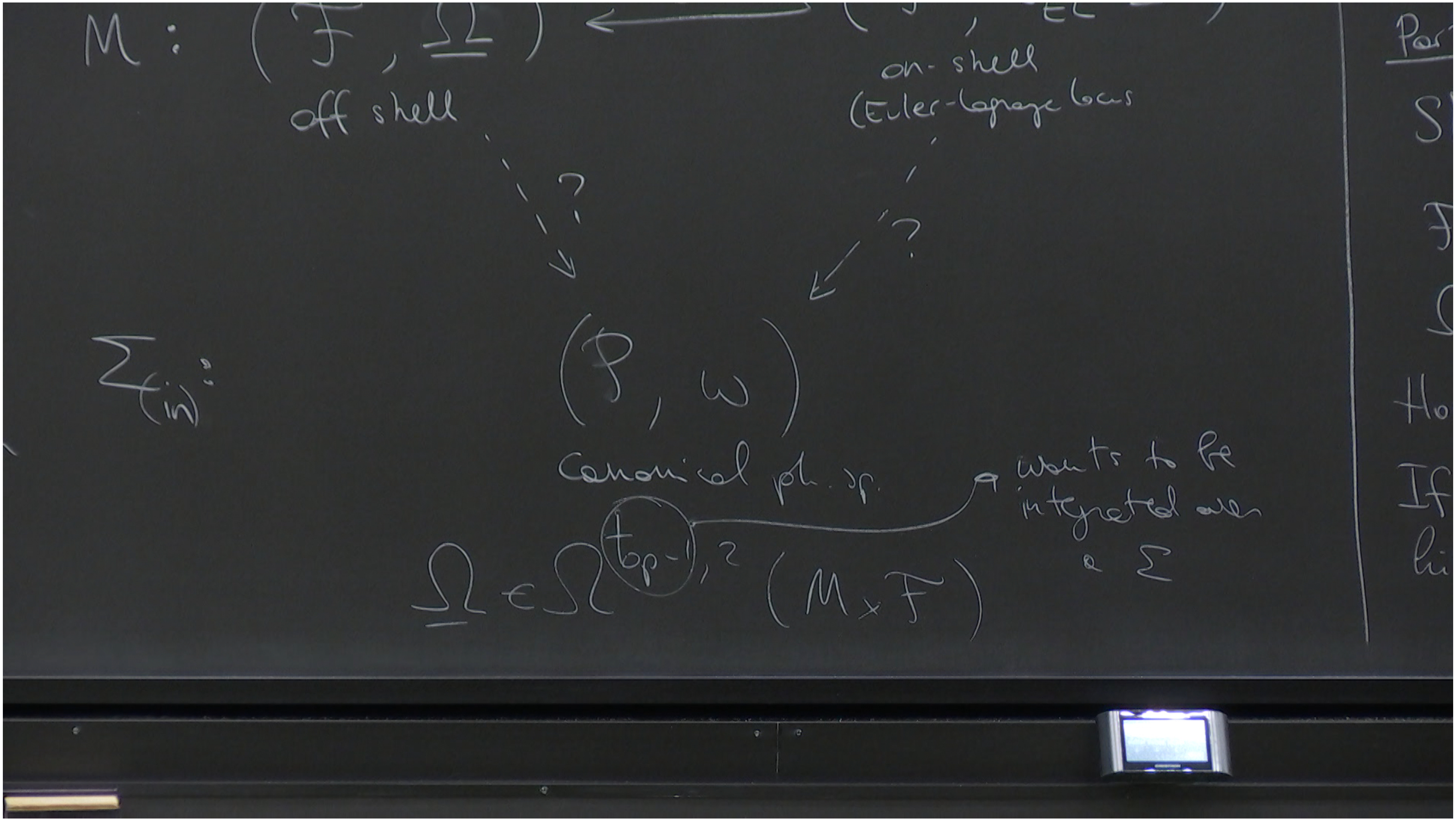
$$S[\gamma(t)] = \int \left(\frac{m}{2}\dot{\gamma}^2 - V(\gamma(t))\right) dt$$

$$\mathcal{F} = \mathrm{Maps}(\mathbb{R}, \mathbb{R}^3)$$

$$\Omega = (\partial_t d\gamma(t)) \wedge d\gamma(t)$$

How to describe $\left(\underline{\mathcal{F}}, \iota^*_{EL}\,\underline{\Omega}\right)$ explicitly?!

