Title: Causal decompositions of unitary maps

Speakers: Robin Lorenz

Series: Quantum Foundations

Date: April 27, 2023 - 11:00 AM

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Abstract: Every unitary map with a factorisation of domain and codomain into subsystems has a well-defined causal structure given by the set of influence relations between its input and output subsystems. A causal decomposition of a unitary map U is, roughly, one that makes all there is to know about U in terms of causal structure evident and understandable in compositional terms. We'll argue that this is more than just about drawing more intuitive pictures for the causal structure of U -- it is about really understanding it at all. Now, it has been known for a while that decompositions in terms of ordinary circuit diagrams do not suffice to this end and that at least so called 'extended circuit diagrams', or 'routed circuit diagrams' are necessary, revealing a close connection between causal structure and algebraic structures that involve a particular interplay of direct sum and tensor product. Though whether or not these sorts of routed circuit diagrams suffice has been an open question since. I will give an introduction and overview of this business of causal decompositions of unitary maps, and share what is an on-going thriller.

Zoom link: https://pitp.zoom.us/j/95689128162?pwd=RFNqWIVHMFV0RjRaakszSTBsWkZkUT09

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Causal decompositions of unitary maps

- an ongoing thriller -

Robin Lorenz

joint work with Jonathan Barrett and Augustin Vanrietvelde







Perimeter Institute, 27. April 2023

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Outline

Part I: The basics – the idea of causal decompositions of unitary transformations

Part II: Quick interlude - the wider context and why care

Part III: The hypothesis – where does it stand?

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Will draw on:

Allen, Barrett, Horsman, Lee, Spekkens (2017). Quantum Common Causes and Quantum Causal Models.

Barrett, RL, Oreshkov (2019). Quantum causal models.

RL, Barrett (2021). Causal and compositional structure of unitary transformations.

Vanrietvelde, Kristjánsson and Barrett (2021). Routed quantum circuits.

Barrett, RL, Oreshkov (2019). Cyclic Quantum causal models.

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Bits of notation

NB finite dimensions throughout (classically and quantum).

Denote system A and unitary maps as:

$$\mathcal{H}_A$$
 U : $\mathcal{H}_A \to \mathcal{H}_B$

$$A := \mathcal{L}(\mathcal{H}_A)$$
 $\mathcal{U} : A \to B$

$$X \mapsto UXU^{\dagger}$$

For CP map $\mathcal{E}: A \to B$ define **CJ operator**:

$$\rho_{\mathcal{B}|A}^{\mathcal{E}} = \sum_{i,j} \mathcal{E}(|i\rangle_{A} \langle j|) \otimes |i\rangle_{A^{*}} \langle j| \in \mathcal{L}(\mathcal{H}_{\mathcal{B}} \otimes \mathcal{H}_{A}^{*})$$

Causal influence

Definition [Causal (no-)influence]: Let $U: \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_C \otimes \mathcal{H}_D$ be a unitary transformation. Then *A does not influence* D, write $A \nrightarrow D$, iff $\exists C$:

$$\begin{array}{c|c}
\hline
 & D \\
\hline
 & 2 \\
\hline
 & A \\
\hline
 & B
\end{array} = \begin{bmatrix}
 & D \\
 & A \\
\hline
 & B
\end{bmatrix}$$

A is a direct cause of D iff A does influence D, i.e. $\neg (A \nrightarrow D)$.

These are all equivalent

$$\begin{array}{c|c}
\hline
C & D \\
\hline
U & = & C \\
\hline
A & B & A & B
\end{array}$$

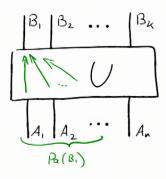
$$\mathsf{Tr}_{\mathcal{C}} \circ \mathcal{U} = \mathcal{C} \circ \mathsf{Tr}_{\mathcal{A}}$$

$$\mathsf{Tr}_{\mathcal{C}}ig[
ho_{\mathcal{C}D|AB}^{\mathcal{U}}ig] =
ho_{D|B}^{\mathcal{C}} \otimes \mathsf{id}_{A^*}$$

$$\begin{array}{c|c}
\hline
\mathcal{U} \\
A \\
B \\
A'
\end{array} = \begin{bmatrix}
C \\
A'
\end{bmatrix}$$

$$\mathcal{U}^{\dagger}(D) \subseteq B$$
 $\mathcal{U}^{\dagger}(\mathrm{id}_{C} \otimes D) \subseteq \mathrm{id}_{A} \otimes B$
 $\left[\mathcal{U}^{\dagger}(D), A\right] = 0$

The special properties of unitary maps I



For each output B_j define:

$$Pa(B_j) := \{A_i \mid A_i \rightarrow B_j\}$$

Theorem ['über-theorem']: The CJ operator of a unitary U factorises as:

$$\rho_{B_1...B_k|A_1...A_n}^U = \prod_{j=1}^k \rho_{B_j|Pa(B_j)} ,$$

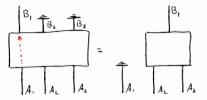
where $\left[
ho_{\mathcal{B}_j \mid Pa(\mathcal{B}_j)} \;,\;
ho_{\mathcal{B}_m \mid Pa(\mathcal{B}_m)}
ight] = 0 \;\; orall \; j, m = 1, ..., k.$

[Allen et al. (2016), Barrett et al. (2019)]

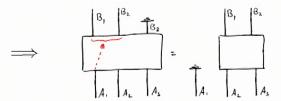
Proof

The special properties of unitary maps II

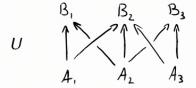
Causal atomicity:
$$A_i \nrightarrow B_j \land A_i \nrightarrow B_k \Rightarrow A_i \nrightarrow B_j B_k$$







