Title: Spinors and geometric structures

Speakers: Kirill Krasnov

Date: May 01, 2023 - 3:30 PM

URL: https://pirsa.org/23040164

Abstract: I will describe a construction that allows to understand spinors in an arbitrary number of dimensions, with arbitrary signature. I will describe what pure spinors are, and how in low dimensions all spinors are pure. The first impure spinors arise in 8 dimensions, and "purest" impure spinors are octonions. I will describe how a spinor in an arbitrary dimension defines a set of geometric structures. The easiest example of this is how a pure spinor defines a complex structure. As one increases the dimension, the types of geometric structures that are described by spinors become more and more exotic. If time permits, I will describe some examples in 14 and 16 dimensions. Almost nothing is known about spinors in dimension beyond 16.

Zoom link: https://pitp.zoom.us/j/94776499052?pwd=RGVURlRnaEx6REJwVE10VXhqa1Q5Zz09

Spinors and Geometric Structures

Kirill Krasnov
(Nottingham)

Mathematics (geometry) motivation

Riemannian geometry – metric – is a reduction of the $GL(n,\mathbb{R})$ principal frame bundle to an $O(n)$ bundle.

Often one is interested in adding other geometric structures apart from the metric – e.g. orthogonal complex structure or several such structures (hyperkähler)

In many examples such structures can be encoded by a spinor on the manifold, and when the structure is integrable – parallel spinor. E.g. Calabi-Yau – parallel pure spinor

So, one is often interested in the setup

Metric + Spinor

The purpose of this talk is to explain that there is an equivalent viewpoint

Metric + Spinor = Collection of differential forms on M

Sometimes, there is just a single object in the collection on the right-hand side.

Most of this talk is about algebra. But in all known examples, if one wants the geometric structure described by the spinor to be integrable $\Longleftrightarrow$ differential forms are closed

Physics (unification) motivations

There are 4 different types of fields that are used by physisists to describe the reality

So-called bosonic fields (Forces)

- Scalar fields (e.g. famous Higgs)
- Vector fields (electromagnetic potential, more generally gauge fields)
- Metric = gravity

Fermionic fields (Particles)

- Spinors = matter (particle) fields

The idea of unification, in its simplest form, is to find a principle that realises different fields as components of some other fields

For bosonic fields such an idea is known for a very long time — dimensional reduction

E.g. if $\mathcal{M}^n = X^k \times \mathbb{R}^{n-k}$

- gauge fields on $\mathcal{M}$ are gauge fields + scalars on X
- metric on $\mathcal{M}$ is metric on X + gauge fields + scalars on X

Spinors also behave very nicely under the dimensional reduction

Spin(2n) , S - spinor representation $S = S^+ \oplus S^-$

If take $\mathrm{Spin}(2k) \times \mathrm{Spin}(2(n-k)) \subset \mathrm{Spin}(2n)$

then $S^+_{2n} = S^+_{2k} \otimes S^+_{2(n-k)} \oplus S^-_{2k} \otimes S^-_{2(n-k)}$

Spinors remain spinors under dimensional reduction

So, appears that one just needs a metric and a spinor in sufficiently high number of dimensions to encode all known fields

In this talk I want to explain that there is a large collection of examples where

Metric + Spinor on $\mathbb{R}^{2n}$ $\Longleftrightarrow$ Collection of differential forms on $\mathbb{R}^{2n}$

From the point of view of unification
— ultimate unification in objects of the same type.

Moreover, it appears that this encoding is always possible

Very little new in this talk, just the interpretation

Spinors and complex structures

For the first construction will stay in the more familiar territory of $Spin(2n)$ and complex structures

Restriction to $U(n) \subset Spin(2n)$ arises if one chooses a complex structure on $\mathbb{R}^{2n}$

$J^2 = -\mathbb{1}$

$$\mathbb{R}^{2n}_{\mathbb{C}} = E^+ \oplus E^-$$

$$J E^{\pm} = \pm i E^{\pm}$$

eigenspaces of eigenvalue $\pm i$

$E^{\pm}$ are totally null $\quad (E^+, E^+) = 0$

Choosing a complex structure gives a very concrete and useful model for Cliff(2n) and S

Consider $S := \Delta(\mathbb{C}^n)$ where $\mathbb{C}^n \simeq E^+$

Define a_i^+, a_i $i = 1, \ldots, n$

fermionic creation-annihilation operators $\{a_i, a_j^+\} = \delta_{ij}$

Define $\gamma_i := a_i^+ + a_i$ $\gamma_{i+n} := i(a_i^+ - a_i)$

Easy to check that generate Cliff(2n)

$$\gamma_I \gamma_J + \gamma_J \gamma_I = 2\delta_{IJ} \qquad I = 1, \ldots, 2n$$

Canonical realisation of a_i^+, a_i on S

$$\Psi = \sum \Psi_{i_1 \dots i_k} dz_{i_1} \wedge \dots \wedge dz_{i_k}$$

$$a_i^+ \Psi = dz_i \wedge \Psi$$

$$a_i \Psi = i \partial/\partial z_i \Psi$$

preserves $S = \Lambda^+ \oplus \Lambda^-$

$spin(2n)$ Lie algebra is generated by $\gamma_I \gamma_J$

$u(n)$ is generated by $a_i^+ a_j$

preserves the grading of $S = \Lambda(\mathbb{C}^n)$ by diff. form degree

Charge conjugation

$$\bar{\Psi} = R(\Psi) := \gamma_1 \gamma_2 \cdots \gamma_n \Psi^*$$

← complex conjugation

Either commutes or anti-commutes with all γ_I

so commutes with $spin(2n)$

When $R : S_+ \to S_+$ (n even)

squares to plus or minus identity

Choosing a complex structure gives a very concrete and useful model for Cliff(2n) and S

Consider $S := \Delta(\mathbb{C}^n)$ where $\mathbb{C}^n \simeq E^+$

Define a_i^+, a_i $i = 1, \dots, n$

fermionic creation-annihilation operators $\{a_i, a_j^+\} = \delta_{ij}$

Define $\gamma_i := a_i^+ + a_i$ $\gamma_{i+n} := i(a_i^+ - a_i)$

Easy to check that generate Cliff(2n)

$$\gamma_I \gamma_J + \gamma_J \gamma_I = 2\delta_{IJ} \qquad I = 1, \dots, 2n$$

Canonical realisation of $a_i^\dagger, a_i$ on S

$$\Psi = \sum \Psi_{i_1 \dots i_k} dz_{i_1} \wedge \dots \wedge dz_{i_k}$$

$$a_i^\dagger \Psi = dz_i \wedge \Psi$$

$$a_i \Psi = i \, \partial / \partial z_i \, \Psi$$

preserves $S = \Lambda^+ \oplus \Lambda^-$

Spin(2n) Lie algebra is generated by $\gamma_I \gamma_J$

$u(n)$ is generated by $a_i^\dagger a_j$

preserves the grading of $S = \Lambda(\mathbb{C}^n)$ by diff. form degree

Charge conjugation

$$\bar{\Psi} = R(\Psi) := \gamma_1 \gamma_2 \cdots \gamma_n \Psi^*$$

← complex conjugation

Either commutes or anti-commutes with all γ_I so commutes with spin(2n)

When $R : S_+ \to S_+$ (n even) squares to plus or minus identity

Invariant inner product in S

$$\langle \Psi_1, \Psi_2 \rangle = \tilde{\Psi}_1 \wedge \Psi_2 \Big|_{\text{top form}}$$

