Title: Grad Student Seminar with Hank Chen

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Abstract: Hank Chen, Perimeter Institute and University of Waterloo

Drinfel'd double symmetry of the 4d Kitaev model

We construct the 2-Drinfel'd double associated to a finite 2-group G, and compute its braided 2-category of 2-representations, in order to characterize the topological excitations in the 4d toric code and its spin-Z_2 variant. This work relates the categorical and field theoretical approaches toward characterizing higher-dimensional topological phases in existing literature. In particular, we show that particular twists of the underlying 2-Drinfel'd double is responsible for much of the higher-structural properties that arise in gapped topological phases in 4d. We emphasize that this work also displays the first ever instance of higher Tannakian duality for 2-categories.

2-Drinfel'd double symmetry of the 4d Kitaev model

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Overview

1. Motivation: the story in 3d.
2. Review: higher Drinfel'd doubles $D(BG)$, Postnikov class τ.
3. Specializing to $G = \mathbb{Z}_2$; twists ω_b, ω_f.

Theorem. (H.C.)

There are braided equivalences

$$2\mathrm{Rep}^\tau(D(B\mathbb{Z}_2)^{\omega_b}) \simeq Z_1(\Sigma\,\mathrm{Vect}[\mathbb{Z}_2]) \simeq \text{ 4d toric code,}$$
$$2\mathrm{Rep}^\tau(D(B\mathbb{Z}_2)^{\omega_f}) \simeq Z_1(\Sigma\,\mathrm{sVect}) \simeq \text{ spin-}\mathbb{Z}_2\text{ gauge thy.,}$$

where Z_1 is the Drinfel'd centre functor.

- Example of categorified Tannakian reconstruction.

The story in 3d — braided 1-categories

- 2d toric code: Drinfel'd centre $Z_1(\mathrm{Rep}(\mathbb{Z}_2))$ [KK12] .
- $\mathbb{Z}_2$-BF theory in 3d (3d Kitaev model)

$$S[A, B] = \int BF, \qquad F = dA,$$

excitations form Drinfel'd double $D(\mathbb{Z}_2)$.

- **2d toric code $\sim$ 3d Kitaev model**.
  1. Continuum limit: *Tannaka-Krein* duality [Del18]

$$Z_1(\mathrm{Rep}(\mathbb{Z}_2)) \simeq \mathrm{Rep}(D(\mathbb{Z}_2)),$$

  2. Lattice reconstruction: correspondence with groupoid algebra $\mathbb{C}[\Lambda G] \sim D(G)$ [DDR17] .

What is the story in 4d?

What we know in 4d, and what is missing

- Two approaches to gapped G-symm. 4d topological phases:
  1. 2-categories [KTZ20; Joh20] ,
  2. 2-group field theory [ZLW19; WWH15].
- How are they related? Need *higher* Drinfel'd doubles $\mathcal{D}(G)$.
  1. Continuum limit: Tannaka-Krein for braided 2-categories

$$Z_1(\mathcal{C}_G) \simeq 2\mathrm{Rep}(\mathcal{D}(G)),$$

  2. Lattice reconstruction: correspondence with 2-groupoid algebra $\mathcal{D}(G) \overset{?}{\sim} \mathbb{C}[\Lambda G]$ [BD21] .

This work: solving the first point for $G = \mathbb{Z}_2$.

The 2-Drinfel'd double $\mathcal{D}(G) = D(BG)$

- Pontrjagyn dual pair $\widehat{G} \cong G$.
- 2-Drinfel'd double $D(BG)$:

$$\begin{array}{lccc} \text{deg-1:} & k\widehat{G} & & * \\ & \downarrow 0 & \bar{\bowtie} & \downarrow 0 \\ \text{deg-0:} & * & & kG \end{array}$$

1. 2-bialgebra: coproduct Δ is *grouplike* (for fusion).
2. Classification via **Postnikov class** [KT13]

$$\tau \in H^3(G, \widehat{G}) \cong HH^3(kG, k\widehat{G}).$$

3. Canonical 2-R-matrix $(\mathcal{R}, R)$ (for braiding)

$$\mathcal{R} \in \widehat{G} \otimes G, \qquad R \in G \otimes G.$$

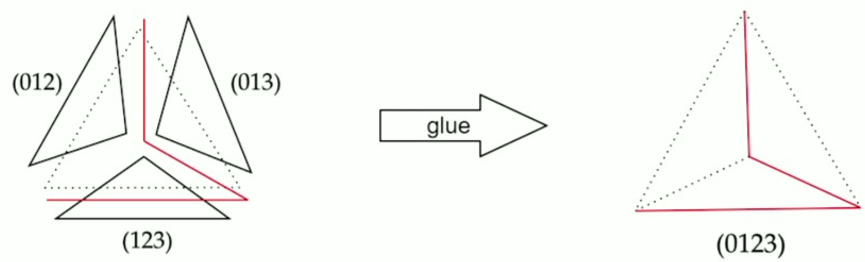
2-BF theory from $D(BG)$

- G-cocycles on 4-manifold X:

$$A \in C^1(X, G), \qquad B \in C^2(X, \widehat{G}).$$

- Kinematics; equations of motion (EoM)

$$F = dA = 0, \qquad dB = \tau(A)$$



- 4d topological 2-BF theory [CG22]:

$$S^0[A, B] = \int_X B \cup F - \tau \cup A.$$

Excitations of Z^0_{Kit}

- Take $G = \mathbb{Z}_2$, "invisible toric code"

$$Z^0_{\text{Kit}}(X) = \sum_{A,B} e^{2\pi i S^0[A,B]}.$$

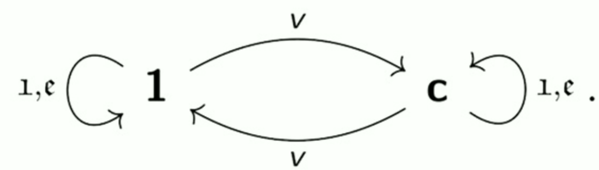
- Excitations = *2-representations* of $D(B\mathbb{Z}_2)$ [CG23]:
  1. *Four* distinct 2-irreps.

	Vacuua	Cheshire
Electric	$\mathbf{1} = (1 \overset{d=1}{\oplus} 1, \rho_1 = 0)$	$\mathbf{c} = (1 \overset{d=0}{\oplus} \text{sgn}, \rho_1 = \hat{1})$
Magnetic	$\mathbf{1}^* = (\text{sgn} \overset{d=1}{\oplus} \text{sgn}, \rho_1 = 0)$	$\mathbf{c}^* = (\text{sgn} \overset{d=0}{\oplus} 1, \rho_1 = \hat{1})$

  2. Postnikov $\tau \rightsquigarrow$ coassoc. $\delta \rightsquigarrow$ fusion associator a.

- **Fact:** no 1-morphisms between elec./mag. sectors,

Identity component Γ (ie. elec. sector) =



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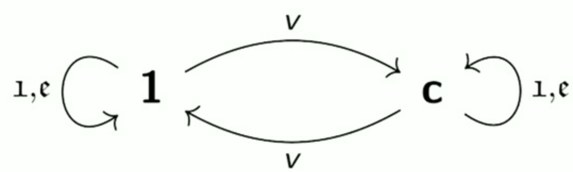

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So. . . why "invisible" toric code?

Proposition. (H.C.)

Functor $2\mathrm{Rep}(D(B\mathbb{Z}_2)) \to Z_1(\Sigma\,\mathrm{Vect}[\mathbb{Z}_2])$ is non-monoidal.

1. Fusions are correct on objects (have Cheshire charge [EN17])...

$$\mathbf{1}^2 \simeq (\mathbf{1}^*)^2 \simeq \mathbf{1}, \qquad \mathbf{c}^2 \simeq \mathbf{c} \oplus \mathbf{c}, \qquad \mathbf{1}^* \otimes \mathbf{c} \simeq \mathbf{c}^*,$$

but not morphisms

$$\mathfrak{1}^2 \simeq \mathfrak{e}^2 \simeq \mathfrak{1}, \qquad \mathfrak{e} \otimes \mathfrak{1} \simeq \mathfrak{1} \otimes \mathfrak{e} \simeq \mathfrak{e}, \qquad v \otimes v \simeq \mathfrak{1}.$$

2. Moreover: violates **remote detectability** [KW14];

$$\mathcal{R}, R = \mathrm{id} \implies \text{all braidings } b \text{ are trivial!}$$

Fix: twist $D(B\mathbb{Z}_2)$.

Cycle twists on $D(B\mathbb{Z}_2)$

1. Extension at deg-0: $\bar{e} \in H^2(\mathbb{Z}_2.\mathbb{Z}_2) \cong H^2(\mathbb{Z}_2, \widehat{\mathbb{Z}_2})$:

$$\rho_0(x)\rho_0(x') = \rho_1(\bar{e}(x,x')) \cdot \rho_0(xx'), \qquad x, x' \in G.$$

   - Carried by $\mathbf{1}^*, \mathbf{c}$.