The special properties of unitary maps III



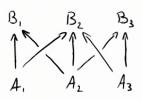
Summary causal structure of unitary maps

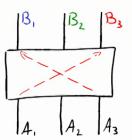
$$\left(Pa(B_j)\right)_{j=1}^k$$

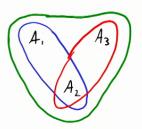
$$\mathcal{U}^{\dagger}(B_j) \subset \mathit{Pa}(B_j)$$

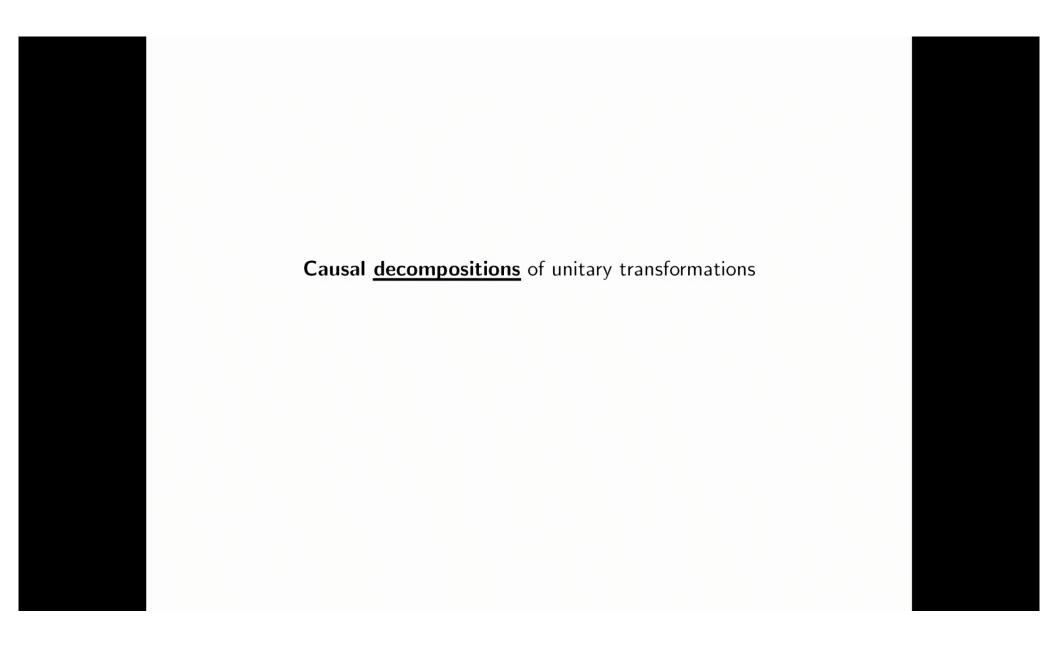
$$\left(Ch(A_i)\right)_{i=1}^n$$

$$\mathcal{U}(A_i) \subset Ch(A_i)$$

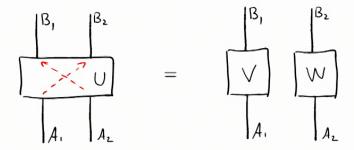






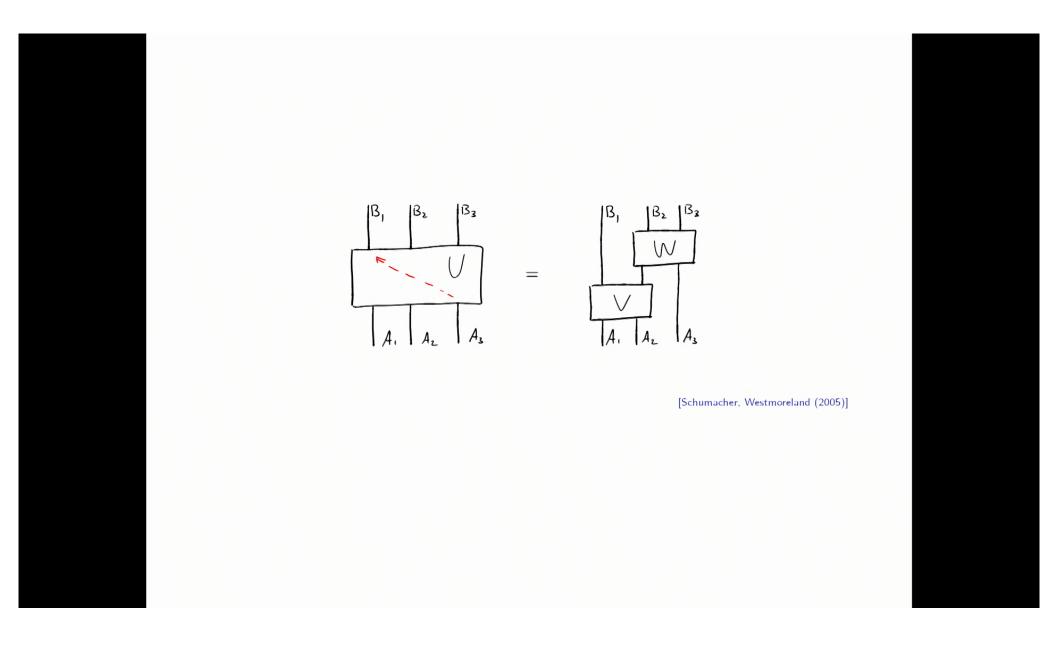


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$$\rho^{U}_{B_{1}B_{2}|A_{1}A_{2}} = \rho_{B_{1}|A_{1}} \rho_{B_{2}|A_{2}} = \rho_{B_{1}|A_{1}} \otimes \rho_{B_{2}|A_{2}}$$

