Title: Quantum matter from algebraic geometry

Speakers: Steven Rayan

Series: Mathematical Physics

Date: April 28, 2023 - 1:30 PM

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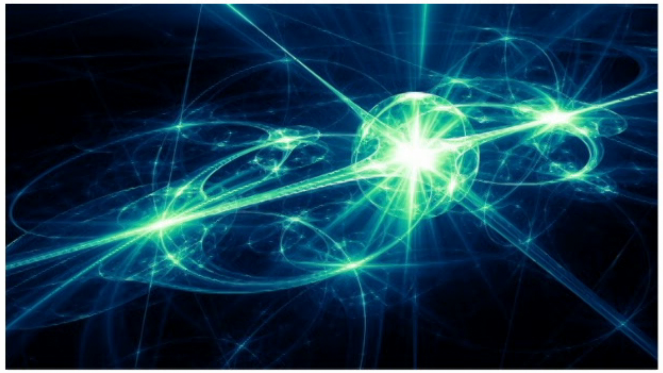
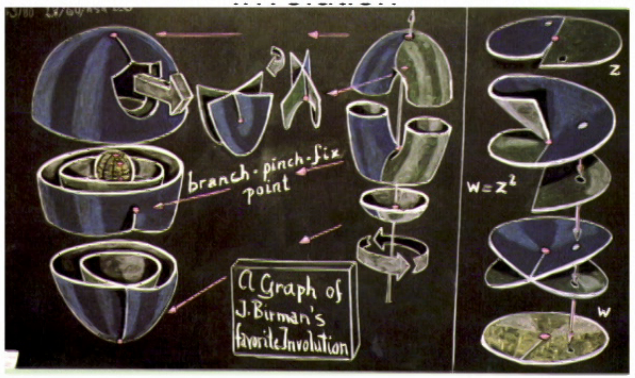
Abstract: The advent of topological materials has brought with it unexpected new connections between physics and pure mathematics. In particular, algebraic topology has played a significant role in the classification of topological materials. In this talk, I will offer a brief look at an emerging chapter in this story in which algebraic geometry -- in particular the algebraic geometry of moduli spaces associated with complex curves -- is used to anticipate new forms of quantum matter arising from 2-dimensional hyperbolic lattices. In the process, I will explain my recent joint works with each of J. Maciejko, E. Kienzle, and A. Nagy that establishes an electronic band theory for 2-dimensional hyperbolic matter.

Zoom link: https://pitp.zoom.us/j/98074477672?pwd=bmVScWx1M09EaGx2ZXZrRit6NXF5dz09

Quantum Matter from Algebraic Geometry





Mathematical Physics Seminar
Perimeter Institute

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April 28, 2023

Algebraic Geometry in Physics

- AG enjoys fruitful interactions with physics, particularly around high-energy physics, often in the form of mirror symmetry
- Algebraic and arithmetic perspectives have led to robust programs for constructing and interpreting mirror pairs in both the Calabi-Yau and Fano settings (e.g. Batyrev-Borisov mirror symmetry, Landau-Ginzburg models, Gross-Schubert program), as well as quiver-theoretic descriptions of gauge theories
- What about solid-state physics?

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Quantum Materials

- Traditionally, *strongly-correlated systems of electrons*
- Perhaps best defined by what they are *not*
- They are materials that exhibit behaviours with no counterpart in the macroscopic world
- These behaviours are emergent ones, resulting from an atypical dependence of the 4 fundamental quantum degrees of freedom: charge, spin, orbit, and lattice
- This dependence is induced under precise and/or extreme conditions (thin highly-engineered crystalline films, supercooling, strong transverse magnetic fields)

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Examples

- *Spin liquids:* crystalline materials in which the spin of atomic components are disordered and dynamically fluctuating
- E.g. titanate pyrochlores (such as $Yb_2Ti_2O_7$)
- *Spin glasses:* magnetic materials in which spins are random
- E.g. IrMnGa

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Topologogy is a field of mathematics that describes properties that are stable and only change in integer steps: 1, 2, 3…
The number of holes is a topological invariant that is always an integer, but never anything in between.
n = 0
n = 1
n = 2

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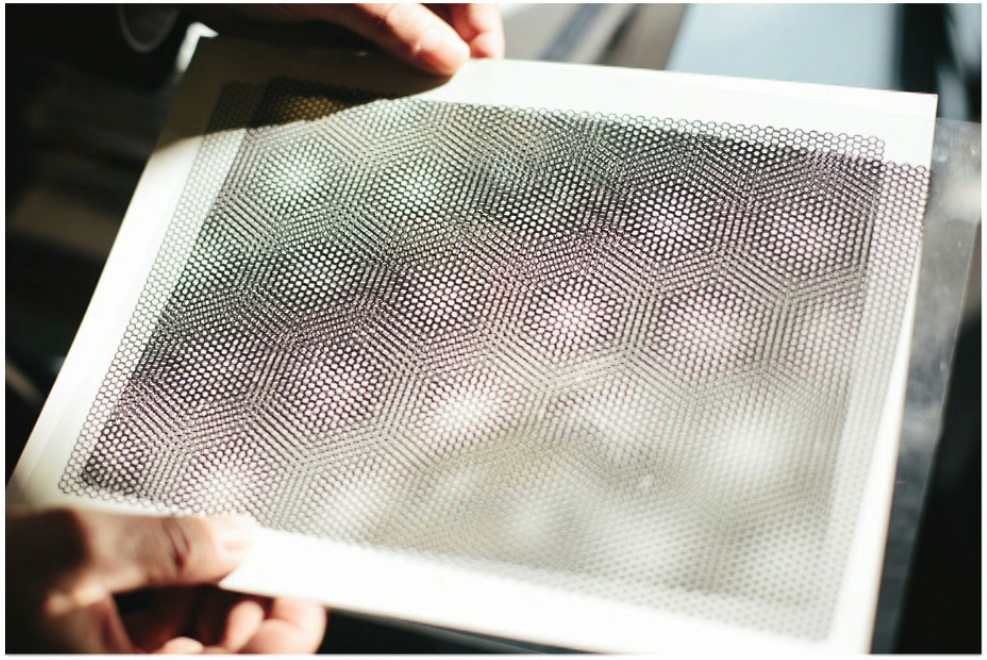
Strange topological materials are popping up everywhere physicists look

'Fragile topology' is the latest addition to a group of quantum phenomena that give materials exotic – and exciting – properties.

Davide Castelvecchi



Misaligned layers of graphene seem to exhibit a phenomenon known as fragile topology. Credit: Juliette Halsey for *Nature*

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Examples

- *Topological materials:* materials with robust band-gap phenomena manifesting in protected insulating or conducting states
- E.g. 2D Mercury telluride sandwiched in cadmium telluride
- E.g. 3D Bismuth antimony
- Hundreds of such materials are known and governed by an algebro-topological periodic table

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Symmetry				Dimension							
AZ	T	C	S	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

A. Kitaev, AIP Conference Proceedings 1134, 22 (2009)

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Periodic Media

- Waves propagating in a periodic medium with a symmetric potential are determined, up to phase, by their behaviour in the fundamental cell
- Felix Bloch (1928)
- Describes behaviour of *any* type of wave in *any* periodic medium, but application to electron motion in crystalline solids anticipates the band theory of electrical conduction and various quantum Hall-type phenomena

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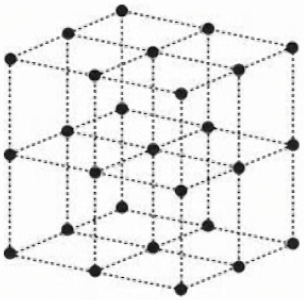
Bloch's Theorem



- Tile Euclidean space $\mathbb{R}^n$ with a regular lattice Λ
- Can obtain any cell by translating the *fundamental* one
- We wish to consider eigenvalue problems that respect the symmetry of the lattice
- If $V : \mathbb{R}^n \to \mathbb{R}$ is a Λ-invariant function, then we consider the operator $H = -\nabla^2 + V$ and the eigenvalue problem

$$H\varphi = E\varphi$$

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Bloch's Theorem

- Hilbert space $\mathbb{L} = L^2(\mathbb{R}^n)$ has an induced action of Λ via translation of coordinates, i.e. $T_\lambda \varphi(x) = \varphi(x+\lambda)$
- $\mathbb{L}$ splits into irreducible representations $\mathbb{L}_k$ of Λ, each of which is 1-dimensional and generated by a quasi-invariant function $\varphi_k(x)$ satisfying

$$\varphi_k(\lambda+x) = e^{ik(\lambda)}\,\varphi_k(x)$$

  for some $k : \Lambda \to U(1)$
- Translation by λ is an isometry and so T_λ is unitary and commutes with ∇^2, while T_λ and V commute by definition
- H diagonalizes in the basis of irreducible representations for $\mathbb{L}$ and so general solutions can be written in terms of quasi-invariant ones, known as **Bloch wave functions**