$\tilde{\Psi}$ differential form with all elementary 1-form factors written in the opposite order

$$\widetilde{dz} = dz$$

$$\widetilde{dz_1 \wedge dz_2} = - dz_1 \wedge dz_2$$

$$\widetilde{dz_1 \wedge dz_2 \wedge dz_3} = - dz_1 \wedge dz_2 \wedge dz_3$$

$$\widetilde{dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4} = dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4$$

Pure spinors

Definition $M(\Psi) \subset \mathbb{R}^{2n}_{\mathbb{C}}$ subspace spanned by all Clifford generators that annihilate Ψ (empty inhomogeneous diff. form)

Example: $\Psi = 1$ is annihilated by all annihilation operators

$$a_i = \frac{1}{2}(\gamma_i + i\gamma_{i+n})$$

and so $M(1) = \mathrm{Span}(\gamma_i + i\gamma_{i+n}) = E^+$

When $J(\gamma_i) = -\gamma_{i+n}$
$J(\gamma_{i+n}) = \gamma_i$

Definition: A spinor whose annihilator subspace has the maximal dimension possible – n – is called a pure spinor

Proposition: Lines of pure spinors (i.e. pure spinors up to scale) are in one-to-one correspondence with orthogonal complex structures

In one direction this is because one can declare $M(\Psi_{pure})$ to be E^+ of the complex structure. Thus pure spinors "parametrise" complex structures

Proposition: Pure spinors are Weyl spinors (i.e. $\Psi_{pure} \in S_\pm$)

Pure spinors in S_+, S_- encode complex structures giving different orientations of $\mathbb{R}^{2n}$

Proposition: Weyl spinors in dimensions $2n = 2, 4, 6$ are always pure.

The first impure spinors arise for Spin(8) (and are related to octonions)

Geometric map
(or generalised Hopf map)

A Weyl spinor Ψ can be inserted between a set of γ-matrices, generating elements of $\Lambda_{\mathbb{C}}(\mathbb{R}^{2n})$

$$\langle \Psi, \gamma_{I_1} \cdots \gamma_{I_k} \Psi \rangle = B_{I_1 \cdots I_k} \equiv B_k(\Psi) \in \Lambda^k_{\mathbb{C}}(\mathbb{R}^{2n})$$

One can also use $\Psi, \bar{\Psi}$ for this purpose

$$\langle \bar{\Psi}, \gamma_{I_1} \cdots \gamma_{I_k} \Psi \rangle$$

The arising geometric objects are squares of Ψ, and this is why $\Psi = \sqrt{\text{geometry}}$

Talk by Atiyah "What is a spinor"

Proposition (Cartan): A spinor Ψ is pure if and only if

$$\mathcal{B}_k(\Psi) = 0 \quad \forall k < n$$

and $\mathcal{B}_n(\Psi)$ is decomposable and given by the product of directions in $M(\Psi)$

Incidentally, this gives a way to see that spinors in 2,4,6 dimensions are pure

$2n = 2$ can only construct $\langle \Psi, \gamma_I \Psi \rangle$

$$\langle S_+, S_- \rangle \neq 0$$

$2n = 4$ $\langle S_+, S_+ \rangle \neq 0$ but anti-symmetric,

so $\langle \Psi, \Psi \rangle = 0$

$2n = 6$ $\langle S_+, S_- \rangle \neq 0$ but γ-matrices anti-symmetric

$\langle \Psi, \gamma_I \Psi \rangle = 0$

The first interesting case is $2n = 8$.

The obstruction to purity is $\langle \Psi, \Psi \rangle = B_0(\Psi)$

because $\langle \Psi, \gamma_I \gamma_J \Psi \rangle = 0$ identically

$S_+ \approx \mathbb{C}^8$ and so pure spinors — null quadric in $\mathbb{C}^8$

Example of Spin(4) more closely $\quad \Psi = \underbrace{\alpha + \beta dz_{12}}_{S_+} + \underbrace{\gamma dz_1 + \delta dz_2}_{S_-}$

$$Spin(4) = SU_L(2) \times SU_R(2)$$

S_+ — two-component columns, fundamental rep of $SU_L(2)$

$S_+ \ni \Psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \alpha, \beta \in \mathbb{C}$

$R : S_+ \to S_+$

$\langle \bar{\Psi}, \Psi \rangle = |\alpha|^2 + |\beta|^2$

$\langle S_+, S_+ \rangle \neq 0$

(but anti-symmetric)

The geometric objects one can construct from Ψ

$$\langle \Psi, \gamma_I \gamma_J \Psi \rangle = i \Sigma^i V^i$$

$$V^i := (2 \mathrm{Re}(\alpha^* \beta), 2 \mathrm{Im}(\alpha^* \beta), |\alpha|^2 - |\beta|^2)$$

$$\Sigma^i = dx^4 dx^i - \frac{1}{2} \epsilon^{ijk} dx^j dx^k$$ self-dual 2-forms

$$(J_\Psi)_I{}^J := \frac{1}{|V|} V^i \Sigma^i{}_I{}^J$$ squares to $-\mathbb{1}$

complex structure corresponding to Ψ

$\langle \Psi, \gamma_I \gamma_J \Psi \rangle$ – decomposable, given by the product of two eigendirections of J_Ψ

Metric + Spinors → Differential Forms

We have seen that a spinor (for simplicity assume $\dim M = 2n$ and $\Psi \in S^+$)

defines a set of differential forms $\langle \Psi, \gamma \cdots \gamma \Psi \rangle$ complex

$\langle \bar{\Psi}, \gamma \cdots \gamma \Psi \rangle$ real

these will not be general, and satisfy algebraic constraints by their very origin

The question is to what degree the metric in M and the spinor Ψ are characterised by the differential forms they produce

In the example $\dim M = 4$ the complex structure J can be recovered from

either $\langle \Psi, \gamma_I \gamma_J \Psi \rangle$ or from $\langle \bar{\Psi}, \gamma_I \gamma_J \Psi \rangle$

E.g. when $\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \langle \bar{\Psi}, \gamma\gamma\Psi \rangle = i\Sigma^3 = i\omega$

$$\langle \Psi, \gamma\gamma\Psi \rangle = i(\Sigma^1 + i\Sigma^2) = i\Omega$$

raising the index of Σ^3 gives J directly,
alternatively $\Sigma^1 + i\Sigma^2$ is decomposable, and gives the eigenvectors of J