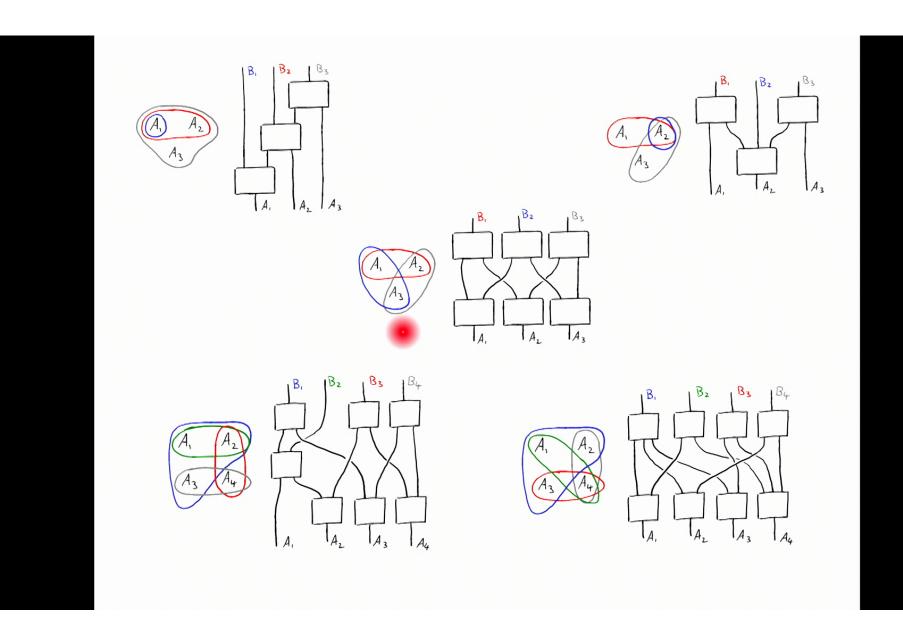
$$\mathcal{U}(A_1) \subset B_1 \quad \wedge \quad \mathcal{U}(A_2) \subset B_2 \quad \Rightarrow \quad \mathcal{U}(A_1) = B_1 \quad \wedge \quad \mathcal{U}(A_2) = B_2$$



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NB path connectivity in circuit diagram is sound for causal influence relations.

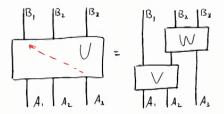
A unitary circuit diagram representing U is called **causally faithful** iff there is path from A_i to B_j in the diagram $\Leftrightarrow A_i \to B_j$ in U.

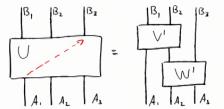


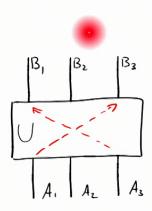
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Hypothesis 1.0: All unitaries have causally faithful circuit decompositions.

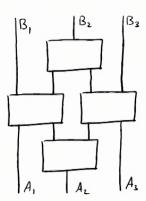
False!











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Theorem. Given unitary $U: \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{A_3} \to \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2} \otimes \mathcal{H}_{B_3}$, if $A_1 \nrightarrow B_3$ and $A_3 \nrightarrow B_1$, then

$$U = (\operatorname{id}_{B_1} \otimes T \otimes \operatorname{id}_{B_3}) \left(\bigoplus_i V_i \otimes W_i \right) \left(\operatorname{id}_{A_1} \otimes S \otimes \operatorname{id}_{A_3} \right)$$

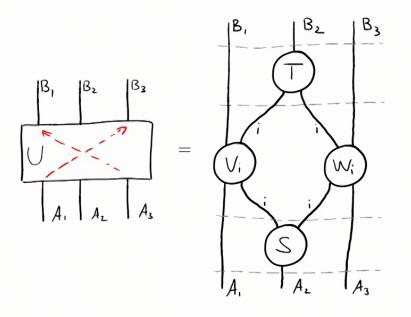
for some (families of) Hilbert spaces and unitaries

$$S: \mathcal{H}_{A_2} \rightarrow \bigoplus_i \mathcal{H}_{X_i^L} \otimes \mathcal{H}_{X_i^R} ,$$

$$S: \mathcal{H}_{A_2} \rightarrow \bigoplus_i \mathcal{H}_{X_i^L} \otimes \mathcal{H}_{X_i^R} , \qquad V_i: \mathcal{H}_{A_1} \otimes \mathcal{H}_{X_i^L} \rightarrow \mathcal{H}_{B_1} \otimes \mathcal{H}_{Y_i^L} ,$$

$$T: \bigoplus_i \mathcal{H}_{Y_i^L} \otimes \mathcal{H}_{Y_i^R} \rightarrow \mathcal{H}_{B_2}$$
,

$$W_i : \mathcal{H}_{X_i^R} \otimes \mathcal{H}_{A_3} \rightarrow \mathcal{H}_{Y_i^R} \otimes \mathcal{H}_{B_3}$$
.



