

Title: Grad Student Seminar with Hank Chen

Speakers: Hank Chen

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Abstract: Hank Chen, Perimeter Institute and University of Waterloo

Drinfel'd double symmetry of the 4d Kitaev model

We construct the 2-Drinfel'd double associated to a finite 2-group G , and compute its braided 2-category of 2-representations, in order to characterize the topological excitations in the 4d toric code and its spin-Z_2 variant. This work relates the categorical and field theoretical approaches toward characterizing higher-dimensional topological phases in existing literature. In particular, we show that particular twists of the underlying 2-Drinfel'd double is responsible for much of the higher-structural properties that arise in gapped topological phases in 4d. We emphasize that this work also displays the first ever instance of higher Tannakian duality for 2-categories.



2-Drinfel'd double symmetry of the 4d Kitaev model

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Overview

- 1 Motivation: the story in 3d.
- 2 Review: higher Drinfel'd doubles $D(BG)$, Postnikov class τ .
- 3 Specializing to $G = \mathbb{Z}_2$; twists ω_b, ω_f .

Theorem. (H.C.)

There are braided equivalences

$$\begin{aligned} 2\text{Rep}^\tau(D(B\mathbb{Z}_2)^{\omega_b}) &\simeq Z_1(\Sigma \text{Vect}[\mathbb{Z}_2]) \simeq \text{4d toric code,} \\ 2\text{Rep}^\tau(D(B\mathbb{Z}_2)^{\omega_f}) &\simeq Z_1(\Sigma \text{sVect}) \simeq \text{spin-}\mathbb{Z}_2 \text{ gauge thy.,} \end{aligned}$$

where Z_1 is the Drinfel'd centre functor.

- Example of categorified Tannakian reconstruction.

The story in 3d — braided 1-categories

- 2d toric code: Drinfel'd centre $Z_1(\text{Rep}(\mathbb{Z}_2))$ [KK12] .
- \mathbb{Z}_2 -BF theory in 3d (3d Kitaev model)

$$S[A, B] = \int BF, \quad F = dA,$$

excitations form Drinfel'd double $D(\mathbb{Z}_2)$.

- **2d toric code \sim 3d Kitaev model.**

- 1 Continuum limit: *Tannaka-Krein* duality [Del18]

$$Z_1(\text{Rep}(\mathbb{Z}_2)) \simeq \text{Rep}(D(\mathbb{Z}_2)),$$

- 2 Lattice reconstruction: correspondence with groupoid algebra $\mathbb{C}[\Lambda G] \sim D(G)$ [DDR17] .

What is the story in 4d?

What we know in 4d, and what is missing

- Two approaches to gapped G -symm. 4d topological phases:
 - ① 2-categories [KTZ20; Joh20] ,
 - ② 2-group field theory [ZLW19; WWH15].
- How are they related? Need *higher* Drinfel'd doubles $\mathcal{D}(G)$.
 - ① Continuum limit: **Tannaka-Krein** for braided 2-categories

$$Z_1(\mathcal{C}_G) \simeq 2\text{Rep}(\mathcal{D}(G)),$$

- ② Lattice reconstruction: **correspondence** with 2-groupoid algebra $\mathcal{D}(G) \stackrel{?}{\sim} \mathbb{C}[\wedge G]$ [BD21] .

This work: solving the first point for $G = \mathbb{Z}_2$.

The 2-Drinfel'd double $\mathcal{D}(G) = D(BG)$

- Pontrjagyn dual pair $\widehat{G} \cong G$.
- 2-Drinfel'd double $D(BG)$:

$$\begin{array}{ccc}
 \text{deg-1:} & k\widehat{G} & * \\
 & \downarrow 0 & \downarrow 0 \\
 & & \boxtimes \\
 \text{deg-0:} & * & kG
 \end{array}$$

- 1 2-bialgebra: coproduct Δ is *grouplike* (for fusion).
- 2 Classification via **Postnikov class** [KT13]

$$\tau \in H^3(G, \widehat{G}) \cong HH^3(kG, k\widehat{G}).$$

- 3 Canonical 2-R-matrix (\mathcal{R}, R) (for braiding)