2. Extension at deg-1: $\bar{c} \in H^2(\widehat{\mathbb{Z}_2}, \mathbb{C}^\times)$:

$$\rho_1(y)\rho_1(y') = \bar{c}(y,y)\rho_1(yy'), \qquad y, y' \in \widehat{G}.$$

   - Carried by $\mathbf{c}, \mathbf{c}^*$.

2-reps.	Twists	Name
$2\mathrm{Rep}^\tau(D(B\mathbb{Z}_2)^{\omega_b})$	$\omega_b = \bar{e}$	"Bosonic"
$2\mathrm{Rep}^\tau(D(B\mathbb{Z}_2)^{\omega_f})$	$\omega_f = \bar{e} + \bar{c}[1]$	"Fermionic"

Excitations in 4d spin-Kitaev models

- $\bar{e}$ occurs when:
  1. using naturality of fusion

$$v^2 = \mathrm{id}_{\mathbf{1} \text{ or } \mathbf{c}} \otimes v \circ v \otimes \mathrm{id}_{\mathbf{c} \text{ or } \mathbf{1}} \simeq \mathbb{1} \oplus \mathfrak{e},$$

  2. computing *full*-braids $B_{iW} = b_{iW} \circ b_{Wi} \propto \mathcal{R}^2$:

$$B_{\mathfrak{e}\mathbf{1}^*} \simeq B_{\mathfrak{e}\mathbf{c}^*} \simeq -1 \cdot \mathrm{id},$$

  implements remote detectability.
- $\bar{c}$ occurs when:
  1. computing fusion; lifting Chershires to sign reps. of $\mathbb{Z}_2 \subset \mathbb{Z}_4$:

$$\mathbf{c}^2 \simeq \mathbf{1} \simeq (\mathbf{c}^*)^2.$$

  Cheshires $\mathbf{c}, \mathbf{c}^*$ now invertible and *self-braidable*,
  2. computing self-braids $b_{\mathbf{c}} \propto R^2, b_{\mathfrak{e}} \propto \mathcal{R}^2$:

$$b_{\mathbf{c}} \simeq \mathfrak{e}, \qquad b_{\mathfrak{e}} \simeq -1 \cdot \mathrm{id}$$

Proof of the main theorem

- What of self-braiding $b_{\mathbf{1}^*}$? Use fusion rule + ribbon formula

$$\begin{aligned} b_{\mathbf{1}^*} &= b_{\mathbf{c}^* \otimes \mathbf{c}} \\ &\cong b_{\mathbf{c}^*} \otimes b_{\mathbf{c}} \otimes (b_{\mathbf{c}^*\mathbf{c}} \circ b_{\mathbf{c}\mathbf{c}^*}) \\ &\simeq \mathbb{1} \otimes \mathfrak{e} \otimes \mathfrak{e} \simeq \mathbb{1}. \end{aligned}$$

- *All* braiding/fusion properties now match:

Theorem. (H.C.)

There are braided equivalences

$$2\mathrm{Rep}^{\tau}(D(B\mathbb{Z}_2)^{\omega_b}) \simeq \text{4d toric code} \simeq Z_1(\Sigma\,\mathrm{Vect}[\mathbb{Z}_2]),$$

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where second $\simeq$'s are from [Joh20] .

Tannakian duality and the continuum limit

- Twists introduce cohomological (non-dynamical) terms:

$$Z^{s}_{\mathrm{Kit}}(X) \cong \sum_{\substack{dA=0 \\ dB=\tau(A)}} \exp\left[i2\pi \int_X \bar{c}(B,B) + B \cup \bar{e}(A,A) \right].$$

1. Appeared as topological non-linear σ-models [ZLW19] .
2. Reproduce the right deg-5 anomalies [Joh20] .

- Success in establishing Tannaka-Krein:

$$\text{(2-)category} \overset{\text{2-Drinfel'd double}}{\Longleftrightarrow} \text{(2-group) field theory}$$

Outlook

1. 4d boundary of the $w_2 w_3$ grav. anomaly [Joh20].
   - Requires *adjunctions* $\implies$ *(weak) ribbon 2-Hopf* structure.
2. Morita duality $D(BG)^\omega \sim \mathbb{C}^\omega[\Lambda BG]$ [BD21] .
   - Lattice reconstruction from field theory.
3. Applications to *compact* groups, eg. $G = SU(2)$.
   - Study Baez's conjecture: 4d BF-BB theory $\overset{?}{=}$ Crane-Yetter.