$$\mathcal{H}_{\mathcal{B}_1} \otimes \mathcal{H}_{\mathcal{B}_2} \otimes \mathcal{H}_{\mathcal{B}_3}$$

$$\uparrow \operatorname{id}_{\mathcal{B}_1} \otimes \mathcal{T} \otimes \operatorname{id}_{\mathcal{B}_3}$$

$$\mathcal{H}_{\mathcal{B}_1} \otimes (\bigoplus_i \mathcal{H}_{Y_i^L} \otimes \mathcal{H}_{Y_i^R}) \otimes \mathcal{H}_{\mathcal{B}_3}$$

$$\uparrow \bigoplus_i V_i \otimes W_i$$

$$\mathcal{H}_{\mathcal{A}_1} \otimes (\bigoplus_i \mathcal{H}_{X_i^L} \otimes \mathcal{H}_{X_i^R}) \otimes \mathcal{H}_{\mathcal{A}_3}$$

$$\uparrow \operatorname{id}_{\mathcal{A}_1} \otimes \mathcal{S} \otimes \operatorname{id}_{\mathcal{A}_3}$$

$$\mathcal{H}_{\mathcal{A}_1} \otimes \mathcal{H}_{\mathcal{A}_2} \otimes \mathcal{H}_{\mathcal{A}_3}$$

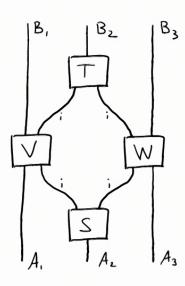
The graphical language

Extended circuit diagrams

[RL, Barrett (2021)]

(Index-matching) routed circuits

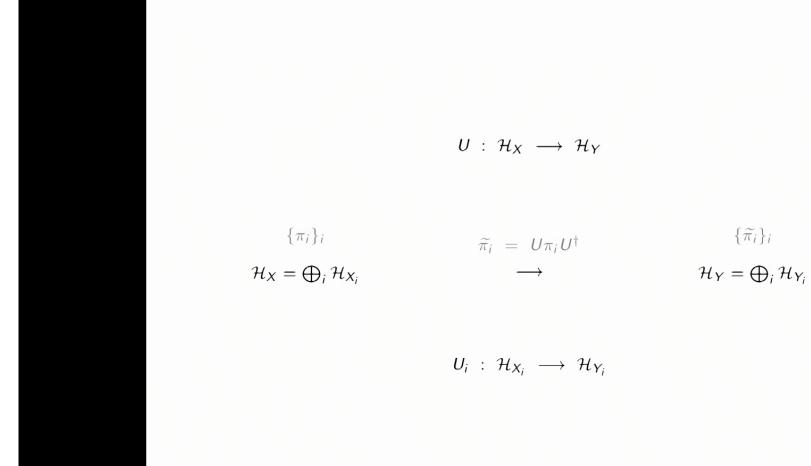
[Vanrietvelde, Kristjánsson and Barrett (2021)]



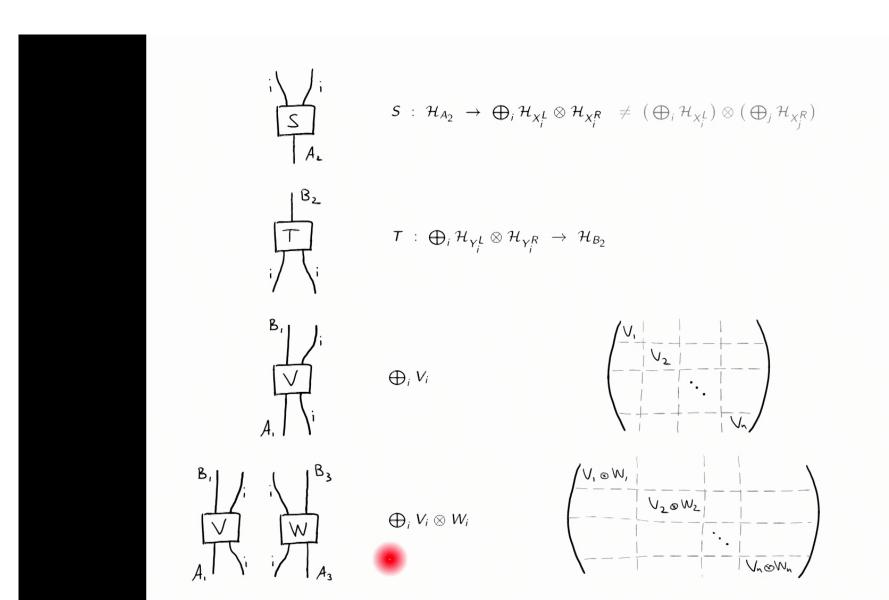
Well-behaved compositional language: symmetric monoidal category

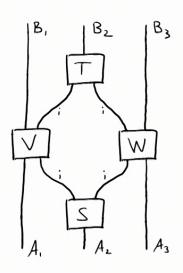
Notion of **causally faithful** decompositions in terms of index-matching routed circuits makes sense.

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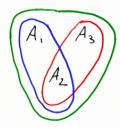
 $\{\widetilde{\pi_i}\}_i$





 $U \ = \ \left(\mathsf{id}_{B_1} \otimes \mathcal{T} \otimes \mathsf{id}_{B_3}\right) \ \left(\ \bigoplus_i V_i \otimes W_i \right) \ \left(\mathsf{id}_{A_1} \otimes \mathcal{S} \otimes \mathsf{id}_{A_3}\right)$

The core ideas behind the proof



Proofs in [RL, Barrett (2021)] exploited the factorisation of CJ operators. In this case:

$$\rho^U_{B_1B_2B_3|A_1A_2A_3} \quad = \quad \rho_{B_1|A_1A_2} \; \rho_{B_2|A_1A_2A_3} \; \rho_{B_3|A_2A_3} \; .$$

But let's stay closer to the heart of things.