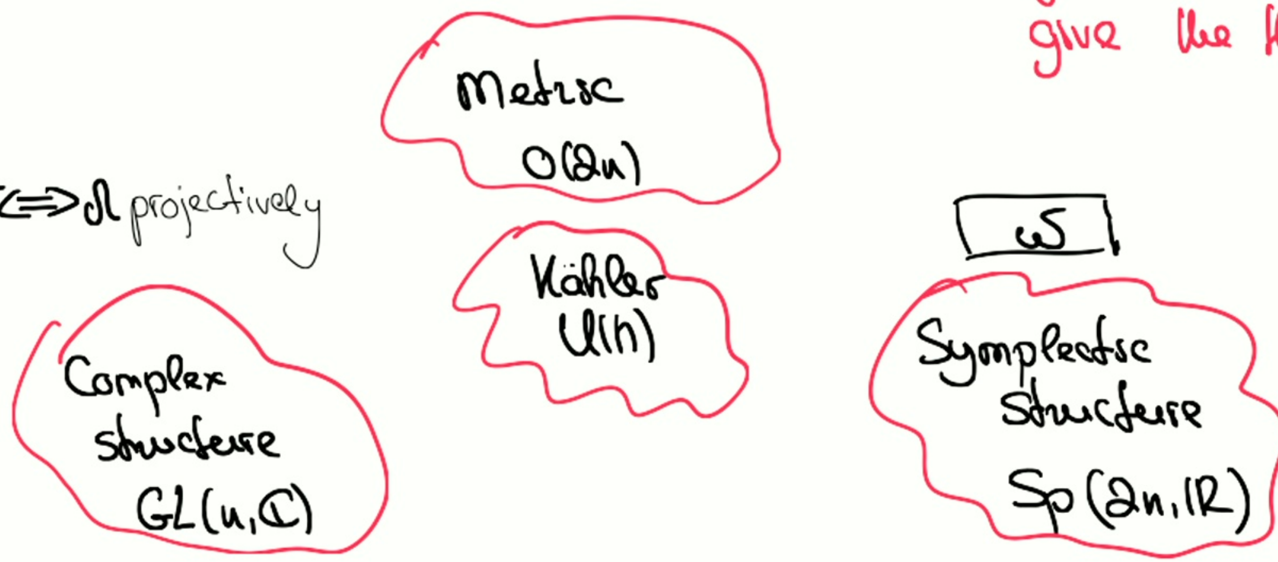
Thus, the knowledge of either of the two and the metric recovers the spinor, at least projectively

Alternatively, when know both $\langle \bar{\Psi}, \gamma\gamma\Psi \rangle$ and $\langle \Psi, \gamma\gamma\Psi \rangle$ can recover both the spinor and the metric

Indeed, we have the familiar story of reductions of G-structures

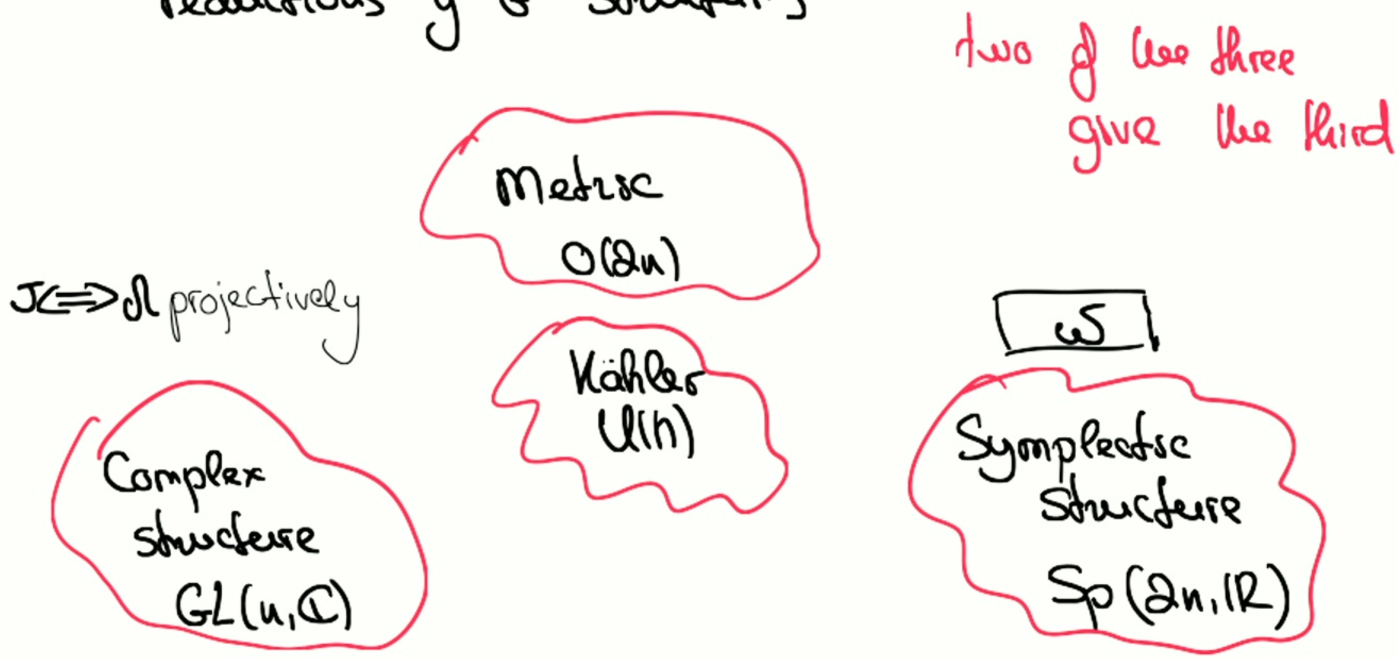
two of the three give the third

$J \Longleftrightarrow \Omega$ projectively



Alternatively, when know both $\langle \bar{\Psi}, \gamma\gamma\Psi \rangle$ and $\langle \Psi, \gamma\gamma\Psi \rangle$ can recover both the spinor and the metric

Indeed, we have the familiar story of reductions of G-structures



In the example $\dim M = 4$ the complex structure J can be recovered from

either $\langle \Psi, \gamma_I \gamma_J \Psi \rangle$ or from $\langle \bar\Psi, \gamma_I \gamma_J \Psi \rangle$

E.g. when $\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \langle \bar\Psi, \gamma\gamma\Psi \rangle = i\Sigma^3 = i\omega$

$$\langle \Psi, \gamma\gamma\Psi \rangle = i(\Sigma^1 + i\Sigma^2) = i\Omega$$

raising the index of Σ^3 gives J directly,

alternatively $\Sigma^1 + i\Sigma^2$ is decomposable, and gives the eigenvectors of J

Thus, the knowledge of either of the two and the metric recovers the spinor, at least projectively

Impure spinors

These first arise for Spin(8).

In this dimension we also have $R: S_+ \to S_+$ and $R^2 = +\mathbb{1}$

So, meaningful to impose $R\Psi = \Psi$ Majorana condition

Majorana spinors of Spin(8) can be identified with octonions

$\langle \Psi_-, \gamma \Psi_+ \rangle$ map $S_- \times S_+ \to \mathbb{R}^8$

$$S_\pm^{\text{Majorana}} = \mathbb{O}$$

Non-zero Majorana-Weyl spinors of Spin(8) are never pure, and so octonions give the "purest" impure spinors

$$\langle \Psi^M, \Psi^M \rangle = |q|^2$$

where q is the octonion representing Ψ^M

For Majorana spinors $R(\Psi) \equiv \bar{\Psi} = \Psi$,

so the only geometric object we can construct is

$$\langle \Psi, \gamma \cdots \gamma \Psi \rangle$$

In the case of Spin(8) the non-vanishing objects are

$$|\Psi|^2 \equiv \langle \Psi, \Psi \rangle \quad \text{and} \quad \langle \Psi, \gamma\gamma\gamma\gamma \Psi \rangle = B_4(\Psi)$$

According to our general principle, the knowledge of this 4-form should recover both the metric and the spinor

This indeed holds. The 4-form produced by an impure spinor is of a special algebraic type — Cayley form in $\mathbb{R}^8$

Its stabiliser in $GL(8, \mathbb{R})$ is $Spin(7)$

Cayley forms $= GL(8,\mathbb{R})/Spin(7)$

dimension $= 64 - 21 = 43$

Metric in $\mathbb{R}^8$ + unit Majorana spinor

$$\frac{8\cdot 9}{2} + 7 = 43$$

This case is particularly striking, because the metric + spinor information is encoded by a single differential form rather than a collection of such forms

Impure spinors

These first arise for Spin(8).