If eom. are nice, $\exists$ map btw on-shell histories & initial conditions @ Σ_{in}.


M : (F , Ω) ⟵ (, EL)
off shell
on-shell
(Euler-Lagrange locus
?
?
(P , ω)
canonical ph. sp.
Σ(in) :
Ω ∈ Ω^(top-1, 2) (M × F)
wants to be
integrated over
a Σ

Exemple 1

Particle (0+1 d field theory)

$$S[\gamma(t)] = \int \left(\frac{m}{2}\dot{\gamma}^2 - V(\gamma(t)) \right) dt$$

$$\mathcal{F} = \mathrm{Maps}(\mathbb{R}, \mathbb{R}^3)$$

$$\Omega = (\partial_t d\gamma(t)) \wedge d\gamma(t)$$

How to describe $(\overline{\mathcal{F}}, \iota_{EL}\underline{\Omega})$ explicitly ?!

If eom are nice, $\exists$ map btw on-shell histories & initial conditions @ Σ_{in}.

$$\Sigma_{in} = \{ t = t_0 \}$$

$$\Omega_\Sigma := \Omega|_{t=t_0} = d\dot{\gamma}(t_0) \wedge d\gamma(t_0)$$

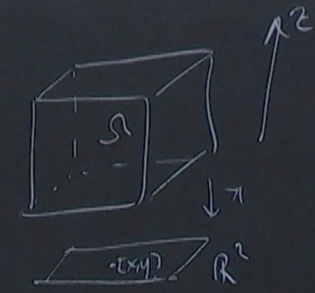
Rmk Ω_Σ has a huge kernel in $\mathcal{F}$:

$\mathbb{X} \in \mathrm{Ker}(\Omega_\Sigma)$ iff $i_{\mathbb{X}}\Omega_\Sigma = 0$

$$\left[\mathbb{X} = \int dt\, (\delta_{\mathbb{X}}\gamma)(t) \frac{\delta}{\delta\gamma(t)} \right]$$

iff $\delta_{\mathbb{X}}\gamma(t_0) = (\partial_t \delta_{\mathbb{X}}\gamma)(t_0) = 0$

$$X = X^1 \frac{\partial}{\partial x} + X^2 \frac{\partial}{\partial y} + X^3 \frac{\partial}{\partial z}$$



$$\Omega = dx \wedge dy$$

$$0 \stackrel{!}{=} i_X \Omega = X^1 dy - X^2 dx$$

$$X^1 = X^2 = 0$$

$$\Sigma_{in} = \{ t = t_0 \}$$

$$\Omega_\Sigma \equiv \Omega\Big|_{t=t_0} = m(\partial_t d\gamma)(t_0) \wedge d\gamma(t_0)$$

Rmk Ω_Σ has a huge kernel in $\mathcal{F}$:

$\mathbb{X} \in \mathrm{Ker}(\Omega_\Sigma)$ iff $i_{\mathbb{X}}\Omega_\Sigma = 0$ as a 1 form

$$\left[\mathbb{X} = \int dt (\delta_{\mathbb{X}}\gamma)(t) \frac{\delta}{\delta\gamma(t)} \right]$$

iff $\delta_{\mathbb{X}}\gamma(t_0) = (\partial_t \delta_{\mathbb{X}}\gamma)(t_0) = 0$

...gy ?!