Theorem ['Wedderburn decomposition']: Let $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$ be a *-subalgebra. Then there exists a Hilbert space $\widetilde{\mathcal{H}} = \bigoplus_i \mathcal{H}_{X_i^L} \otimes \mathcal{H}_{X_i^R}$ and a unitary map $\mathcal{S}: \mathcal{H} \to \widetilde{\mathcal{H}}$ such that

$$SAS^{\dagger} = \bigoplus_{i} \mathcal{L}(\mathcal{H}_{X_{i}^{L}}) \otimes id_{X_{i}^{R}}.$$

$$\mathcal{L}(\mathcal{H}) \supseteq \mathcal{A} \cong \bigoplus_{i} X_{i}^{L} \otimes \mathrm{id}_{X_{i}^{R}}$$
 $\mathcal{A}' \cong \bigoplus_{i} \mathrm{id}_{X_{i}^{L}} \otimes X_{i}^{R}$

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Theorem ['Wedderburn decomposition']: Let $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$ be a *-subalgebra. Then there exists a Hilbert space $\widetilde{\mathcal{H}} = \bigoplus_i \mathcal{H}_{X_i^L} \otimes \mathcal{H}_{X_i^R}$ and a unitary map $\mathcal{S}: \mathcal{H} \to \widetilde{\mathcal{H}}$ such that

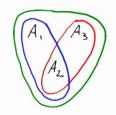
$$SAS^{\dagger} = \bigoplus_{i} \mathcal{L}(\mathcal{H}_{X_{i}^{L}}) \otimes id_{X_{i}^{R}}.$$

$$\mathcal{L}(\mathcal{H}) \supseteq \mathcal{A} \cong \bigoplus_{i} X_{i}^{L} \otimes \operatorname{id}_{X_{i}^{R}}$$

$$\mathcal{A}' \cong \bigoplus_{i} \operatorname{id}_{X_{i}^{L}} \otimes X_{i}^{R}$$

$$= \left\{ x \in \mathcal{L}(\mathcal{H}) \mid [x, \mathcal{A}] = 0 \right\}$$

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$$\mathcal{U}(A_1) \subset B_1B_2$$

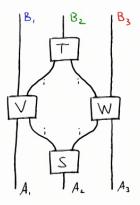
 $\mathcal{U}(A_3) \subset B_2B_3$

$$F := \mathcal{U}(A_1A_2) \cap B_2 \stackrel{\mathcal{T}}{\cong} \bigoplus_i Y_i^L \otimes \mathrm{id}_{Y_i^R}$$

$$\mathcal{U}^{\dagger}\big(\ Z[F]\ \big)\subset A_2$$

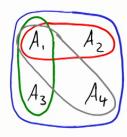
$$\begin{array}{cccc} U_i: \mathcal{H}_{A_1} \otimes \mathcal{H}_{(A_2)_i} \otimes \mathcal{H}_{A_3} & \to & \mathcal{H}_{B_1} \otimes \mathcal{H}_{Y_i^L} \otimes \mathcal{H}_{Y_i^R} \otimes \mathcal{H}_{B_3} \\ \\ A_1 \nrightarrow Y_i^R B_3, & A_3 \nrightarrow B_1 Y_i^L \end{array},$$

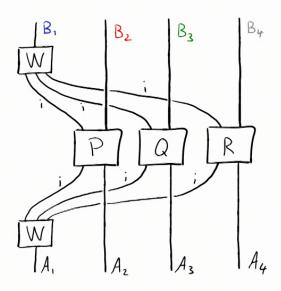
 $U_i = V_i \otimes W_i$.



Centre of a subalgebra: $Z[A] = A \cap A'$.

Some more decompositions





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Diagrammatics for causal models

Classical causal models

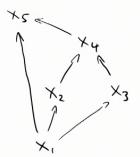
$$P(X_1, X_2, X_3, X_4, X_5)$$

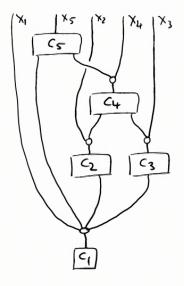
$$= P(X_5 | X_1, X_4)$$

$$P(X_4 | X_2, X_3)$$

$$P(X_2 | X_1) P(X_3 | X_1)$$

$$P(X_1)$$

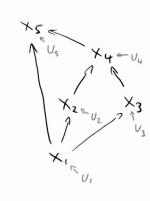




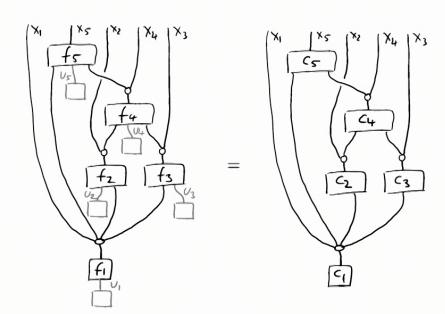
Causal models as *network diagrams* composed of single-output boxes and copy maps in *FStoch*.

[Fong (2013); Jacobs *et al.* (2021); Fritz and Klingler (2023)] [RL, Sean Tull (2023)]

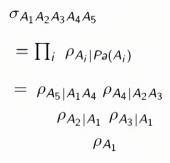
In particular for *functional* causal models:

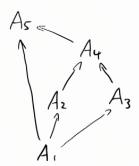


$$\left\{ \begin{array}{l} f_i: Pa(X_i) \times U_i
ightarrow X_i \end{array}
ight\}_i \ \left\{ \begin{array}{l} P(U_i) \end{array}
ight\}_i$$

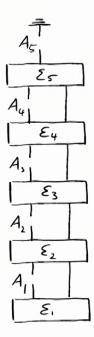


Now, what of quantum causal models?