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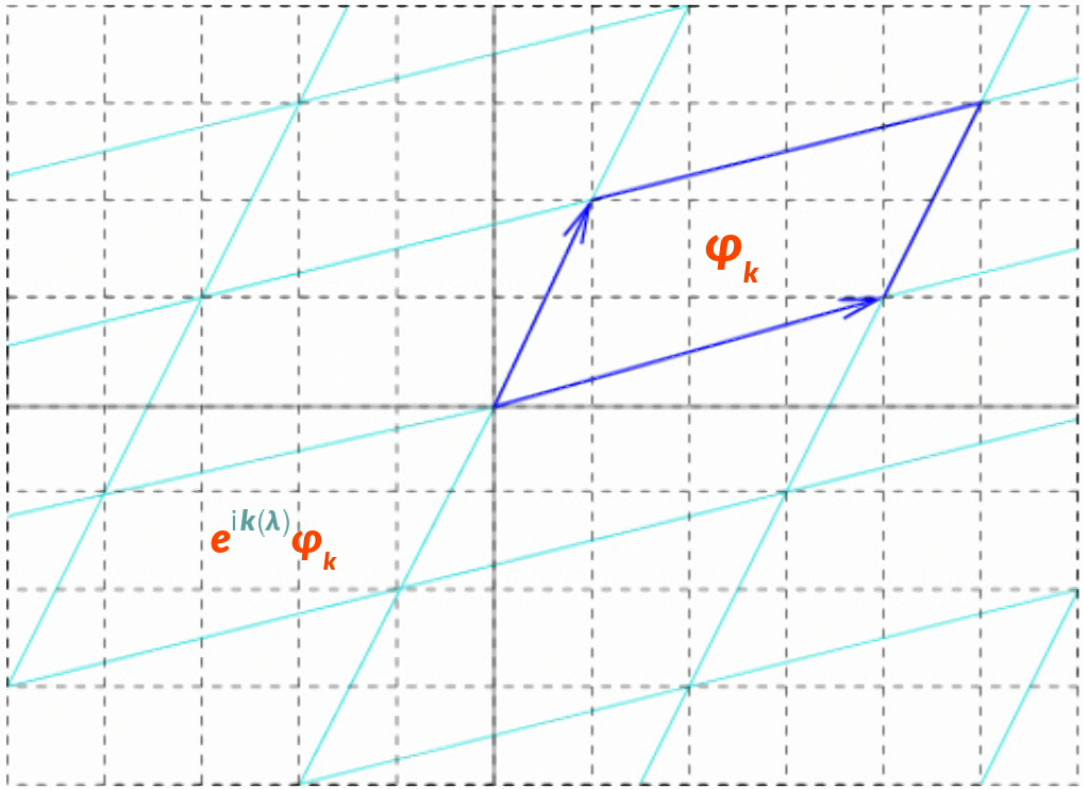

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Bloch's Theorem

- When the eigenvalue problem is regarded as a Schrödinger equation for electrons propogating in a periodic medium with a symmetric potential, Bloch's theorem gives us a description of their wavefunctions in terms of data that depends only on the lattice

- Equivalently, only on the topology of the quotient $\mathbb{R}^n/\Lambda$, which is the effective space where the electrons are propagating

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Bloch's Theorem in 2D

- Consider the Euclidean plane $\mathbb{R}^2 = \mathbb{C}$ with lattice Λ
- Equip with Λ-periodic potential $V : \mathbb{C} \to \mathbb{R}$
- The eigenvalue problem is solved in the basis of representations

$$k : \Lambda \to U(1)$$

  which are parametrized by the dual torus $(\mathbb{C}/\Lambda)^*$
- Can Fourier transform so that the problem is written in k coordinates on the dual torus, known as the (crystal) momentum space or the Brillouin zone, where the spectra or energy bands of H are studied

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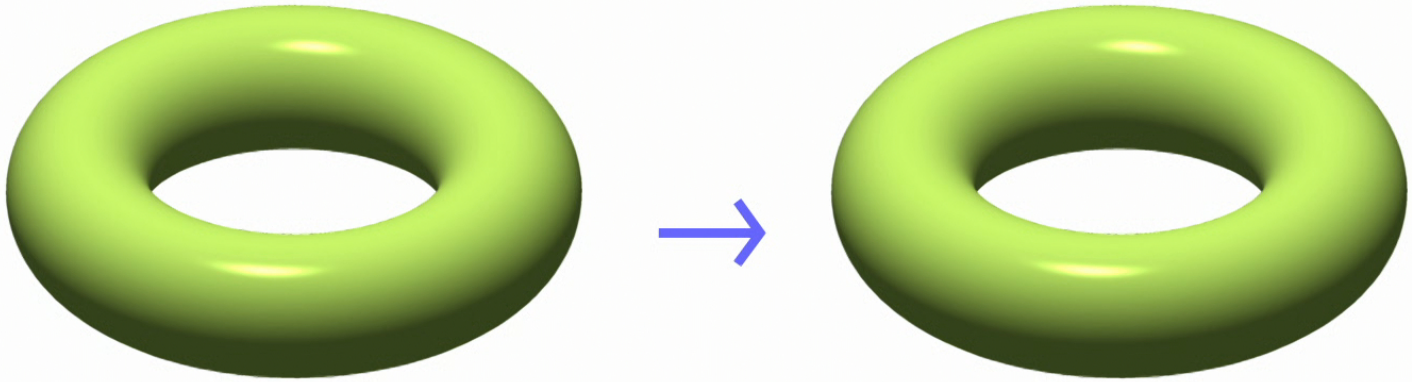

position (real space, *x*) **momentum (phase space, *k*)**

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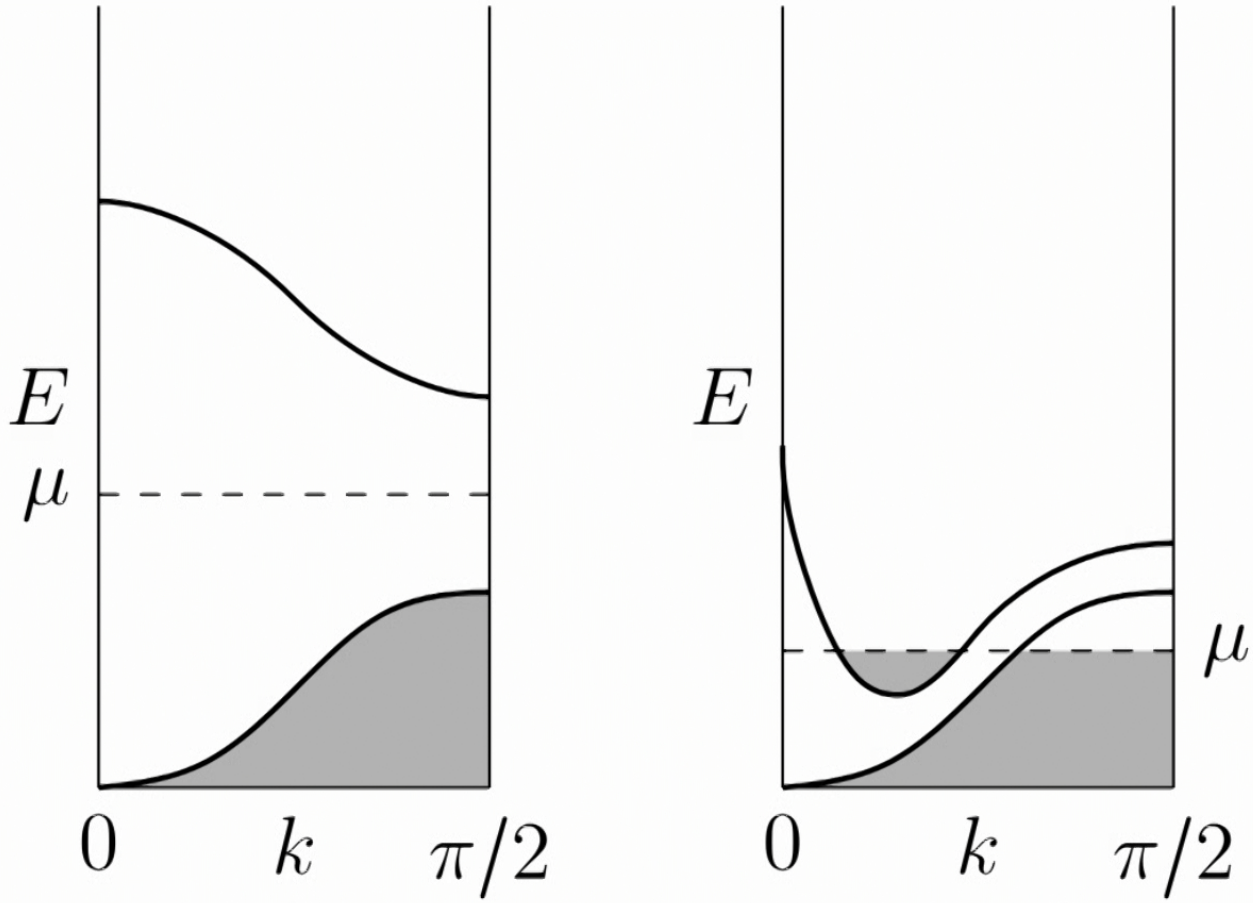