In this dimension we also have $R: S_+ \to S_+$ and $R^2 = +\mathbb{1}$

So, meaningful to impose $R\Psi = \Psi$ Majorana condition

Majorana spinors of Spin(8) can be identified with octonions

$\langle \Psi_-, \gamma \Psi_+ \rangle$ map $S_- \times S_+ \to \mathbb{R}^8$

$$S_\pm^{\text{Majorana}} = \mathbb{O}$$

Non-zero Majorana-Weyl spinors of Spin(8) are never pure, and so octonions give the "purest" impure spinors

$$\langle \Psi^M, \Psi^M \rangle = |q|^2$$

where q is the octonion representing Ψ^M

Higher dimensions

A general Weyl spinor can always be represented as a sum of pure spinors (non-uniquely)

Can define "parity" as the minimal # of pure spinors needed to represent a given spinor

The higher the dimension, the higher parities one gets.

Complex spinors of $Spin(2n, \mathbb{C})$ are understood and classified up to and including $2n = 14$

The case of $2n = 16$, where there are again Majorana-Weyl spinors of $Spin(16)$ presents new difficulties and only partially understood

It is quite interesting to consider Spin groups for which real spinors exist.

In this case there is a smaller number of differential forms that one can construct.

For example Spin(16)

$\underline{\Psi}_M \in S^+$ – there are many different orbits now (unlike the case of Spin(8))

But always has

$$\Psi_M \otimes \Psi_M = \mathbb{R} \oplus \Lambda^4(\mathbb{R}^{16}) \oplus \Lambda^8(\mathbb{R}^{16})$$

self-dual 8-form

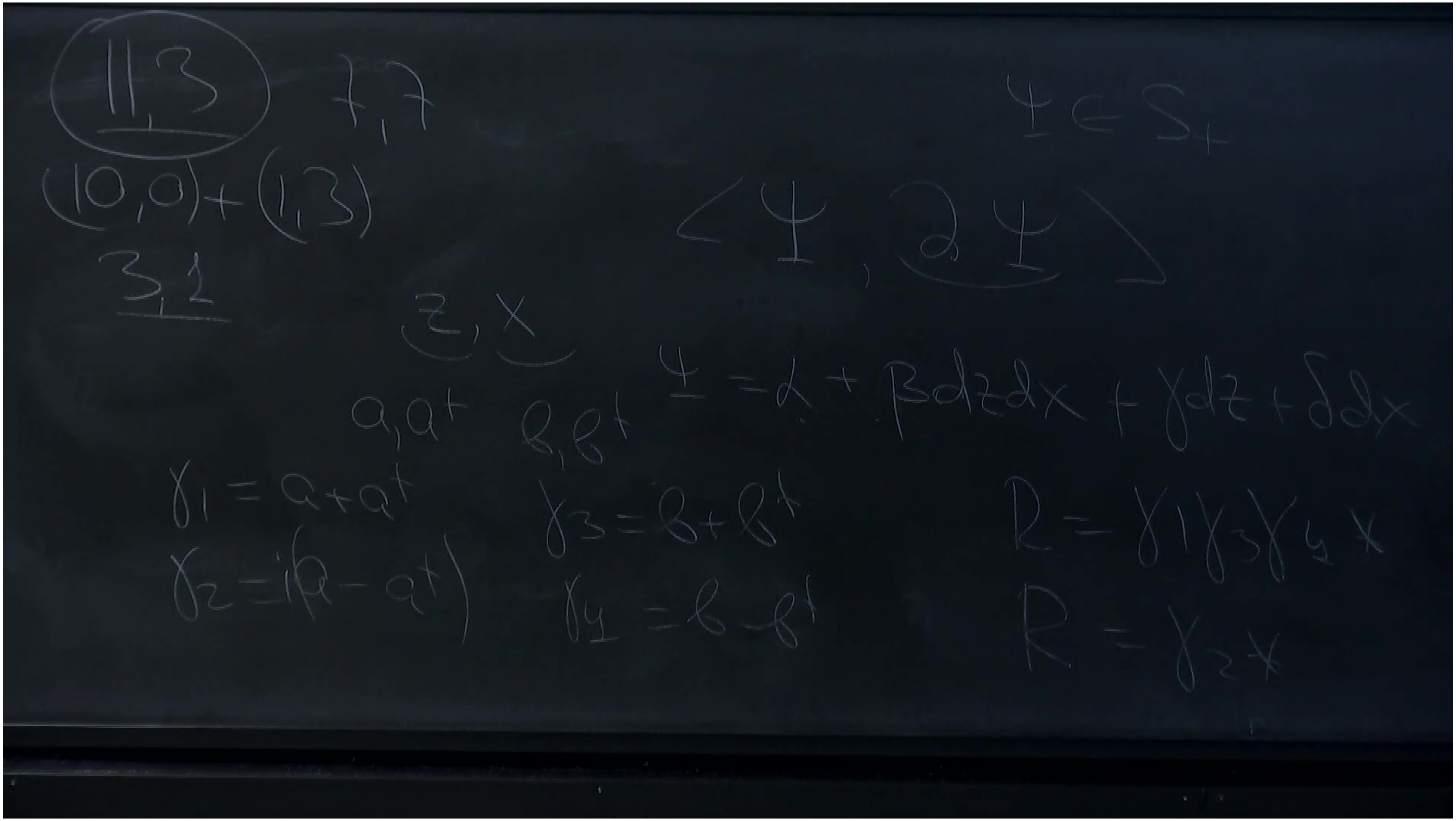
$B_0 = \langle \Psi_M, \Psi_M \rangle$ $B_4 = \langle \Psi_M, \gamma\gamma\gamma\gamma \Psi_M \rangle$ $B_8 = \langle \Psi_M, \underbrace{\gamma \cdots \gamma}_{8\ \text{times}} \Psi_M \rangle$