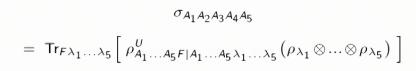
[Allen et al. (2017)], [Barret, RL, Oreshkov (2019)]

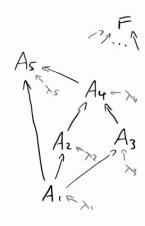


Defines a comb. [Chiribella et al. (2009)]

Though **unsatisfactory** as representation of a causal model!

Quantum causal models are grounded in causal structure of unitary transformations

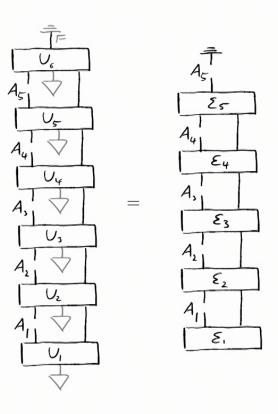


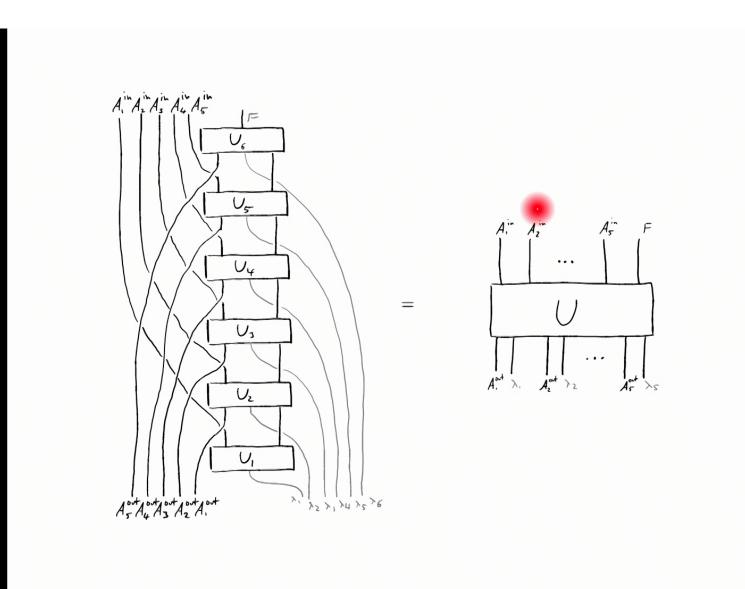


$$A_i \nrightarrow A_j \text{ for } A_i \notin Pa(A_i)$$

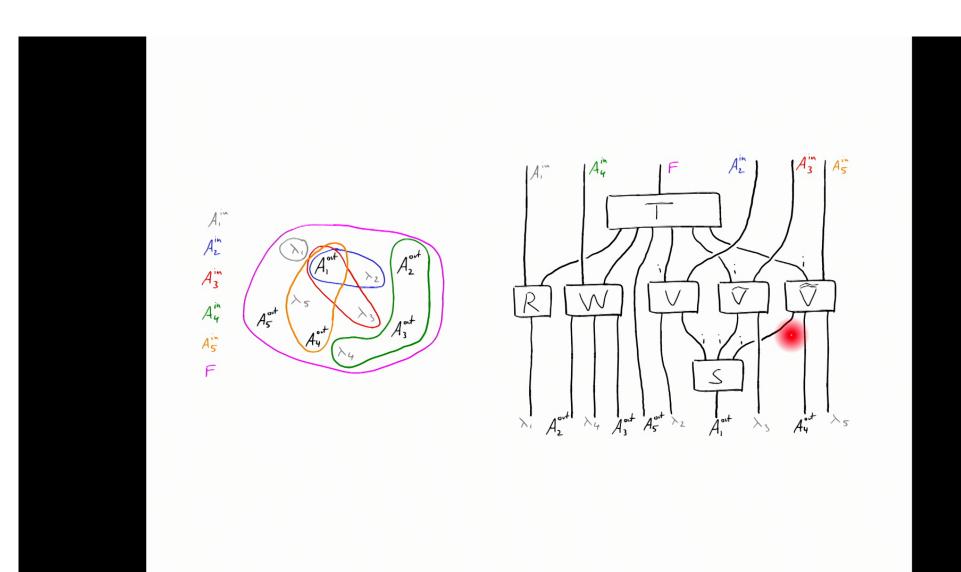
 $\lambda_i \nrightarrow A_j \text{ for } i \neq j$

A bit better:

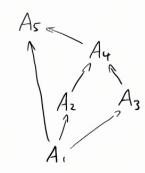




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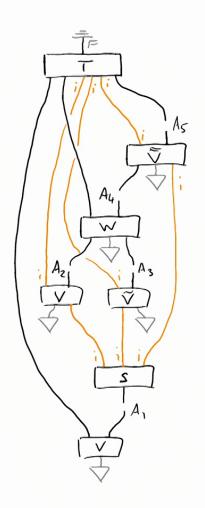


$$\sigma_{A_1 A_2 A_3 A_4 A_5}$$

$$= \rho_{A_5 | A_1 A_4} \rho_{A_4 | A_2 A_3}$$

$$\rho_{A_2 | A_1} \rho_{A_3 | A_1}$$

$$\rho_{A_1}$$



Processes with indefinite causal order

 Insightful decomposition of causally non-separable processes like the Quantum SWITCH and the Lugano process.

[Barrett, RL, Oreshkov (2021), Ormrod et al. (2022)]

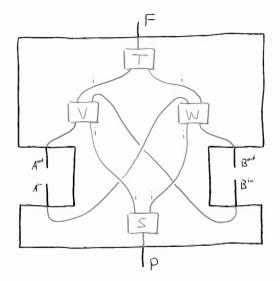
 Behind results like all bipartite unitarily extendible processes are causally separable.