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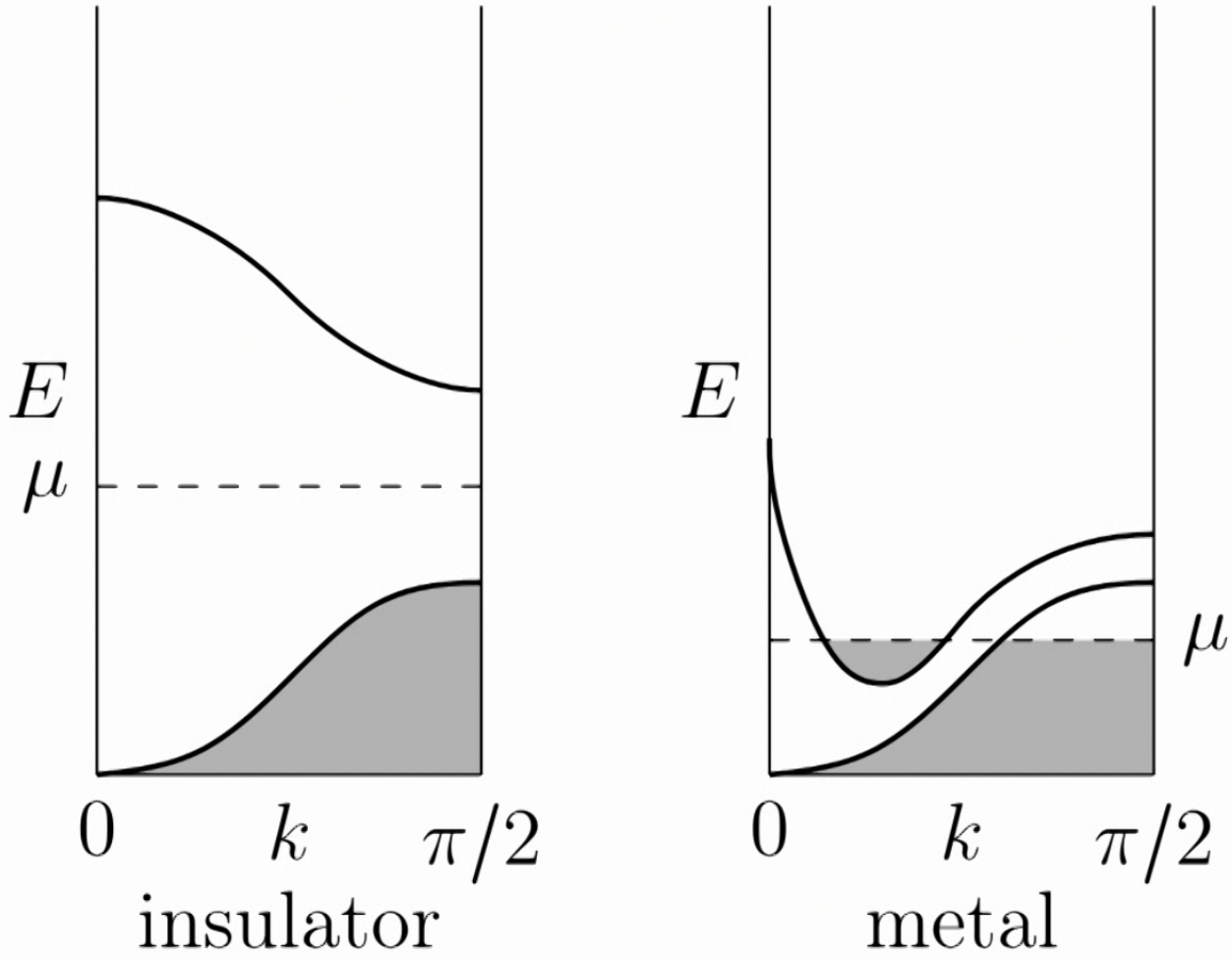

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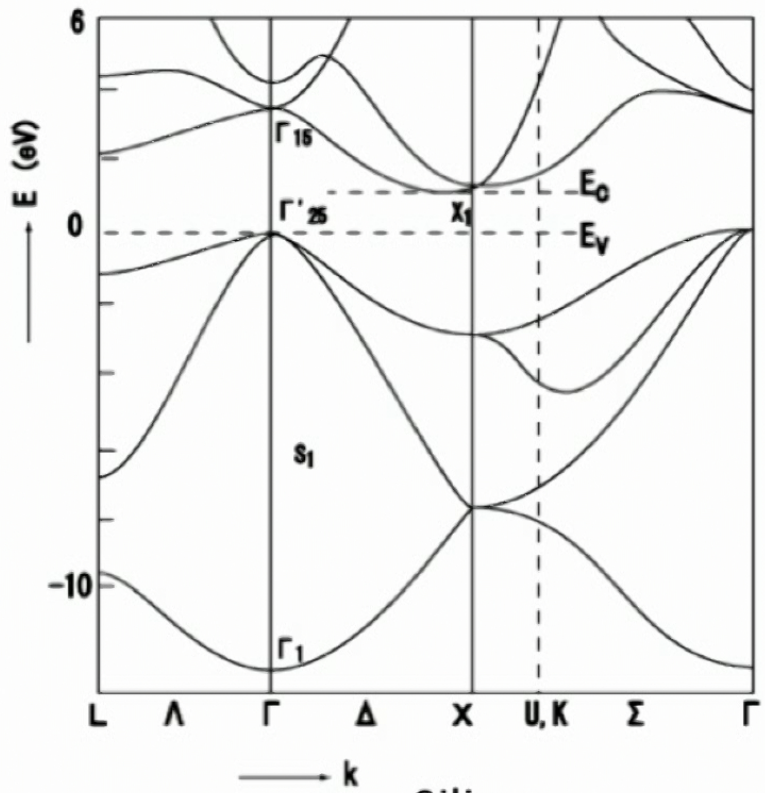

Silicon

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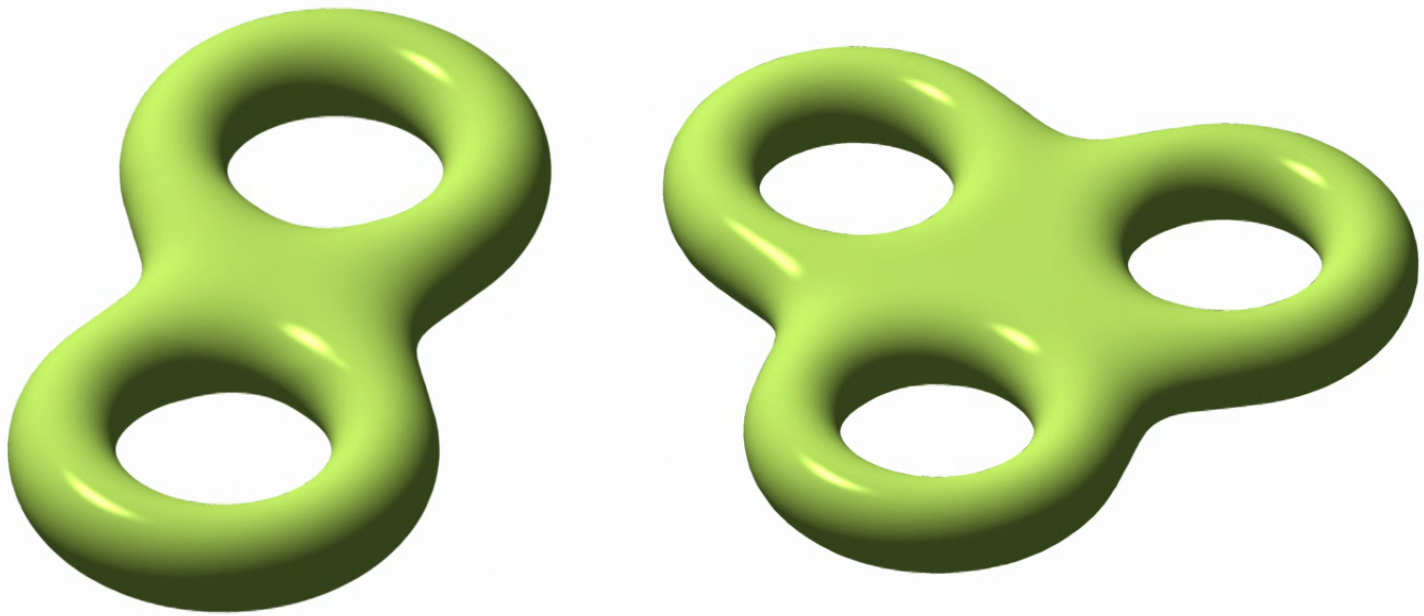

Yasuhiro Hatsugai. Edge states in the integer quantum Hall effect and the Riemann surface of the Bloch function. *Phys. Rev. B* **48**:11851–11862, 1993.