$$\mathcal{R} \in \widehat{G} \otimes G, \quad R \in G \otimes G.$$

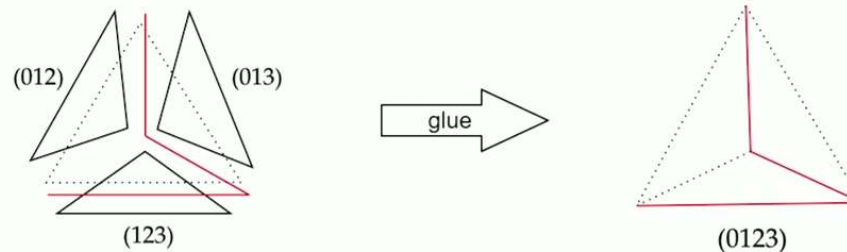
2-BF theory from $D(BG)$

- G -cocycles on 4-manifold X :

$$A \in C^1(X, G), \quad B \in C^2(X, \widehat{G}).$$

- Kinematics; equations of motion (EoM)

$$F = dA = 0, \quad dB = \tau(A)$$



- 4d topological 2-BF theory [CG22]:

$$S^0[A, B] = \int_X B \cup F - \tau \cup A.$$

Excitations of Z_{Kit}^0

- Take $G = \mathbb{Z}_2$, "invisible toric code"

$$Z_{\text{Kit}}^0(X) = \sum_{A,B} e^{2\pi i S^0[A,B]}.$$

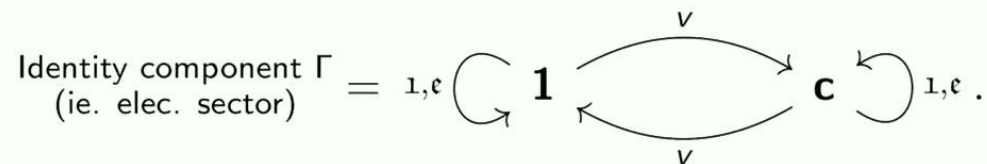
- Excitations = 2-representations of $D(B\mathbb{Z}_2)$ [CG23]:

- 1 Four distinct 2-irreps.

	Vacua	Cheshire
Electric	$\mathbf{1} = (1 \overset{d=1}{\oplus} 1, \rho_1 = 0)$	$\mathbf{c} = (1 \overset{d=0}{\oplus} \text{sgn}, \rho_1 = \hat{1})$
Magnetic	$\mathbf{1}^* = (\text{sgn} \overset{d=1}{\oplus} \text{sgn}, \rho_1 = 0)$	$\mathbf{c}^* = (\text{sgn} \overset{d=0}{\oplus} 1, \rho_1 = \hat{1})$

- 2 Postnikov $\tau \rightsquigarrow$ coassoc. $\delta \rightsquigarrow$ fusion associator a .

- **Fact:** no 1-morphisms between elec./mag. sectors,



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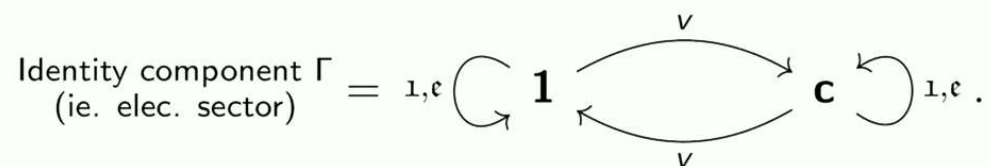
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So... why "invisible" toric code?

Proposition. (H.C.)

Functor $2\text{Rep}(D(B\mathbb{Z}_2)) \rightarrow Z_1(\Sigma \text{Vect}[\mathbb{Z}_2])$ is **non-monoidal**.

- 1 Fusions are correct on objects (have Cheshire charge [EN17])...

$$\mathbf{1}^2 \simeq (\mathbf{1}^*)^2 \simeq \mathbf{1}, \quad \mathbf{c}^2 \simeq \mathbf{c} \oplus \mathbf{c}, \quad \mathbf{1}^* \otimes \mathbf{c} \simeq \mathbf{c}^*,$$

but not morphisms

$$\mathbf{1}^2 \simeq \mathbf{e}^2 \simeq \mathbf{1}, \quad \mathbf{e} \otimes \mathbf{1} \simeq \mathbf{1} \otimes \mathbf{e} \simeq \mathbf{e}, \quad \mathbf{v} \otimes \mathbf{v} \simeq \mathbf{1}.$$

- 2 Moreover: violates **remote detectability** [KW14];

$$\mathcal{R}, R = \text{id} \implies \text{all braidings } b \text{ are trivial!}$$

Fix: twist $D(B\mathbb{Z}_2)$.