[Barrett, RL, Oreshkov (2021)]

• Construction recipe for *Consistent circuits with* indefinite causal order.

[Vanrietvelde, Ormrod, Kristjánsson and Barrett (2022)]

- Hope for progress with technical open questions (equivalence Markov condition and compatibility in cyclic case.)
- Natural question: all unitary processes direct sums over acyclic processes?



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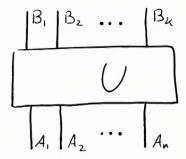
Causal decompositions have been and likely will be useful for:

- (1) Diagrammatics for causal models & understanding of causal mechanisms
- (2) Handle on processes with indefinite causal order (cyclic causal structure)
- (3) Inspiration for a quantum realist causal-inferential theory?
- (4) Inspiration for foundational works on interpretations and counterfactuals?
- (5) Other not explicitly 'causal' quantum information theoretical problems?

:

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What is known



Processes with indefinite causal order

 Insightful decomposition of causally non-separable processes like the Quantum SWITCH and the Lugano process.

[Barrett, RL, Oreshkov (2021), Ormrod et al. (2022)]

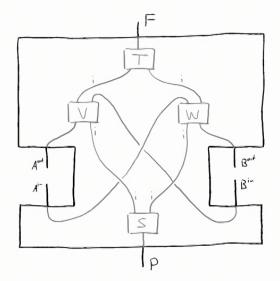
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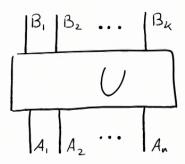


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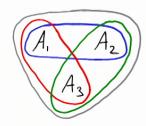
What is known

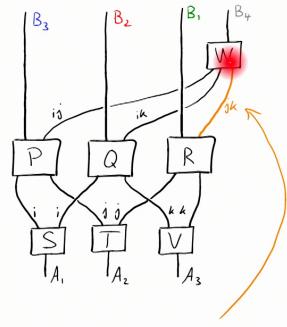


A causally faithful decomposition is known for any \boldsymbol{U} with:

- $n \in \mathbb{N}$, k = 2
- $n = 2, k \in \mathbb{N}$
- $n = 3, k \in \mathbb{N}$
- $n \in \mathbb{N}$, k = 3

A particularly interesting example for n=3, k=4



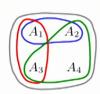


$$\mathcal{U}(A_2A_3)\cap B_4 = \bigoplus_{i,j,k} \operatorname{id}_{N_{ij}^{(1)}} \otimes \operatorname{id}_{N_{ik}^{(2)}} \otimes N_{jk}^{(3)}$$

$$W: \bigoplus_{i,j,k} \mathcal{H}_{N_{ij}^{(1)}} \otimes \mathcal{H}_{N_{ik}^{(2)}} \otimes \mathcal{H}_{N_{jk}^{(3)}} \rightarrow \mathcal{H}_{\mathcal{B}_4}$$

The (4,4) cases





















2







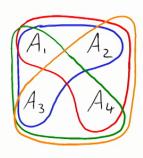






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A particularly interesting case



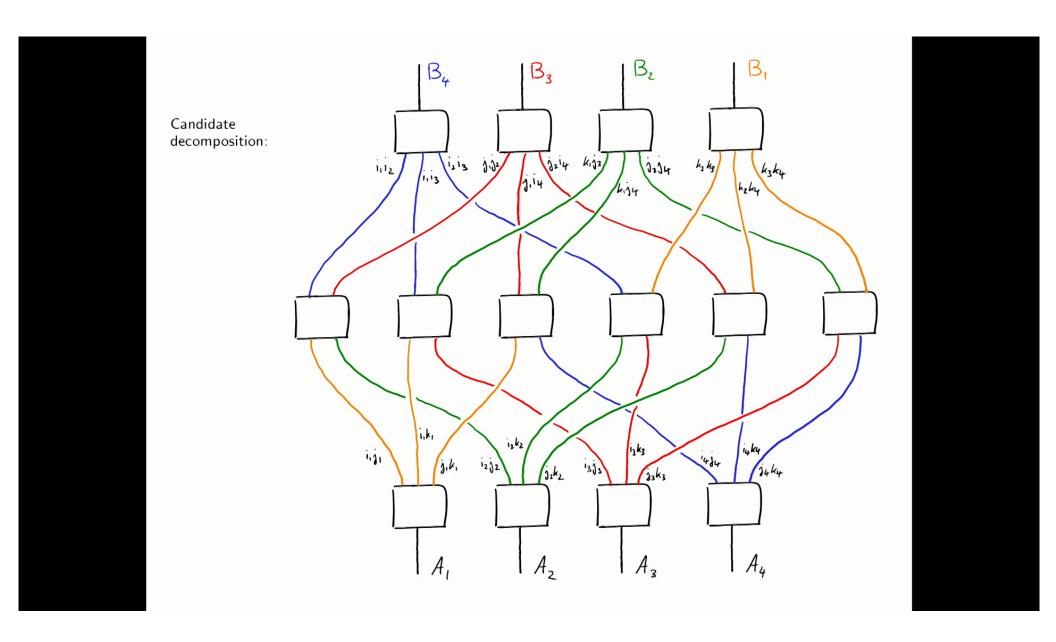
$$\forall i=1,...4: A_i \nrightarrow B_i$$

 $\mathcal{U}(A_1) \subset B_2B_3B_4$

 $\mathcal{U}(A_2) \subset B_1B_3B_4$

 $\mathcal{U}(A_3) \subset B_1B_2B_4$

 $\mathcal{U}(A_4) \subset B_1B_2B_3$



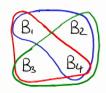
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Return to the interesting (3,4) case – a glimpse of hope?