...ell

Any two histories $\gamma_1(t)$, $\gamma_2(t)$

with the same $\gamma_1(t_0) = \gamma_2(t_0)$

$\dot\gamma_1(t_0) = \dot\gamma_2(t_0)$

lie along the kernel of a $\mathbb{X} \in$

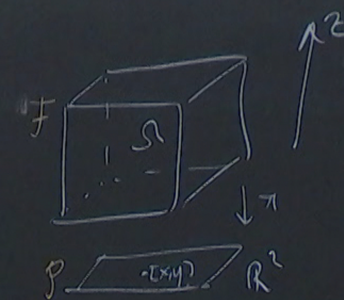
$\Rightarrow$ If I want a symplectic (

Space $\mathcal{P}$ I need to ident...

$\rightarrow$ ... $[\gamma(t)]_{q,p} = \{ \gamma \in$

$\rightarrow$ $\mathcal{P} := \mathcal{F} / \mathrm{Ker}(\Omega_\Sigma)$

$$X = X^1 \frac{\partial}{\partial x} + X^2 \frac{\partial}{\partial y} + X^3 \frac{\partial}{\partial z}$$



$$\Omega = dx \wedge dy$$

$$0 \overset{!}{=} i_X \Omega = X^1 dy - X^2 dx$$

$$X^1 = X^2 = 0$$

Any two histories $\gamma_1(t)$, $\gamma_2(t)$

with the same $\gamma_{1,2}(t_0) = \gamma_2(t_0)$

$$\dot{\gamma}_1(t_0) = \dot{\gamma}_2(t_0)$$

lie along the kernel of a $X \in \mathrm{Ker}(\Omega_\Sigma)$

$\Rightarrow$ If I want a <u>symplectic</u> (non degenerate) space P^{off} I need to identify <u>all</u> these config.

$\rightarrow$ $[\gamma(t)]_{q,p} = \{\gamma \in \mathcal{F} : \gamma(t_0) = q, m\dot{\gamma}(t_0) = p\}$

$\rightarrow$ $\boxed{P^{\text{off}} := \mathcal{F}/\mathrm{Ker}(\Omega_\Sigma) \simeq \{(q,p) \in \mathbb{R}^2\}}$ presymplectic reduction

TO CANONICAL PH. SPACE

$$M: (\mathcal{F}, \underline{\Omega}) \xleftarrow{E=0} (\mathcal{F}, \iota_{EL}^* \underline{\Omega})$$

off shell

on-shell (Euler-Lagrange locus

presympl. red.

$/\ker(\Omega)$

?

Σ_{in}

$\Sigma_{(in)}$:

$$(\mathcal{P}^{\partial}, \omega^{\partial})$$

canonical ph. sp.

wants to be integrated over a Σ

$$\underline{\Omega} \in \Omega^{(top-1),2}(M \times F)$$

Example 1

Particle (0+1 d field the...

$$S[\gamma(t)] = \int \left(\frac{m}{2} \dot{\gamma}^2 - \right.$$

$$\mathcal{F} = \mathrm{Maps}(\mathbb{R}, \mathbb{R}^3)$$

$$\Omega = (\partial_t d\gamma(t)) \wedge d\gamma(t)$$

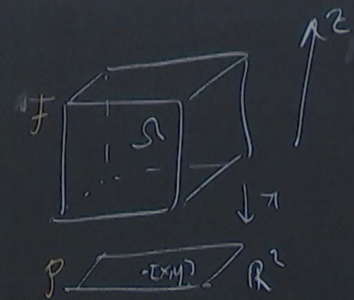
How to describe $(\mathcal{F}, \iota$...

If eom. are nice, $\exists$ m...
histories & initial cond...

$$X = X^1 \frac{\partial}{\partial x} + X^2 \frac{\partial}{\partial y} + X^3 \frac{\partial}{\partial z}$$

$$\Omega = dx \wedge dy$$

$$\theta \doteq i_X \Omega = X^1 dy - X^2 dx$$



$$X^1 = X^2 = 0$$

on $\mathcal{P}^{off}$, Ω_Σ induces a non-degenerate

2form $\omega^{off} := dp \wedge dq$ (d in this ex is on a finite dim space)

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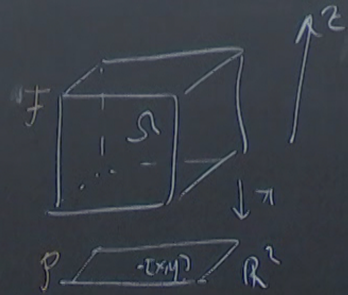

$$\Omega = dx \wedge dy$$

$$0 \overset{!}{=} i_X \Omega = X^1 dy - X^2 dx \qquad X^1 = X^2 = 0$$

on P^{off}, Ω_Σ induces a non-degenerate

2form $\omega^{off} := dp \wedge dq$ (d in this ex is on a finite dim space)

ctic

on

Lucky: (p,q) are also in 1-to-1 correspondence with on-shell histories.