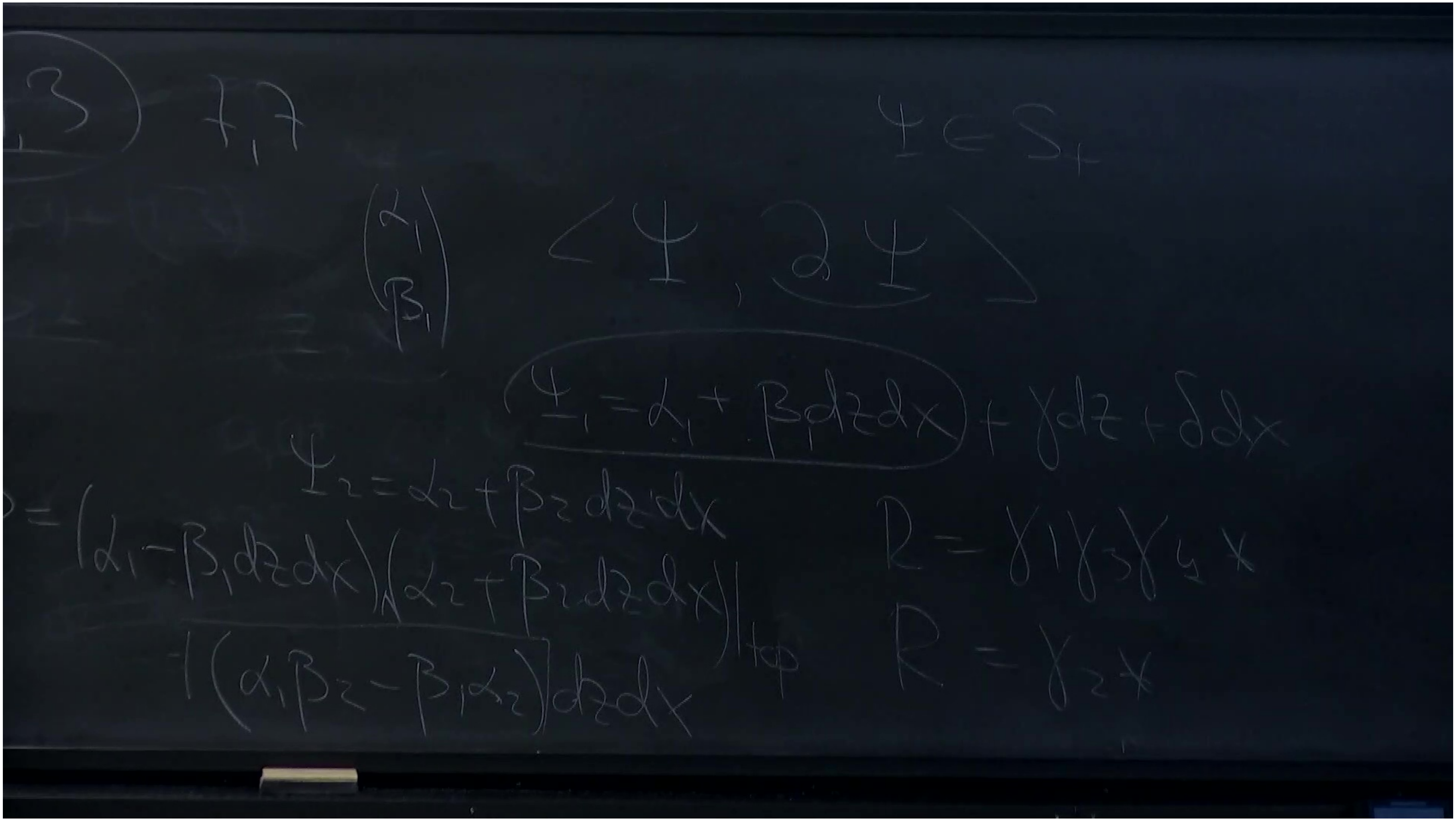
- In all known examples (where spinor gives access to at least two pure spinors)

All differential forms produced by a spinor $\Longleftrightarrow$ Metric + Spinor

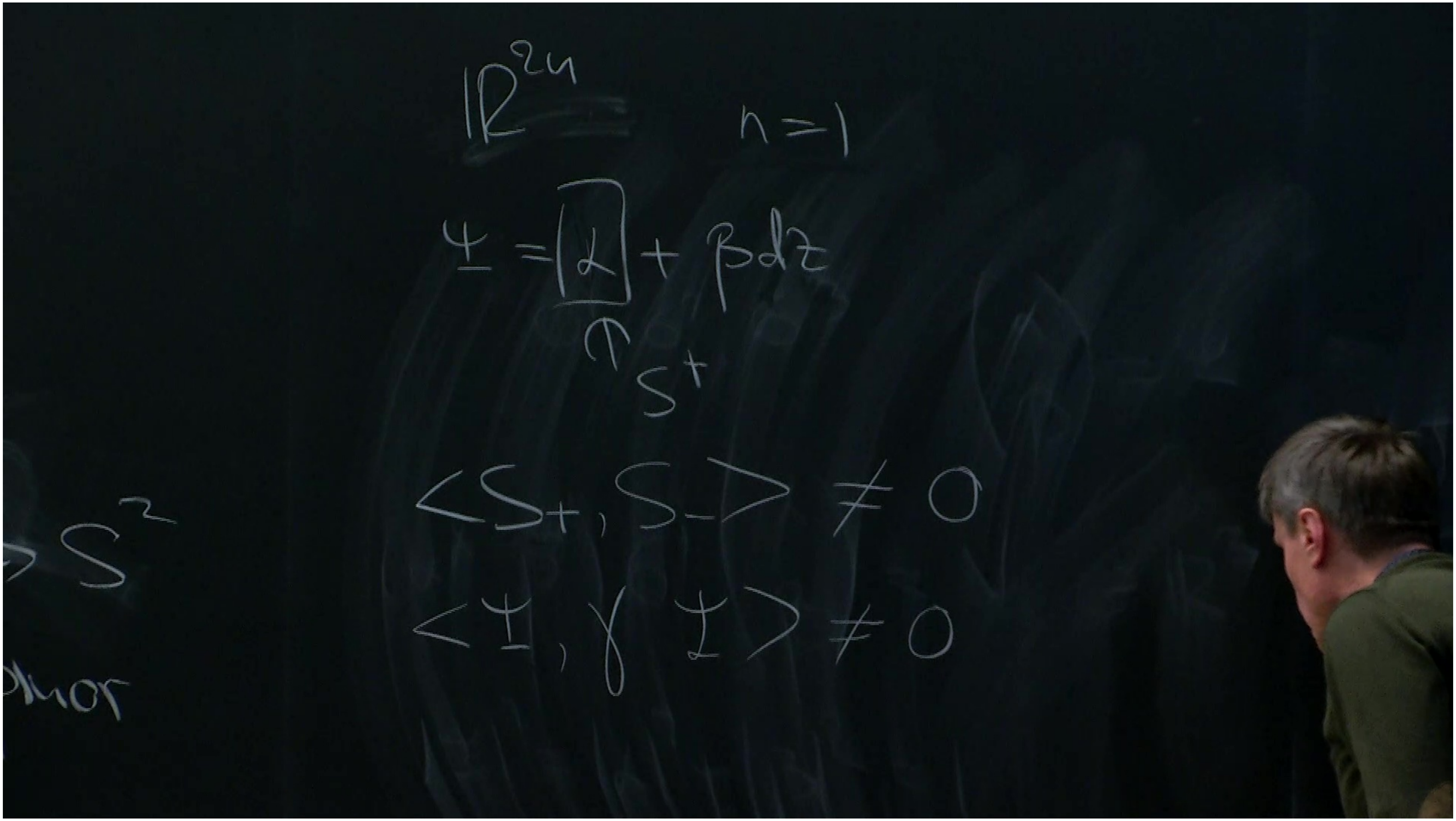
Conjecture – this is always true.