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Naïve Question

- Can the position space be a curve of genus $g > 1$?

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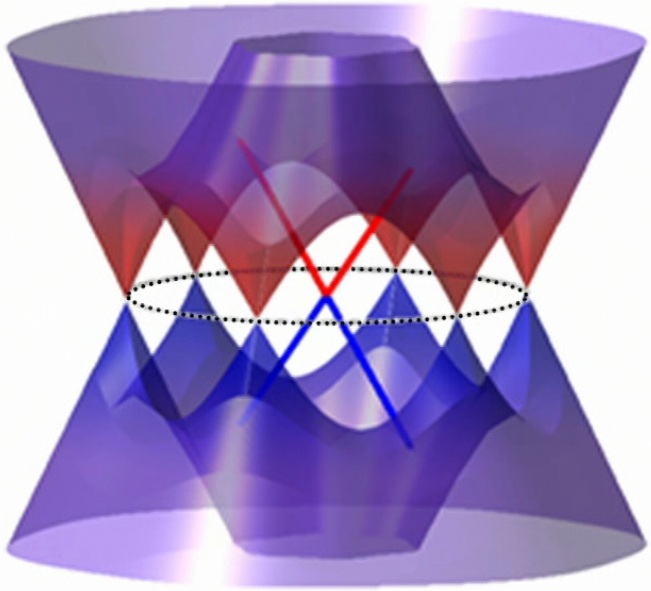

- Credit: © MPI CPfS

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Naïve Question

- Can the position space be a curve of genus $g > 1$?

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→ ?

position (real space) **momentum (phase space)**

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Higher-Genus Band Theory?

- Can we engineer a higher-genus position space?
- Is there a corresponding crystal momentum?
- Is there a band theory?

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Higher-Genus Band Theory?

- To produce a genus $g > 1$ position space, we need a tiling with a higher gonality

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Higher-Genus Band Theory

- To produce a genus g position space, we must tile 2D space by $4g$-gons
- E.g. genus g=2: tile space by regular octagons

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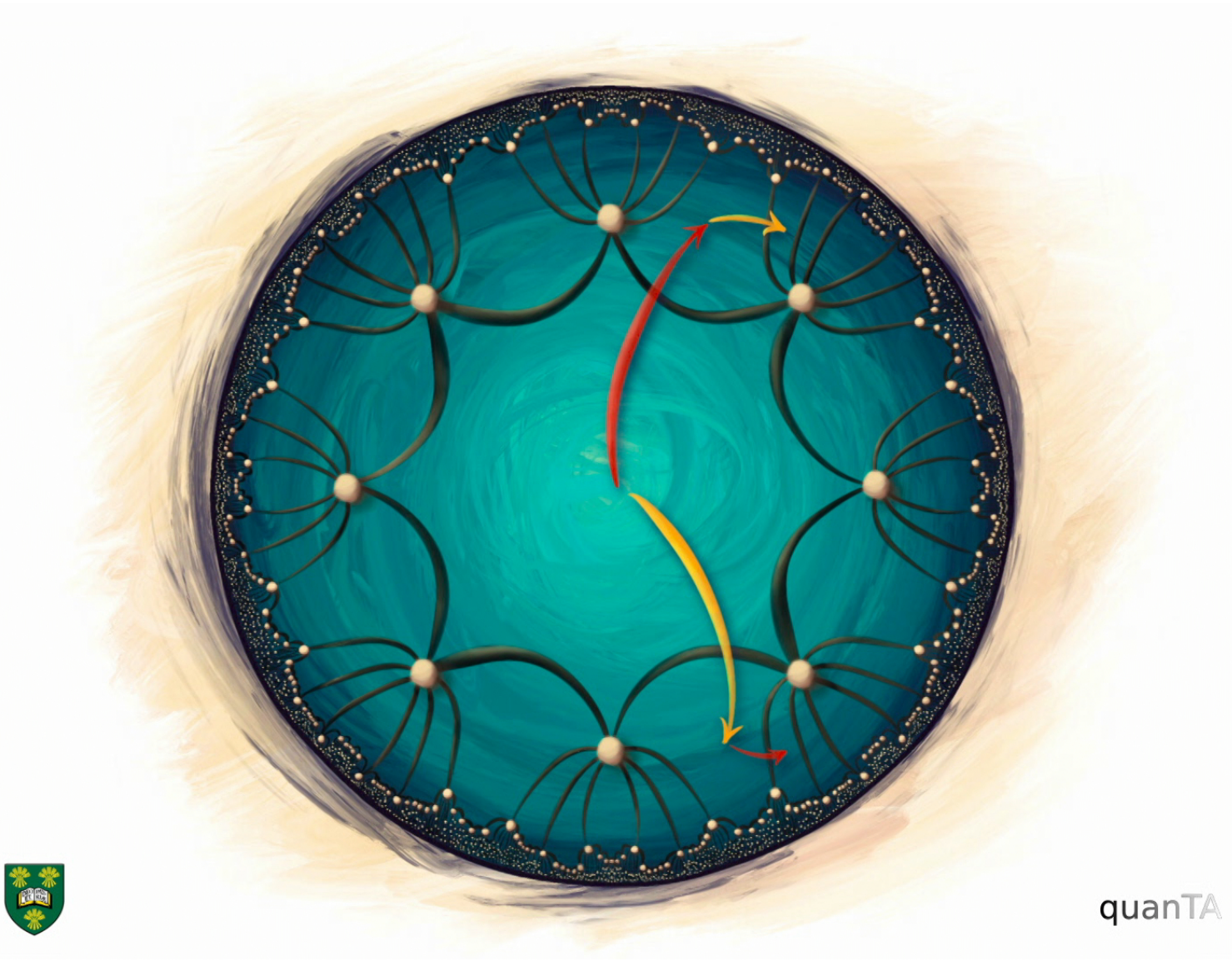
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Hyperbolic Matter

- Tile the hyperbolic plane $\mathbb{H}$ by $4g$-gons and choose a periodic potential V
- Translation group is now nonabelian (a Fuchsian group Γ)

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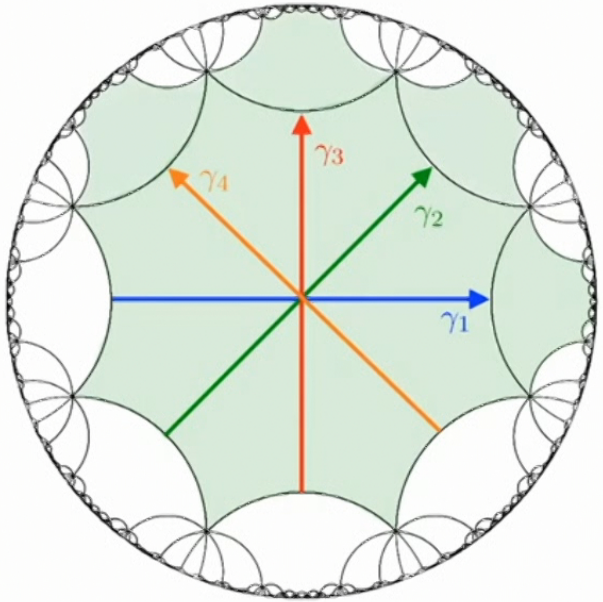

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Hyperbolic Matter

- Tile the hyperbolic plane $\mathbb{H}$ by $4g$-gons and choose a periodic potential V
- Translation group is now a nonabelian Fuchsian group Γ
- Consider $H_P = -\nabla_P^2 + V$ where

$$4\nabla_P = (1-|z|^2)^2\,\nabla$$

  is the Laplacian is adapted to the hyperbolic metric
- Do there exist Bloch waves?