ONICAL PH. SPACE

$: (\mathcal{F}, \underline{\Omega}) \xleftarrow{E=0} (\mathcal{F}, \iota_{EL}^{*}\underline{\Omega})$

off shell

on-shell
(Euler-Lagrange locus

presympl. red.

$/\mathrm{Ker}(\Omega_c)$

lucky case

$(\mathcal{P}^{off}, \omega^{off}) = (\mathcal{P}, \omega)$

Canonical ph. sp.

wants to be integrated over a Σ

$\underline{\Omega} \in \Omega^{top-1, 2}(M \times F)$

$\mathcal{F}_{(in)}$:

Exemple 1

Particle (0+1 d

$S[\gamma(t)] = \int ($

$\mathcal{F} = \mathrm{Maps}(\mathbb{R}$

$\Omega = (\partial_t d\gamma(t)$

How to describe

If e.o.m. on nic

histories & in

$\textcircled{?}$ = an isomorphism.

For particle & scalar field theory

$$
\begin{tikzcd}
\mathcal{M}: & (\mathcal{F}, \underline{\Omega}) \arrow[d, "/\mathrm{Ker}\,\Omega_\Sigma"', "\text{PreSympl red.}"] & & (\mathcal{F}, \iota^*_{EL}\underline{\Omega}) \arrow[ll, hook', "E=0"'] \arrow[d, "\text{presymplectic reduction}"] \\
\Sigma: & (\mathcal{P}^{\mathrm{off}}, \omega^{\mathrm{off}}) \arrow[rr, leftrightsquigarrow, "?"] & & \boxed{(\mathcal{P}, \omega)}
\end{tikzcd}
$$

off shell

Always the "physical" PHASE SPACE

Scalar field theory

$$\mathcal{F} = C^\infty(M, \mathbb{R})$$

$$\Omega_\Sigma = \int_\Sigma \sqrt{h}\left(\nabla_n \, d\varphi(x,t_0)\right) \wedge d\varphi(x,t_0)$$

presympl red: $(\mathcal{P}^{\text{off}}, \omega^{\text{off}}) \simeq (\mathcal{P}, \omega)$

function on Σ density on Σ

$$\mathcal{P}^{\text{off}} = \left\{ [\varphi(x,t)] : \begin{cases} \varphi(t_0,x) = \phi(x) \\ \sqrt{h}\,\nabla_n \varphi(t_0,x) = \pi(x) \end{cases} \right\} = \{ (\phi(x), \pi(x)) \}$$

Maxwell (YM very similar)

$$\mathcal{F} = \Omega^1(M)$$

$$S = \frac{1}{2}\int *F \wedge F \quad , \quad F = dA$$

$$\underline{\Omega} = *\mathbb{d}F \wedge \mathbb{d}A$$

$$\Omega_\Sigma = \int_\Sigma *\mathbb{d}F(A) \wedge \mathbb{d}A = \int_\Sigma \sqrt{h}\ \mathbb{d}F^{0i}(A) \wedge \mathbb{d}A_i$$

$$F^{0i} \sim \dot{A}_i - \nabla^i A_0$$

Rmks

1) A_0 only appears "mixed" with $\dot{A}_i$

2) $\dot{A}_i$ does not appear in isolation

$$\left[\Longleftrightarrow \frac{\partial \mathcal{L}}{\partial(\partial_t A_0)} \equiv 0 \Longleftrightarrow \right.$$