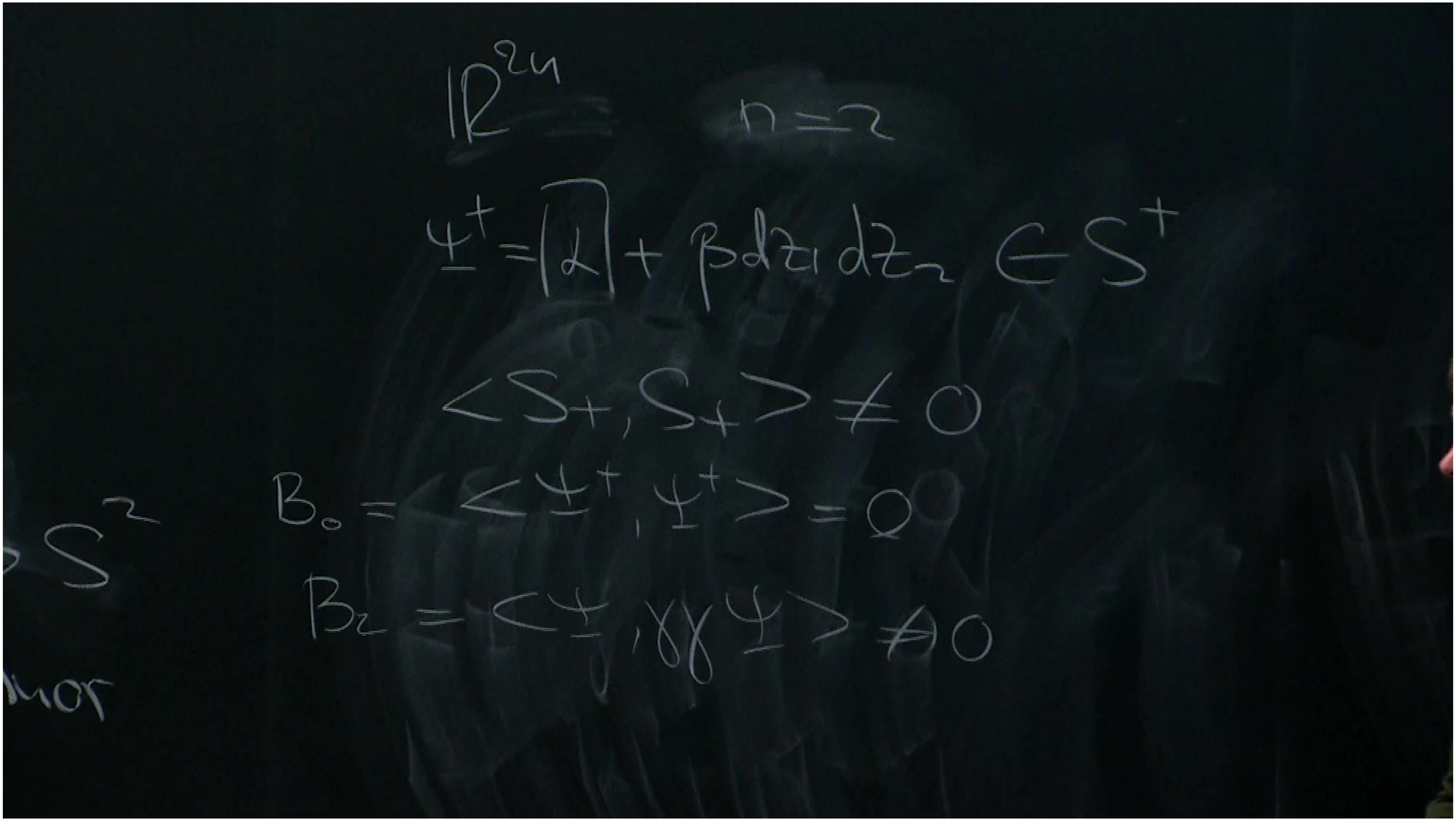
It would be very interesting to work out the details and understand the types of geometry that can arise, e.g. in 16 D


11,3
7,7
Ψ ∈ S+
(10,0)+(1,3)
⟨Ψ, ∂Ψ⟩
3,1
z, x
Ψ = α + β dz dx + γ dz + δ dx
a, a†
b, b†
γ1 = a + a†
γ2 = i(a − a†)
γ3 = b + b†
γ4 = b − b†
R = γ1γ3γ4 X
R = γ2 X



7, 7
Ψ ∈ S+
< Ψ, ∂Ψ >
Ψ1 = α1 + β1 dz dx + γ dz + δ dx
Ψ2 = α2 + β2 dz dx
= (α1 − β1 dz dx) ∧ (α2 + β2 dz dx) |top
(α1β2 − β1α2) dz dx
R = γ1γ3γ4 x
R = γ2 x


n=1
Ψ = α + β dz
S+
< S+ , S- > ≠ 0
< Ψ , γ Ψ > ≠ 0
S²



$\mathbb{R}^{24}$
$n=2$
$\Psi^+ = \alpha + \beta\, dz_1 dz_2 \in S^+$
$\langle S_+, S_+ \rangle \neq 0$
$B_0 = \langle \Psi^+, \Psi^+ \rangle = 0$
$B_2 = \langle \Psi, \gamma\gamma \Psi \rangle \neq 0$
S^2
mor