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Hyperbolic Bloch Waves

- **Yes:** there exist states

$$\varphi(\gamma(x)) = \chi(\gamma)\,\varphi(x)$$

  where $\gamma \in \Gamma$ and χ is a $U(1)$-valued function
- These are simply Γ-automorphic forms of weight 0
- The reduced position space is the genus-g surface $\mathbb{H}/\Gamma$ and the (abelian) Brillouin zone is the space of representations χ
- $\mathbb{H}/\Gamma$ carries a complex structure, dependent on Γ
- $\chi : \Gamma \to U(1)$ generalizes the crystal momentum

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Crystal Momentum

- The space of maps $\chi : \Gamma \to U(1)$ is a $2g$-dimensional torus, namely the Jacobian of $\mathbb{H}/\Gamma$
- Jacobian has a distinguished complex structure inherited from $\mathbb{H}/\Gamma$ and is a moduli space of holomorphic line bundles on $\mathbb{H}/\Gamma$
- Momentum space of a 2D hyperbolic material is higher dimensional in general
- Maciejko-R: *Sci. Adv.* 7 (2021), no. 36

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Hyperbolic Band Structures

- Automorphic states give rise to energy bands that can be plotted over the Jacobian
- Studied numerically for a tight-binding model on the Bolza surface, with both the empty lattice (V=0) and with a generalized Eisenstein series for V (using `FreeFEM++`)
- Also in Maciejko-R: *Sci. Adv.* 7 (2021), no. 36

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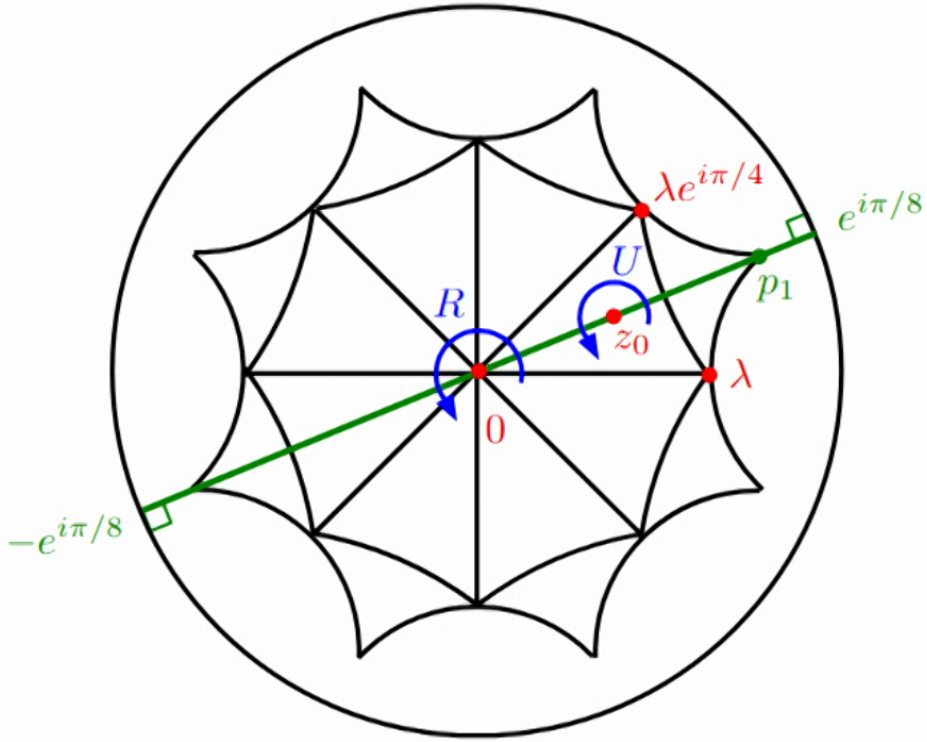

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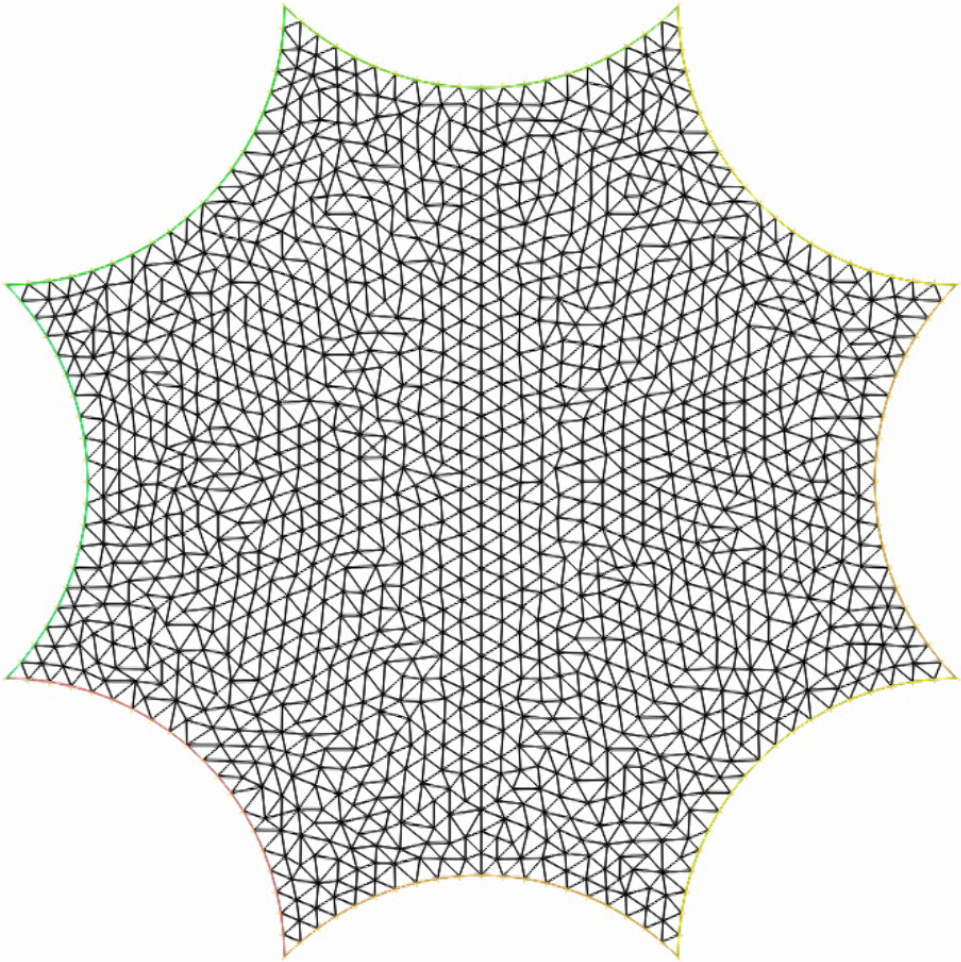

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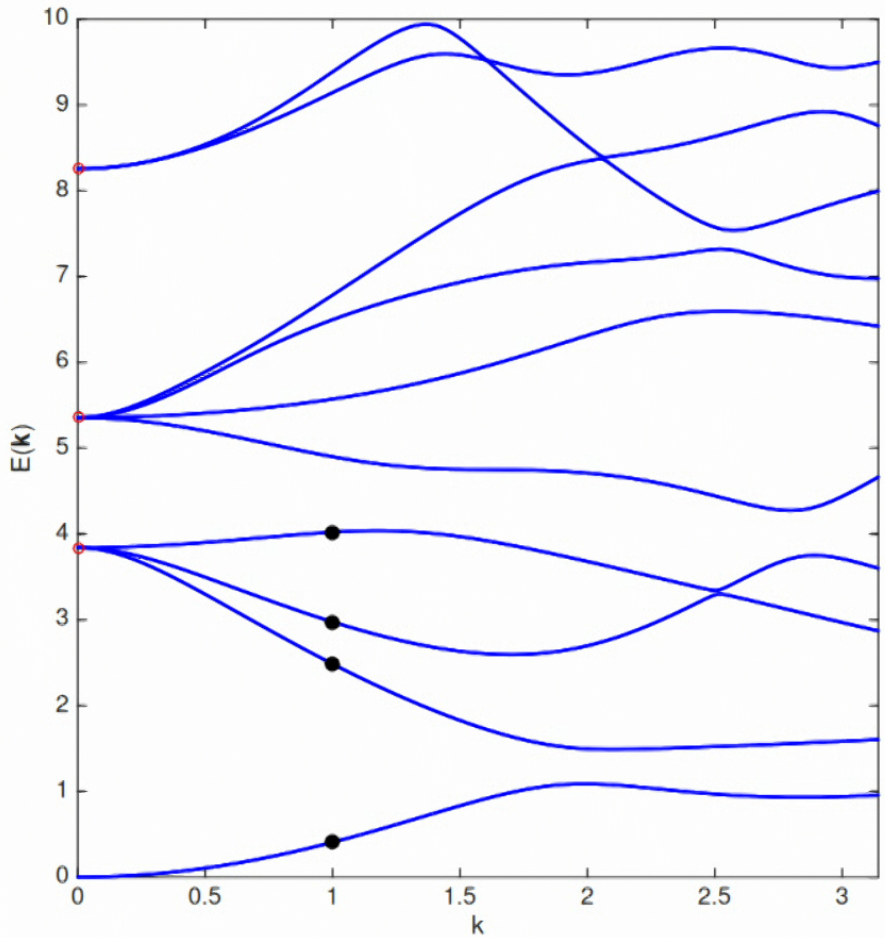

