Title: Lecture 3: Bootstrapping spinning correlators

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Collection: Mini-Course of Numerical Conformal Bootstrap

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References

- Counting Conformal Correlators
  [Kravchuk, Simmons-Duffin '16] arXiv:1612.08987
- Recursion relation for general 3d blocks
  [Erramilli, Iliesiu, Kravchuk '19] arXiv:1907.11247
- blocks_3d: Software for general 3d conformal blocks
  [Erramilli, Iliesiu, Kravchuk, Landry, Poland, Simmons-Duffin '20] arXiv:2011.01959
- The Gross-Neveu-Yukawa archipelago
  [Erramilli, Iliesiu, Kravchuk, AL, Poland, Simmons-Duffin '22] arXiv:2210.02492

General Blocks

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\mathcal{O}_l(x_4)\rangle$$
$$= \sum_I g^I_{ijkl}(u,v)\, T_{I,ijkl}(x_1,x_2,x_3,x_4)$$
$$\sim \sum_{\mathcal{O}} \frac{\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)|\mathcal{O}\rangle\langle\mathcal{O}|\mathcal{O}_k(x_3)\mathcal{O}_l(x_4)\rangle}{\langle\mathcal{O}|\mathcal{O}\rangle}$$

$$g^I_{ijkl}(u,v) = \sum_{\Delta,\rho}\sum_{a,b} \lambda^a_{ij\mathcal{O}}\lambda^b_{kl\mathcal{O}} G^{ab,I}_{ijkl,\Delta,\rho}(u,v), \quad \text{conformal blocks}$$

Spinning Operators $d = 3$

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle \quad \rightarrow \quad \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle$$

We consider Hermitian local primary operators $\mathcal{O}$ with (Δ, j), represented with $2j$ spinor indices,

$$\mathcal{O}^{(\alpha_1 \cdots \alpha_{2j})}(x).$$

where each index $\alpha_i = 1, 2$, e.g. ψ^α and $V^\mu \rightarrow V^{(\alpha_1 \alpha_2)}$.
Index-free notation, contract each index with a single real polarization spinor

$$\mathcal{O}(x, s) = \mathcal{O}^{(\alpha_1 \cdots \alpha_{2j})}(x) s_{\alpha_1} \cdots s_{\alpha_{2j}}, \quad s_{\alpha_i} \equiv \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}.$$

Spinning Operators $d = 3$

$$\mathcal{O}(x, s) = \mathcal{O}^{(\alpha_1 \cdots \alpha_{2j})}(x) s_{\alpha_1} \cdots s_{\alpha_{2j}} \quad s_{\alpha_i} \equiv \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}.$$

The the time-ordered two-point function for space-like separated x_1, x_2 is given by

$$\langle \mathcal{O}(x_1, s_1) \mathcal{O}(x_2, s_2) \rangle = c_{\mathcal{O}} \frac{i^{2j} (s_1^{\alpha} \gamma^{\mu}_{\alpha\beta} s_2^{\beta} x_{12,\mu})^{2j}}{x_{12}^{2\Delta + 2j}},$$

where $x_{12} = x_1 - x_2$, $c_{\mathcal{O}}$ is a positive constant.

Three-point and Four-point functions

- Three-point function, OPE coefficients $\lambda_{ijk}^{(a)}$.

$$\langle \mathcal{O}_i(x_1, s_1)\mathcal{O}_j(x_2, s_2)\mathcal{O}_k(x_3, s_3)\rangle = \sum_{a \in \mathcal{I}_{ijk}} \lambda_{ijk}^{(a)} \boldsymbol{T}_{(a),ijk}(x_1, s_1; x_2, s_2; x_3, s_3),$$

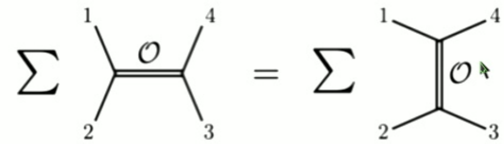
- Four-point function, function coefficients $g_{ijkl}^{(I)}(z, \bar{z})$.