Cocycle twists on $D(B\mathbb{Z}_2)$

- ① Extension at deg-0: $\bar{e} \in H^2(\mathbb{Z}_2, \mathbb{Z}_2) \cong H^2(\mathbb{Z}_2, \widehat{\mathbb{Z}}_2)$:

$$\rho_0(x)\rho_0(x') = \rho_1(\bar{e}(x, x')) \cdot \rho_0(xx'), \quad x, x' \in G.$$

- Carried by $\mathbf{1}^*, \mathbf{c}$.

- ② Extension at deg-1: $\bar{c} \in H^2(\widehat{\mathbb{Z}}_2, \mathbb{C}^\times)$:

$$\rho_1(y)\rho_1(y') = \bar{c}(y, y')\rho_1(yy'), \quad y, y' \in \widehat{G}.$$

- Carried by \mathbf{c}, \mathbf{c}^* .

2-reps.	Twists	Name
$2\text{Rep}^\tau(D(B\mathbb{Z}_2)^{\omega_b})$	$\omega_b = \bar{e}$	"Bosonic"
$2\text{Rep}^\tau(D(B\mathbb{Z}_2)^{\omega_f})$	$\omega_f = \bar{e} + \bar{c}[1]$	"Fermionic"

Excitations in 4d spin-Kitaev models

- \bar{e} occurs when:

- 1 using naturality of fusion

$$v^2 = \text{id}_{\mathbf{1} \text{ or } \mathbf{c}} \otimes v \circ v \otimes \text{id}_{\mathbf{c} \text{ or } \mathbf{1}} \simeq \mathbf{1} \oplus \mathbf{e},$$

- 2 computing *full*-braids $B_{iW} = b_{iW} \circ b_{Wi} \propto \mathcal{R}^2$:

$$B_{\mathbf{c}\mathbf{1}^*} \simeq B_{\mathbf{e}\mathbf{c}^*} \simeq -\mathbf{1} \cdot \text{id},$$

implements remote detectability.

- \bar{c} occurs when:

- 1 computing fusion; lifting Cheshires to sign reps. of $\mathbb{Z}_2 \subset \mathbb{Z}_4$:

$$\mathbf{c}^2 \simeq \mathbf{1} \simeq (\mathbf{c}^*)^2.$$

Cheshires \mathbf{c}, \mathbf{c}^* now invertible and *self-braidable*,

- 2 computing self-braids $b_{\mathbf{c}} \propto R^2, b_{\mathbf{e}} \propto \mathcal{R}^2$:

$$b_{\mathbf{c}} \simeq \mathbf{e}, \quad b_{\mathbf{e}} \simeq -\mathbf{1} \cdot \text{id}$$

Proof of the main theorem

- What of self-braiding b_{1^*} ? Use fusion rule + ribbon formula

$$\begin{aligned} b_{1^*} &= b_{c^* \otimes c} \\ &\cong b_{c^*} \otimes b_c \otimes (b_{c^* c} \circ b_{cc^*}) \\ &\cong \mathbf{1} \otimes \mathbf{e} \otimes \mathbf{e} \simeq \mathbf{1}. \end{aligned}$$

- All braiding/fusion properties now match:

Theorem. (H.C.)

There are braided equivalences

$$2\text{Rep}^\tau(D(B\mathbb{Z}_2)^{\omega_b}) \simeq \text{4d toric code} \simeq Z_1(\Sigma \text{Vect}[\mathbb{Z}_2]),$$

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where second \simeq 's are from [Joh20] .

Tannakian duality and the continuum limit

- Twists introduce cohomological (non-dynamical) terms:

$$Z_{\text{Kit}}^s(X) \cong \sum_{\substack{dA=0 \\ dB=\tau(A)}} \exp \left[i2\pi \int_X \bar{c}(B, B) + B \cup \bar{e}(A, A) \right].$$

- 1 Appeared as topological non-linear σ -models [ZLW19] .
 - 2 Reproduce the right deg-5 anomalies [Joh20] .
- Success in establishing Tannaka-Krein:

$$(2\text{-})\text{category} \xrightleftharpoons{\text{2-Drinfel'd double}} (2\text{-group}) \text{ field theory}$$

Outlook

- 1 4d boundary of the $w_2 w_3$ grav. anomaly [Joh20].
 - Requires *adjunctions* \implies (weak) ribbon 2-Hopf structure.
- 2 Morita duality $D(BG)^\omega \sim \mathbb{C}^\omega[\Lambda BG]$ [BD21].
 - Lattice reconstruction from field theory.
- 3 Applications to *compact* groups, eg. $G = SU(2)$.
 - Study Baez's conjecture: 4d BF-BB theory $\stackrel{?}{=} \text{Crane-Yetter}$.