***V*=0**

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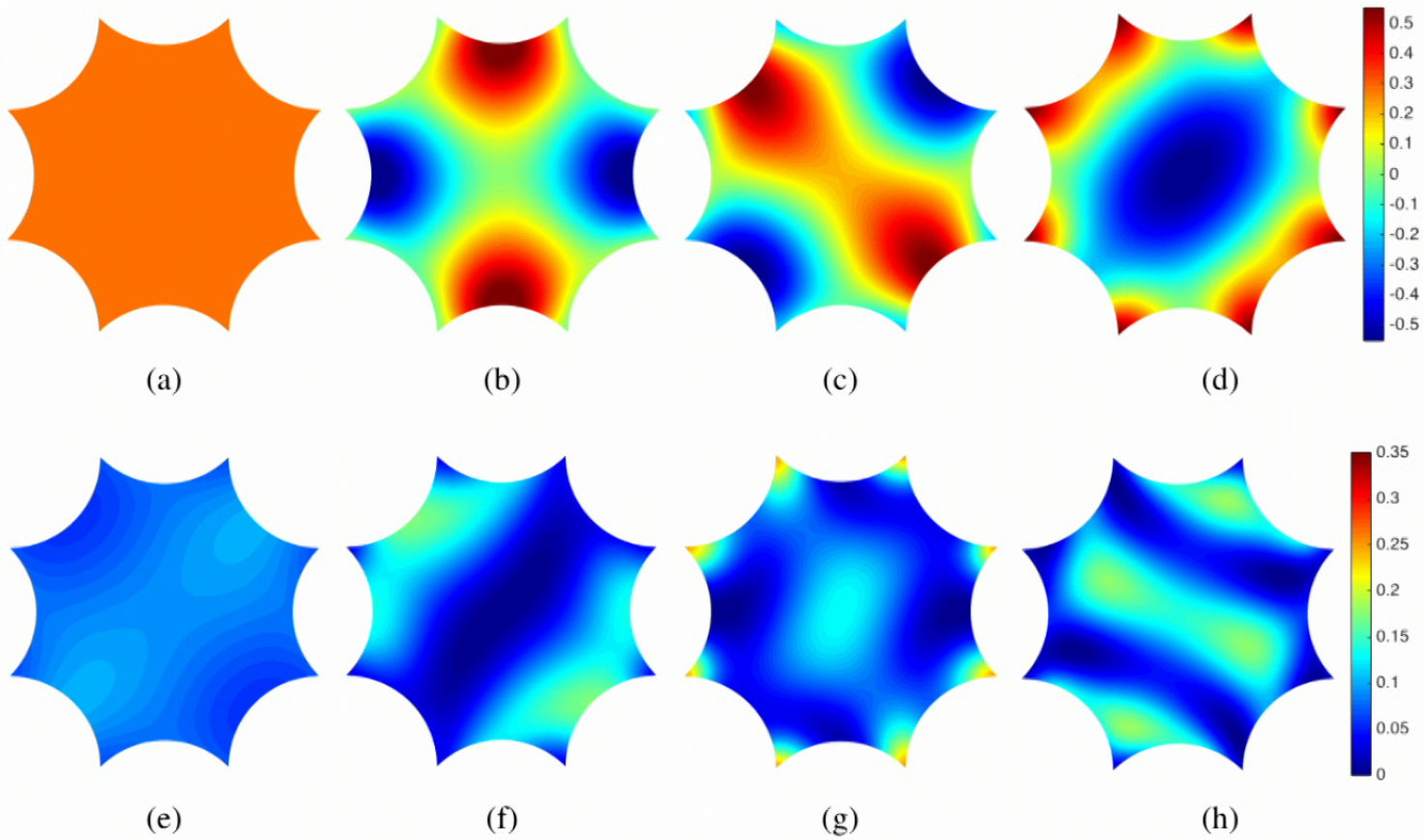

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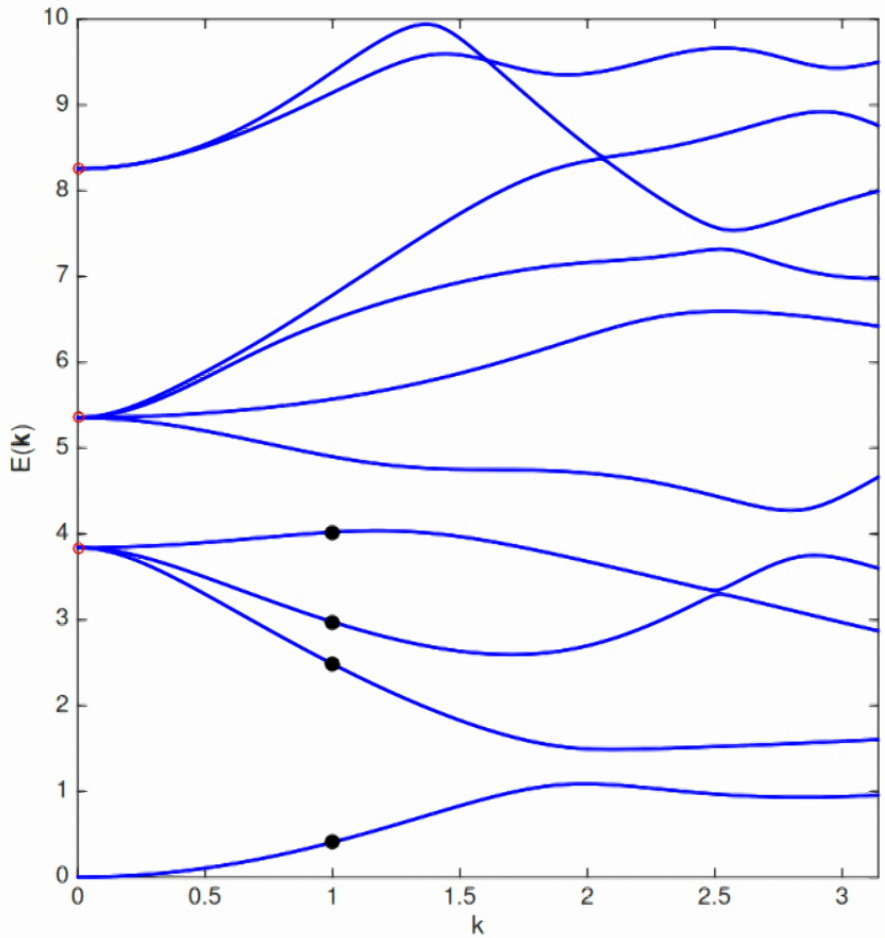

***V*=0**

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Hyperbolic Bloch's Theorem

- Can we decompose *any* wavefunction for the hyperbolic crystal into automorphic Bloch waves?
- In other words, can we diagonalize H_P using quasi-periodic functions?
- We characterize the spectrum for arbitrary symmetric V on *finite* lattices