$$\mathcal{U}(A_1) \subset B_2B_3B_4$$

 $\mathcal{U}(A_2) \quad \subset \quad B_1B_3B_4$

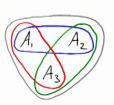
 $\mathcal{U}(A_3) \subset B_1B_2B_4$

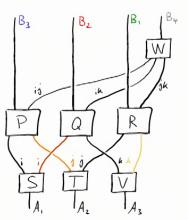


$$\mathcal{U}^{\dagger}(B_1) \subset A_2A_3$$

 $\mathcal{U}^{\dagger}(B_2) \subset A_1A_3$

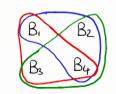
 $\mathcal{U}^{\dagger}(B_3) \subset A_1A_2$

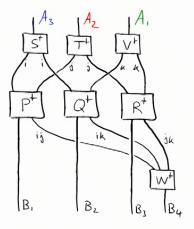




Return to the interesting (3,4) case – a glimpse of hope?

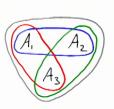
 $\mathcal{U}(A_1) \subset B_2B_3B_4$ $\mathcal{U}(A_2) \subset B_1B_3B_4$ $\mathcal{U}(A_3) \subset B_1B_2B_4$

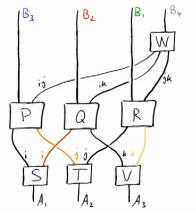




??

 $\mathcal{U}^{\dagger}(B_1) \subset A_2A_3$ $\mathcal{U}^{\dagger}(B_2) \subset A_1A_3$ $\mathcal{U}^{\dagger}(B_3) \subset A_1A_2$

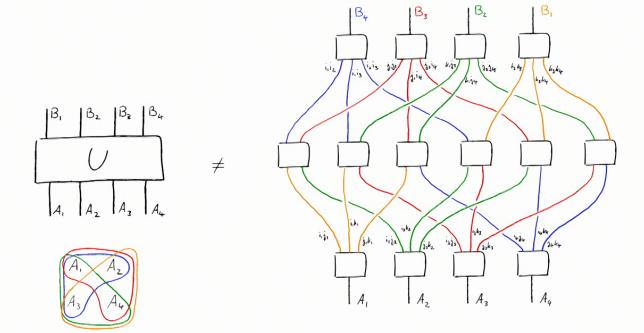




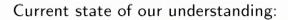
 $\mathcal{U}^{\dagger}(B_2B_4) \cap A_1$ $\mathcal{U}^{\dagger}(B_3B_4) \cap A_2$ $\mathcal{U}^{\dagger}(B_1B_4) \cap A_3$

However....

Augustin recently found a concrete U, which does $\operatorname{\mathbf{not}}$ have expected decomposition:



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That concrete map U does not evade a compositional understanding and there will be a faithful decomposition – we have a few ideas – but it will likely **require extending** the diagrammatic language!

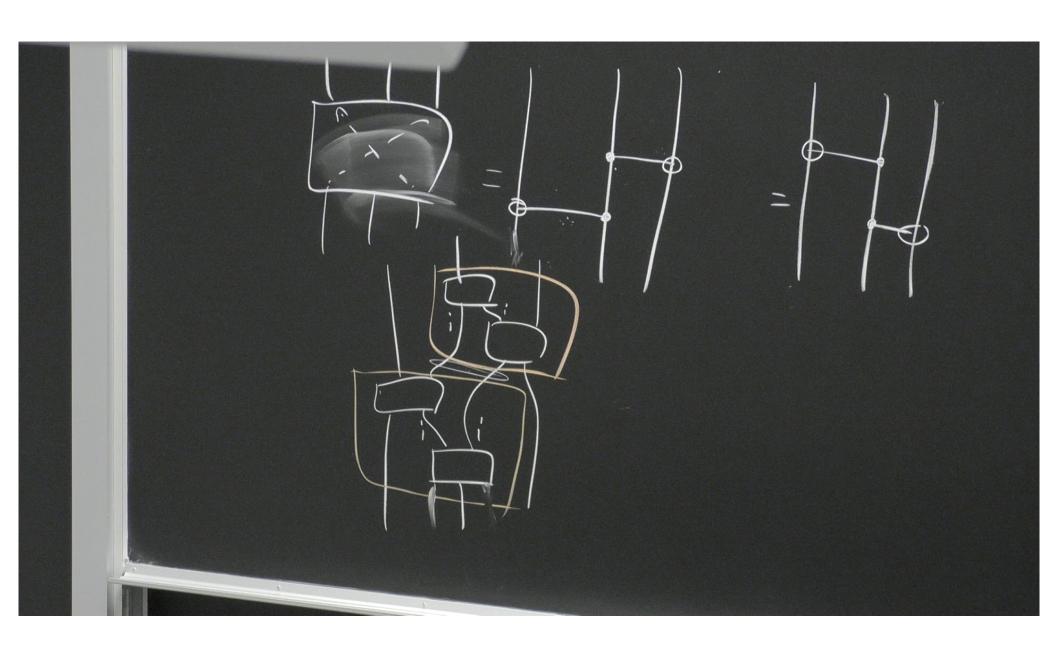
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? ? ? Hypothesis 2.1 Hypothesis 3.0 No-go-theorem Mildly extending A quite different approach, a (index-matching) routed language that isn't a superset circuits suffices. of (index-matching) routed circuits?

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? ? ? Hypothesis 2.1 Hypothesis 3.0 No-go-theorem Mildly extending A quite different approach, a (index-matching) routed language that isn't a superset of (index-matching) routed circuits suffices. circuits? The thriller continues.

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