$$\Psi^+ = |\alpha| + \beta\, dz_1 dz_2$$

$$\gamma\, dz_1 dz_3$$

$$\delta\, dz_2 dz_3 \in S_+$$

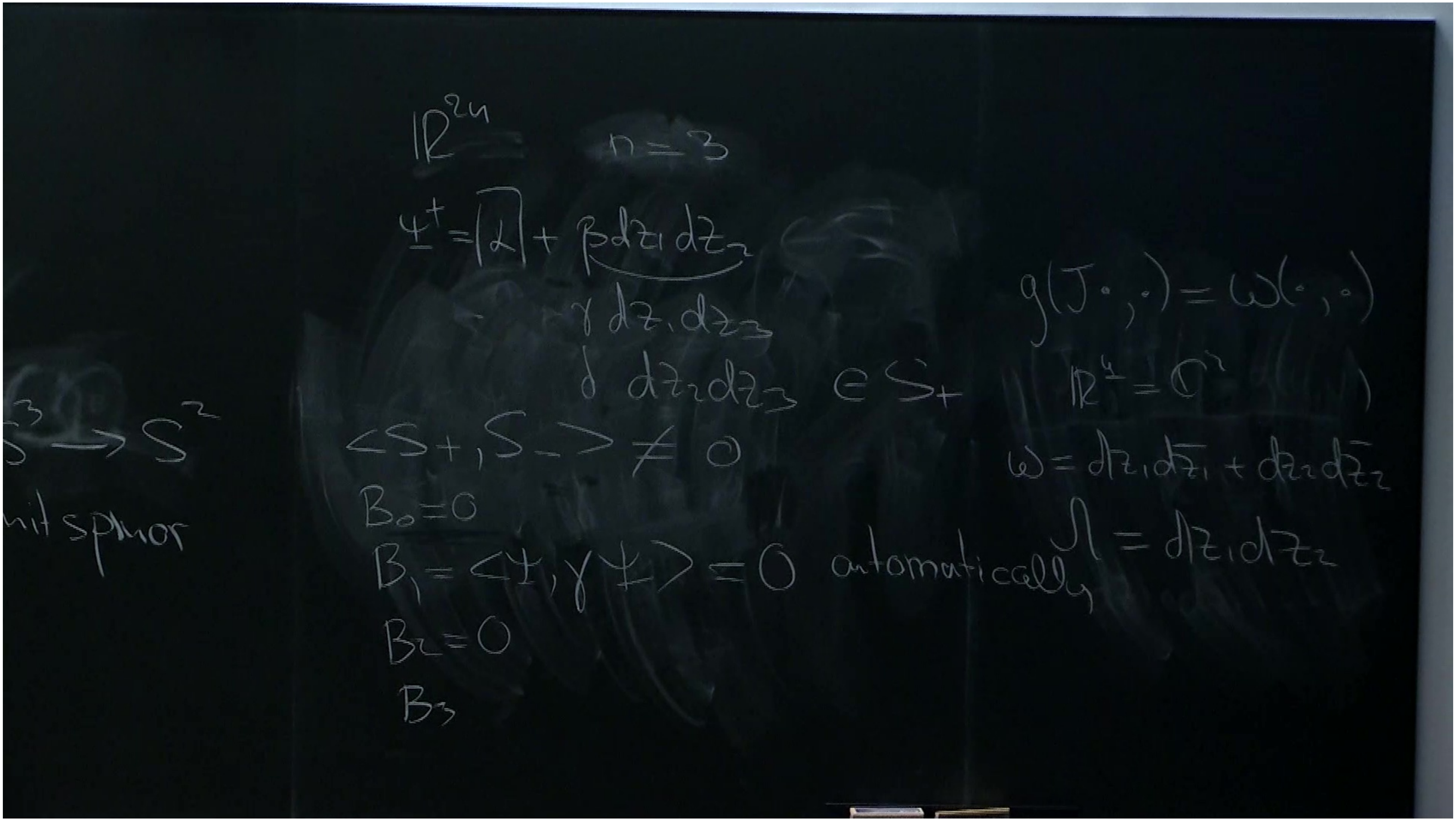
$$\langle S_+, S_- \rangle \neq 0$$

$$B_0 = 0$$

$$B_1 = \langle \Psi, \gamma \Psi \rangle = 0 \quad \text{automatically}$$

$$B_2 = 0$$

$$B_3$$



$\mathbb{R}^{24}$
$n = 3$
$\Psi^+ = \alpha + \beta\, dz_1 dz_2$
$\gamma\, dz_1 dz_3$
$\delta\, dz_2 dz_3 \in S_+$
$S^3 \to S^2$
unit spinor
$\langle S_+, S_- \rangle \neq 0$
$B_0 = 0$
$B_1 = \langle \Psi, \gamma \Psi \rangle = 0$ automatically
$B_2 = 0$
B_3
$g(J\cdot,\cdot) = \omega(\cdot,\cdot)$
$\mathbb{R}^4 = \mathbb{C}^2$
$\omega = dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2$
$\Omega = dz_1 dz_2$