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Hyperbolic Bloch's Theorem

- Consider a tight-binding model with finitely-many cells

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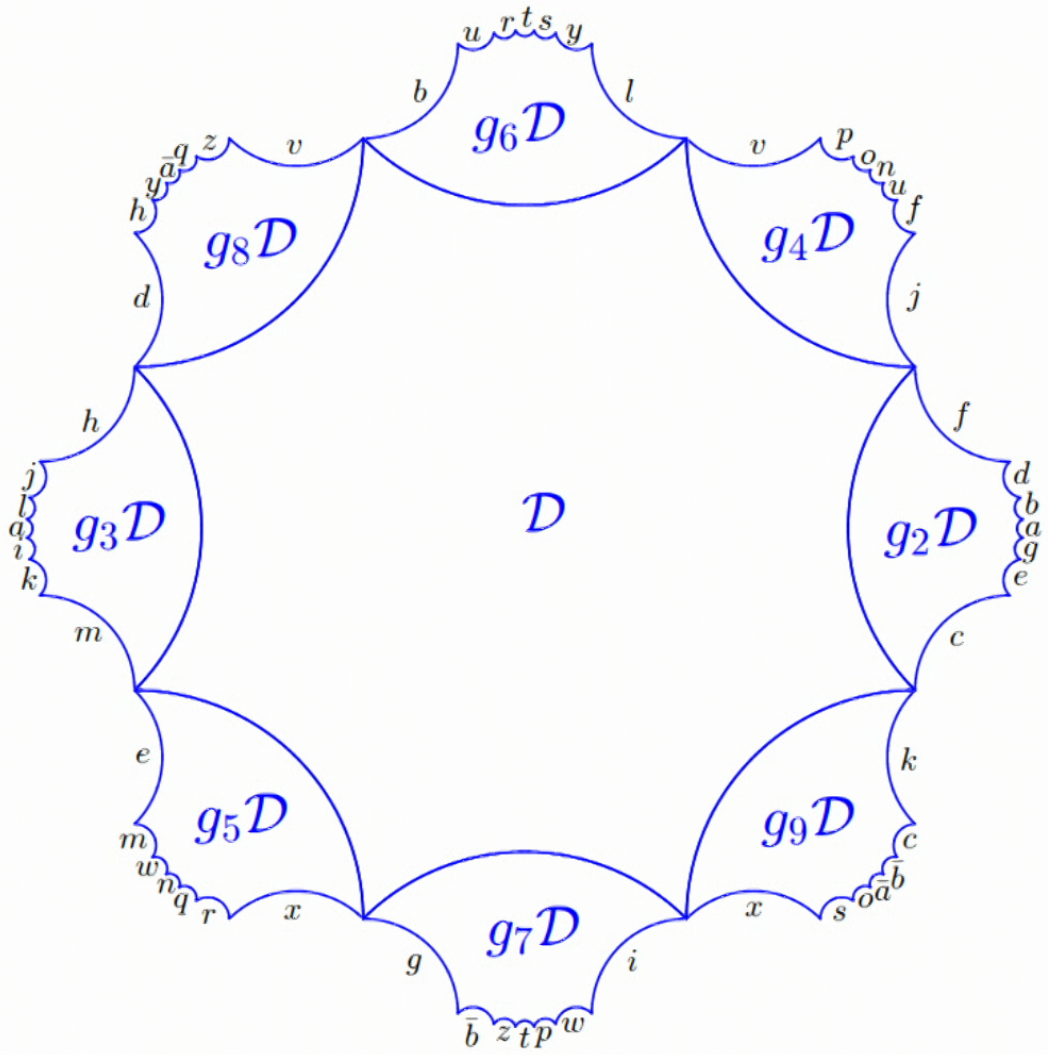
$g_6\mathcal{D}$
$g_8\mathcal{D}$
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$g_3\mathcal{D}$
$\mathcal{D}$
$g_2\mathcal{D}$
$g_5\mathcal{D}$
$g_9\mathcal{D}$
$g_7\mathcal{D}$

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Hyperbolic Bloch's Theorem

- Consider a tight-binding model with finitely-many cells
- We construct by hand a complete basis of quasi-periodic wavefunctions in this case, but see matrix-valued phase factors
- Roughly speaking, these arise from normal subgroups of Γ of different orders
- In general, the phase factor $\chi(\gamma)$ is a $U(r)$-valued function

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Hyperbolic Bloch's Theorem

- All possible ranks *r* of phase factors are possible as number of cells grows
- Infinitely-many Brillouin zones for a single material: one for each *r*
- For $\Sigma = \mathbb{H}/\Gamma$, these spaces of Γ-representations inherit complex structures that recast them as moduli spaces of stable holomorphic bundles
- Maciejko-R: *PNAS* 119 (2022), no. 9 (physics approach, *finite*)
- Nagy-R: arXiv: 2208.02749 (math approach, *infinite*)

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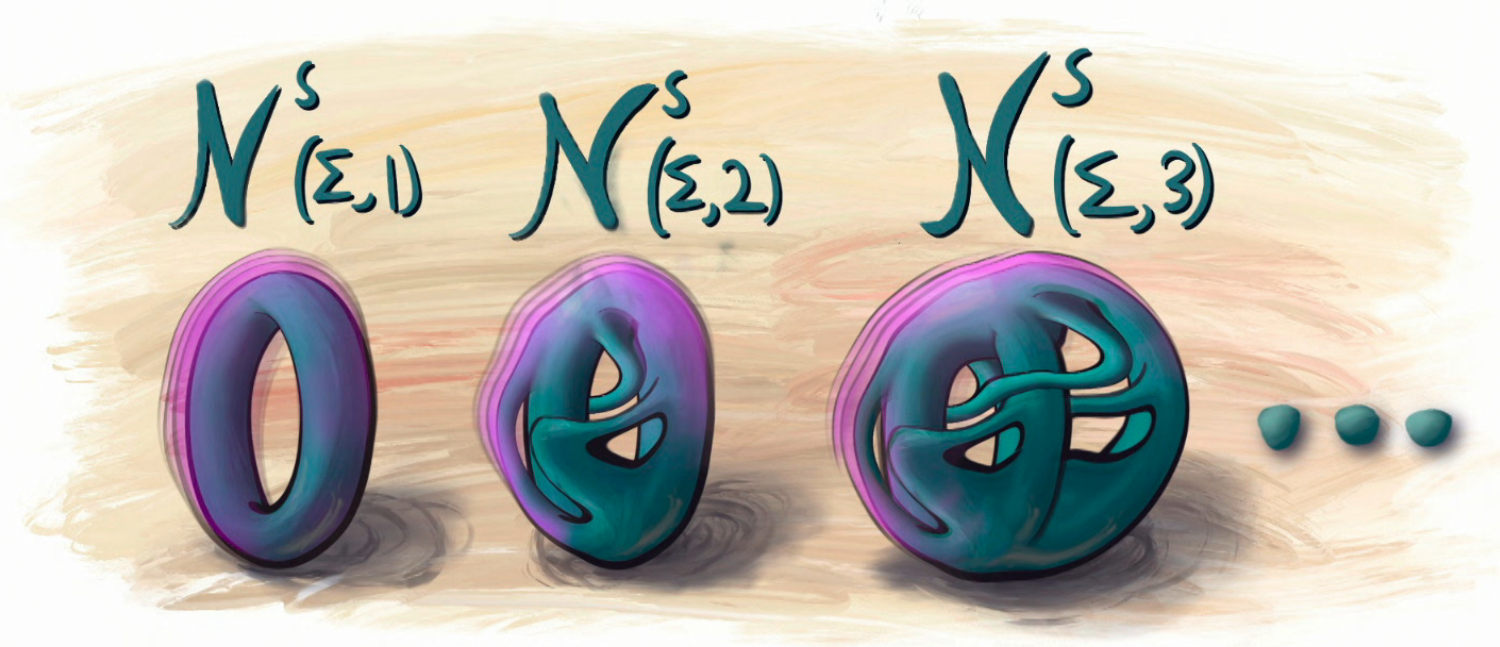

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Nonabelian Brillouin Zones

- E.g. For Σ of genus 2, the rank-2 Brillouin zone is homeomorphic to CP^3 (Narasimhan-Seshadri 1969)
- For general rank, these are compact but typically singular

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Nonabelian Brillouin Zones

- E.g. For Σ of genus 2, the rank-2 Brillouin zone is homeomorphic to CP^3 (Narasimhan-Seshadri 1969)
- For general rank, these are compact but typically singular
- Higher-rank bundles can be treated as infinitesimal corrections to the Bloch data

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Implications

- Hyperbolic materials inhabit a rich $(3g\text{-}3)$-dimensional complex moduli space of algebraic curves
- Coordinates on the moduli space provide new, non-topological invariants
- Compactifications of the moduli space may correspond to materials with enhanced symmetries / degeneracies
- Fourier-Abel-Jacobi map from $\mathbb{H}/\Gamma$ to $\mathrm{Jac}(\mathbb{H}/\Gamma)$ is never an isomorphism
- However, $\mathrm{Sym}^g(\mathbb{H}/\Gamma)$ and $\mathrm{Jac}(\mathbb{H}/\Gamma)$ are birational
- Blown-down loci anticipate high-symmetry regions in the hyperbolic Brillouin zone