Presymplectic reduction:

we identify all $A_\mu(t,x)$ according to:

$$[A_\mu] = \left\{ \underset{\in M}{A_\mu(x)} : \begin{cases} A_i|_\Sigma = a_i \in \Omega^1(\Sigma) \\ \sqrt{h} F^{0i}(A)|_\Sigma = \mathcal{E}^i \end{cases} \right\}$$

$\mathcal{E}^i$ a vector-valued density on Σ

$\mathcal{E}^i = \sqrt{h} E^i$ densitized electric field.

to split $A_\Gamma = (A_0, A_i)$
at all "time"

vector-valued
density on Σ

$\sqrt{h}E^i$ densitized
electric field.

$$\mathcal{P}^{\text{off}}_{\text{Maxwell}} = \{(a_i, \mathcal{E}^i)\} = T^*\Omega^1(\Sigma)$$

$$\omega^{\text{off}}_{\text{Maxwell}} = \int_\Sigma d\mathcal{E}^i(x) \wedge d a_i(x)$$

densitized
electric fields

"spacelike"
1-forms on Σ
(no A_0!)

reduction Lucky: (p,q) are also in 1-to-1

$(\mathcal{P}^{off}, \omega^{off})$ can't be isomorphic to $(\mathcal{P}, \omega)_{\approx}$ on shell!

b.c. on shell: $\sqrt{h}\,\nabla_i E^i = 0$

$\hookrightarrow \partial_i \mathcal{E}^i = 0$

(Gauss Constraint)

Noether II

$\partial\Sigma = \phi$

$$\mathring{\mathbb{I}}_{\rho(\xi)} \Omega_\Sigma = - \mathbb{d} Q_\Sigma(\xi) \qquad Q_\Sigma(\xi) = \int_\Sigma \sqrt{h}\,\underbrace{(\nabla_i F^{i0})}_{\text{Gauss} \approx 0}\,\xi + \cancel{\int_{\partial\Sigma} n_i E^i \xi}$$

presympl reduction

$$\mathring{\mathbb{I}}_{\rho(\xi)} \omega^{off} = - \mathbb{d} \int_\Sigma (\partial_i \mathcal{E}^i)\,\xi \approx 0 \text{ by Noether 2}$$

$\partial \Sigma = \phi$

Constraints (e.g. Gauss)

are e.o.m. which have a

left-over impact on $(P^{\text{eff}}, \omega^{\text{eff}})$

Always present in gauge theories

because of Noether 2:

$$\underline{J}(\xi) = \underbrace{\underline{C}_\alpha}_{\approx 0} \xi^\alpha + dj(\xi)$$

$\uparrow$ $\boxed{N2}$

...pears "mixed" with $\dot{A}_i$

...t appear in isolation

$$\left[\Longleftrightarrow \frac{\partial \mathcal{L}}{\partial(\partial_t A_0)} \equiv 0 \Longleftrightarrow \text{Lagrange multiplier for Gauss constraint} \right]$$

but for this one needs a foliation to split $A_\mu = (A_0, A_i)$ at all "times"

$$\underline{\mathcal{E}} = \iota_\Sigma^*(\star F) \in \Omega^{top}(\Sigma)$$

...eduction:

...all $A_\mu(t,x)$ according to:

$$\ldots = \left\{ A_\mu(x) : \begin{cases} A_i|_\Sigma = a_i \in \Omega^1(\Sigma) \\ \sqrt{h}\, F^{0i}(A)|_\Sigma = \mathcal{E}^i \end{cases} \right\}$$

$x \in M$

$\mathcal{E}^i$ a vector-valued density on Σ

$\mathcal{E}^i = \sqrt{h}\, E^i$ densitized electric field.

$$\mathcal{P}^{off}_{Maxwell} = \{ (a_i, \mathcal{E}^i) \} = T^*\Omega^1(\Sigma)$$

$$\omega^{off}_{Maxwell} = \int_\Sigma d\,\mathcal{E}^i(x) \wedge d\,a_i(x)$$

densitized electric fields

"spacelike" 1-forms on Σ (no A_0!)