$f : S^3 \longrightarrow S^2$

unit spinor

$\mathbb{R}^{2n}$ $\quad n = 3$

$\Psi^+ = \alpha + \beta\, dz_1 dz_2$

$\gamma\, dz_1 dz_3$

$\delta\, dz_2 dz_3 \in S_+$

$\langle S_+, S_- \rangle \neq 0$

$B_0 = 0$

$B_1 = \langle \Psi, \gamma \Psi \rangle = 0$ automatically

$B_2 = 0$

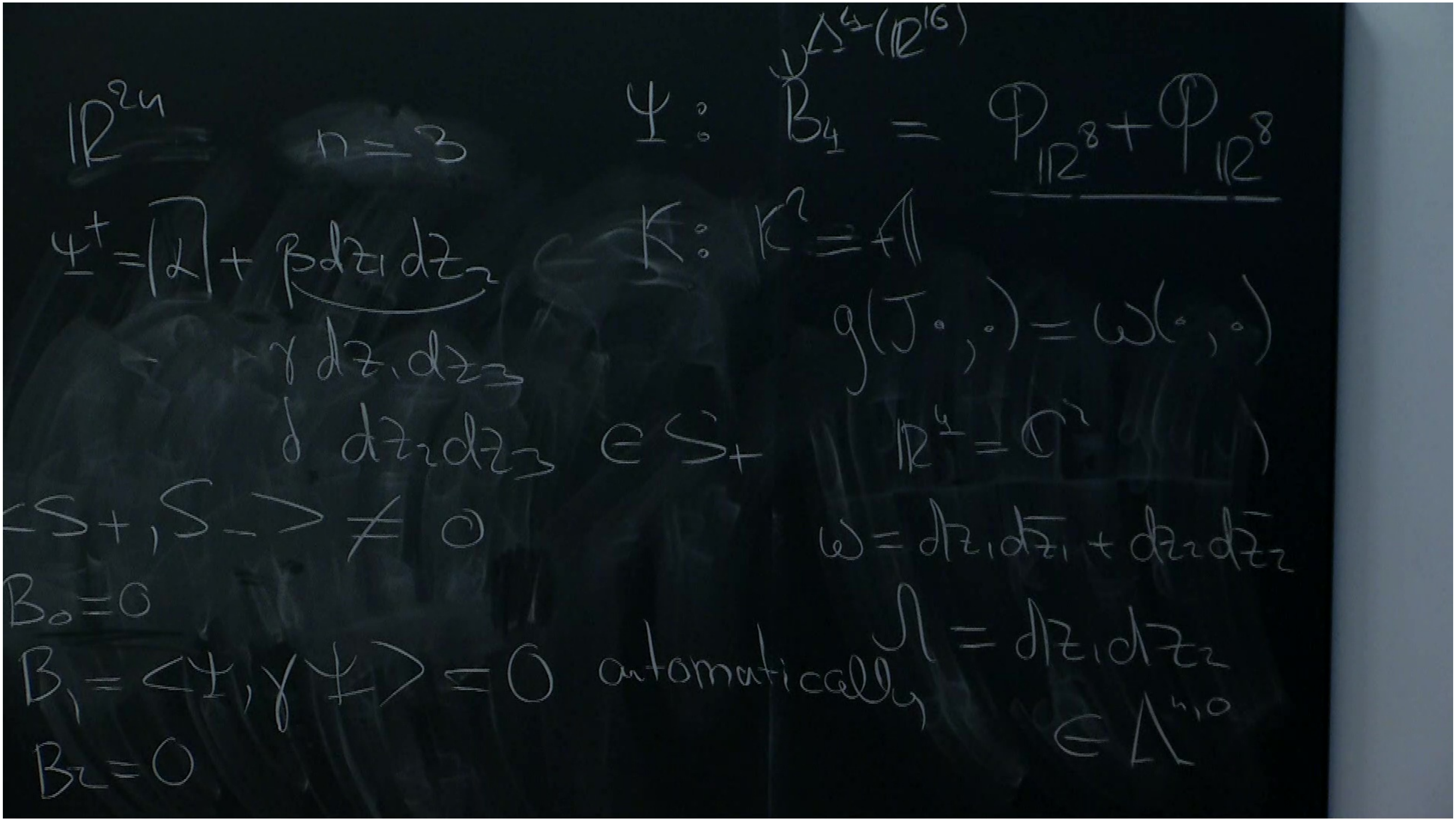
B_3

$g(J\cdot, \cdot) = \omega$

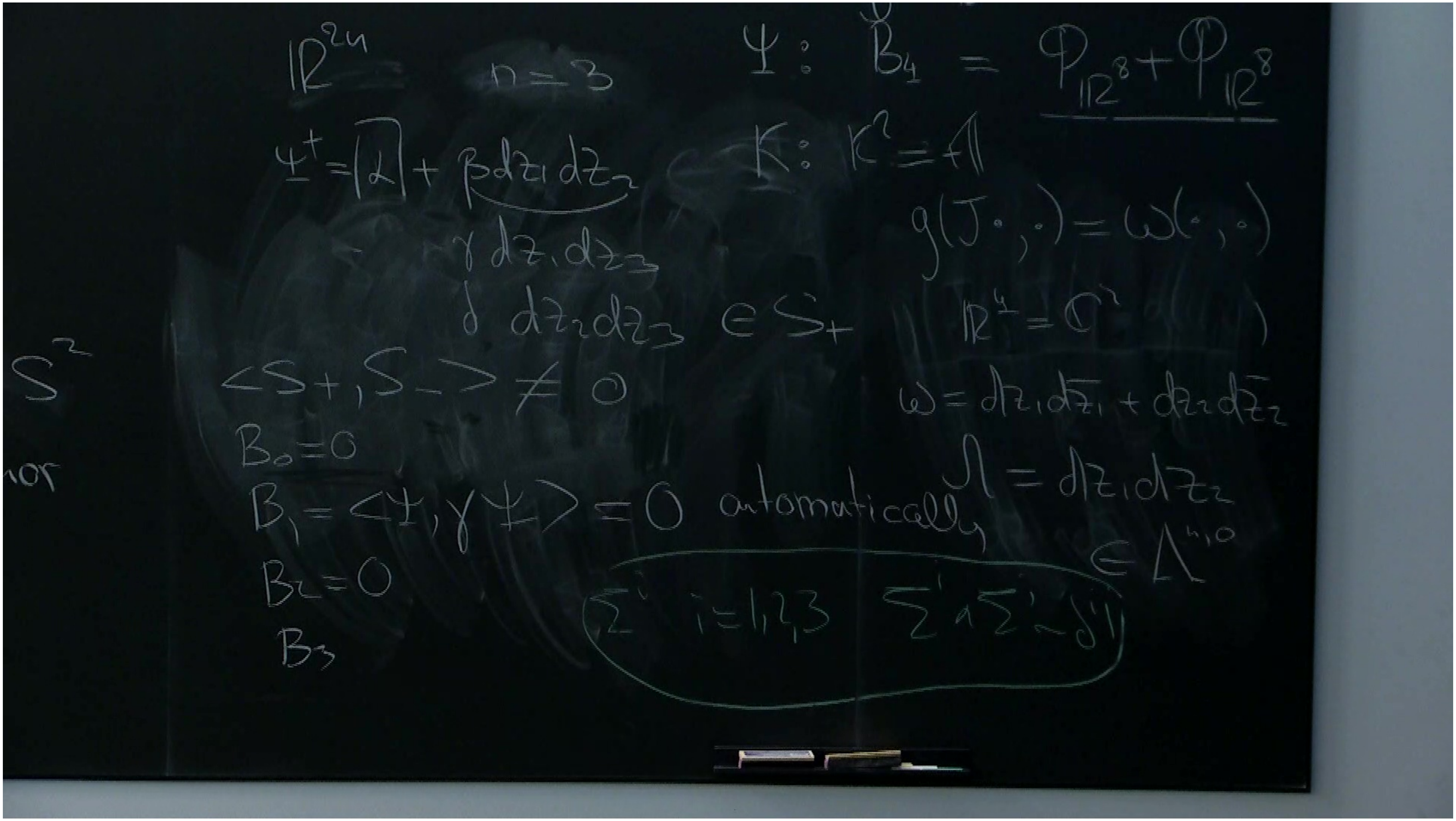
$\mathbb{R}^4 = \mathbb{C}^2$

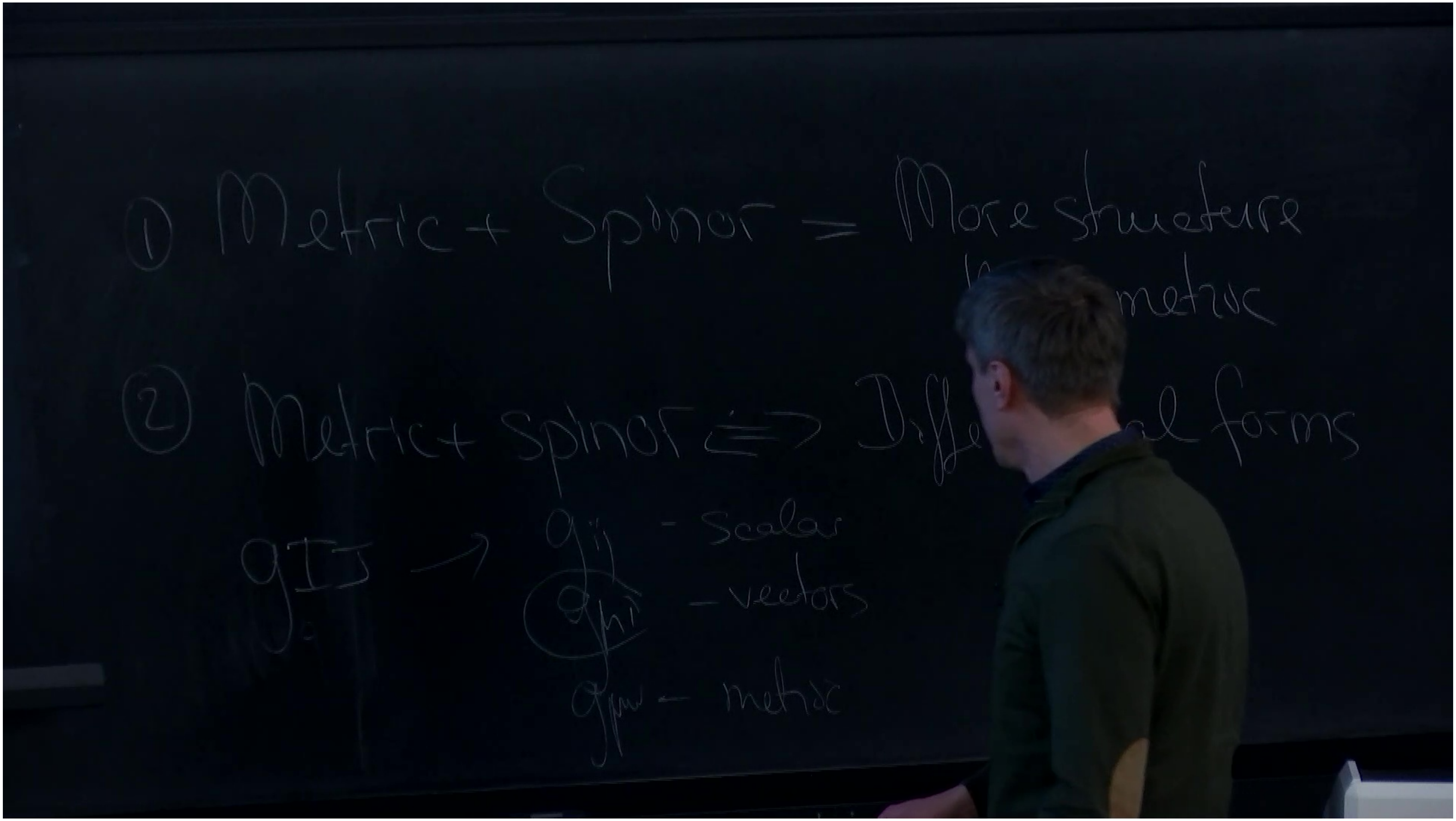
$\omega = dz_1 d\bar{z}_1 + \ldots$

$\Omega = dz_1 d\ldots$



$\mathbb{R}^{24}$
$n = 3$
$\Psi^+ = \alpha + \beta dz_1 dz_2$
$\gamma dz_1 dz_3$
$\delta dz_2 dz_3 \in S_+$
$\langle S_+, S_- \rangle \neq 0$
$B_0 = 0$
$B_1 = \langle \Psi, \gamma \Psi \rangle = 0$ automatically
$B_2 = 0$
$\Lambda^4(\mathbb{R}^{16})$
$\Psi : B_4 = \Phi_{\mathbb{R}^8} + \Phi_{\mathbb{R}^8}$
$K : K^2 = \mathbb{1}$
$g(J\cdot, \cdot) = \omega(\cdot, \cdot)$
$\mathbb{R}^4 = \mathbb{C}^2$
$\omega = dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2$
$\Omega = dz_1 dz_2$
$\in \Lambda^{2,0}$


n = 3
automatically
i = 1, 2, 3


① Metric + Spinor ≥ More structure
metric
② Metric + spinor ⟺ Diffe al forms
$g_{IJ} \to g_{ij}$ - scalar
$g_{\mu i}$ - vectors
$g_{\mu\nu}$ - metric