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Integrable Systems

- Restricting to the abelian Brillouin zone, we note that first-order deformations of the complex structure of $\mathbb{H}/\Gamma$ correspond with those of $\mathrm{Jac}(\mathbb{H}/\Gamma)$
- Hyperelliptic $\mathbb{H}/\Gamma$ arise as double covers of $\mathbf{P}^1$, given by degree $2g+2$ polynomials in a coordinate on $\mathbb{C} \subset \mathbf{P}^1$
- Polynomials comprise an affine space that supports a nontrivial fibration by tori corresponding to $\mathrm{Jac}(\mathbb{H}/\Gamma)$
- Pushforward to $\mathbf{P}^1$ of line bundles L in $\mathrm{Jac}(\mathbb{H}/\Gamma)$ produces not only a rank-2 vector bundle E but also a map $E \to E \otimes \mathcal{O}(g+1)$ whose spectrum is precisely $\mathbb{H}/\Gamma$

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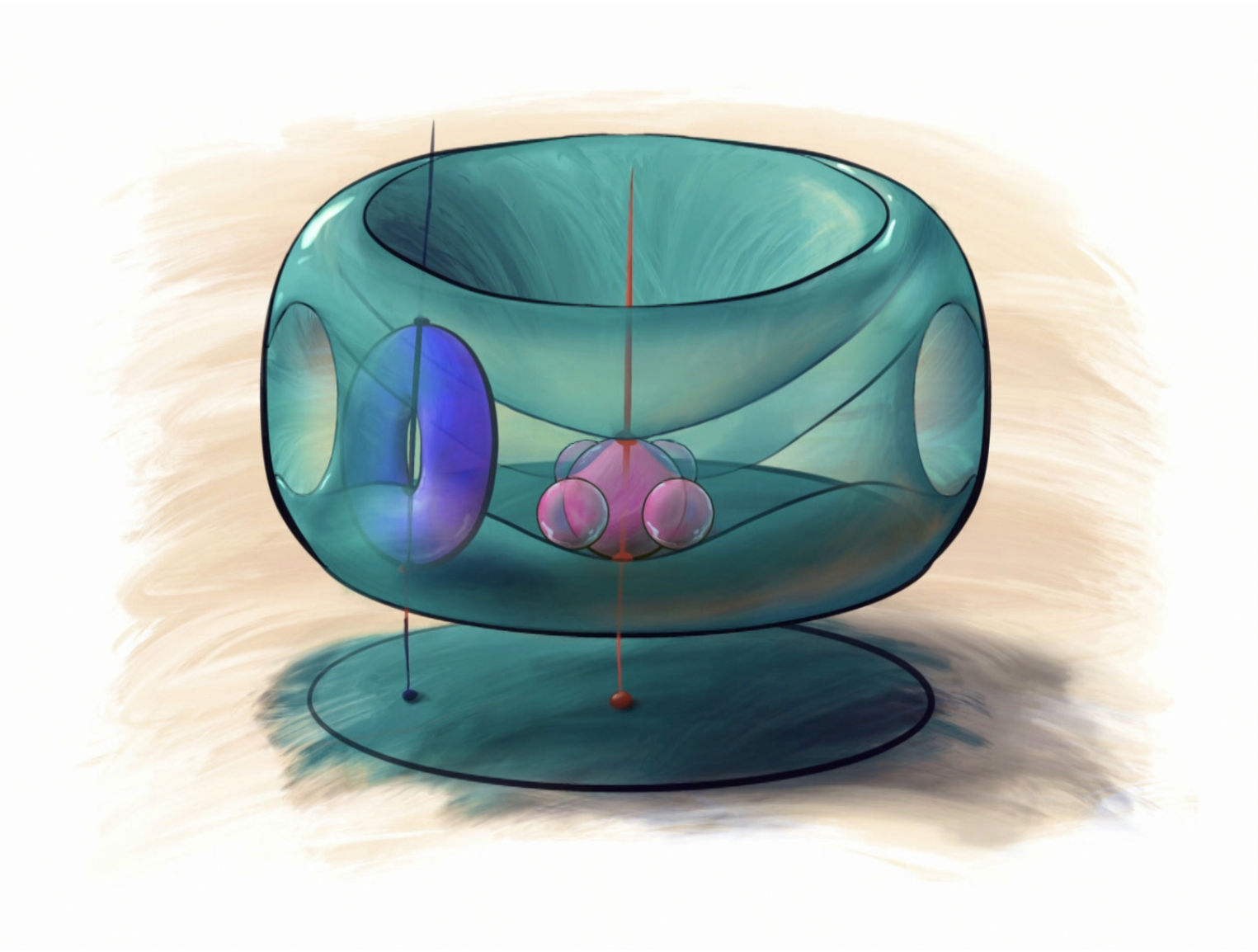

Making Hyperbolic Matter

- Is it possible to artificially engineer hyperbolic states in circuits or devices?

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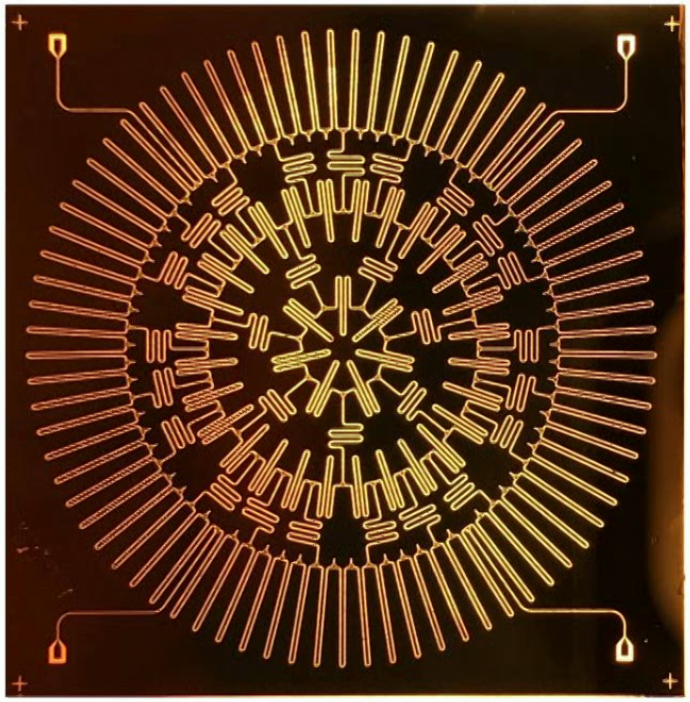
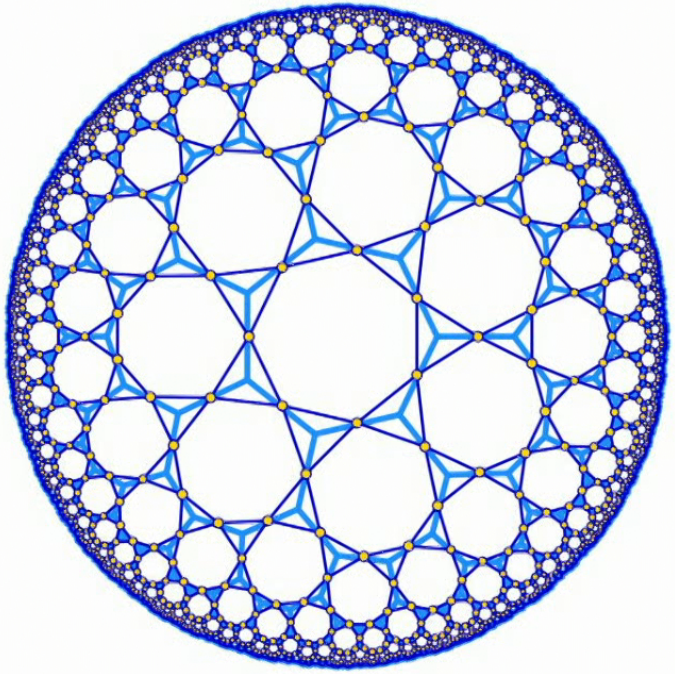
Yes!





- Kollár, Fitzpatrick, Houck, Nature 571 (2019): 45–50 (Copyright Springer-Nature 2019)

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Problems in Hyperbolic Matter

- Is there a full Bloch theorem for the hyperbolic crystal spectrum?
- What is the meaning of the various symmetries induced by moduli spaces?
- Is there a hyperbolic crystallography? (Boettcher-Gorshkov-Kollár-Maciejko-R-Thomale: *Phys. Rev.* B 105 (2022), no. 12)
- Can electronic hyperbolic circuits can be engineered?
- Can they be grown or printed as films (2D) or other structures (3D)?

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Thank You

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