$$\langle \mathcal{O}_i(x_1, s_1)\mathcal{O}_j(x_2, s_2)\mathcal{O}_k(x_3, s_3)\mathcal{O}_l(x_4, s_4)\rangle = \sum_{I \in \mathcal{I}_{ijkl}} g_{ijkl}^{(I)}(z, \bar{z}) \boldsymbol{T}_{(I),ijkl}(x_1, s_1; x_2, s_2; x_3, s_3; x_4, s_4),$$

- Block decomposition of each function coefficient.

$$g_{ijkl}^{I}(z, \bar{z}) = \sum_{\Delta,j} \sum_{a,b} \lambda_{ij\mathcal{O}}^{a} \lambda_{kl\mathcal{O}}^{b} G_{ijkl,\Delta,j}^{ab,I}(z, \bar{z})$$

Crossing symmetry



$$\langle \mathcal{O}_i(x_1, s_1)\mathcal{O}_j(x_2, s_2)|\mathcal{O}_k(x_3, s_3)\mathcal{O}_l(x_4, s_4)\rangle$$

$$= \pm \langle \mathcal{O}_k(x_3, s_3)\mathcal{O}_j(x_2, s_2)|\mathcal{O}_i(x_1, s_1)\mathcal{O}_l(x_4, s_4)\rangle.$$

Factor out the tensor structures,

$$g^I_{ijkl}(z, \bar{z}) \mp \sum_J M^I_J g^J_{kjil}(1-z, 1-\bar{z}) = 0.$$

$$g^I_{ijkl}(z, \bar{z}) = \sum_{\Delta,\rho} \sum_{a,b} \lambda^a_{ij\mathcal{O}} \lambda^b_{kl\mathcal{O}} G^{ab,I}_{ijkl,\Delta,\rho}(z, \bar{z})$$

Blocks_3d

$$G^{ab,I}_{\Delta,j}(z,\bar{z}) \propto r^{\Delta} h^{ab,I}_{\Delta,j}(z,\bar{z})$$

$$h^{ab,I}_{\Delta,j}(z,\bar{z}) = h^{ab}_{\infty,j,I}(z,\bar{z}) + \sum_{i\in\mathcal{P}_j} \frac{(a_0 r)^{n_{j,i}}}{\Delta - \Delta_{j,i}} (\mathcal{L}_{j,i})^{a}_{a'} (\mathcal{R}_{j,i})^{b}_{b'} h^{a'b'}_{\Delta'_{j,i}, j'_{j,i}, I}(z,\bar{z}).$$

`Blocks_3d` computes blocks assuming that

- the three-point structures are defined in $SO(3)$ basis, $a = (j_{12}, j_{12\mathcal{O}})$, $b = (j_{43}, j_{43\mathcal{O}})$.
- the four-point structures defined in q-basis, $I = [q_1 q_2 q_3 q_4]$

Three-Point Structures: q-basis

Given quantum numbers (Δ_i, j_i), the q-basis structures are

$$T^{ijk}_{a=[q_1,q_2,q_3]},\ q_i \in \{-j_i, -j_i+1, \cdots, j_i\},\ q_1+q_2+q_3=0.$$

The structures $T^{ijk}_{[q_1,q_2,q_3]}(x_1,s_1;x_2,s_2;x_3,s_3)$ are defined by requiring that in the following configuration,

$$x_1=(0,0,0), x_2=(0,0,1), x_3(L)=(0,0,L),$$

the following identity holds

$$\lim_{L\to+\infty} L^{2\Delta_3} T^{ijk}_{[q_1,q_2,q_3]}(x_1,s_1;x_2,s_2;x_3,s_3) = \prod_{i=1}^{3}((s_i)_1)^{j_i+q_i}((s_i)_2)^{j_i-q_i}.$$

Three-Point Structures: q-basis

Given quantum numbers (Δ_i, j_i), the q-basis structures are

$$T^{ijk}_{a=[q_1,q_2,q_3]},\ q_i \in \{-j_i, -j_i+1, \cdots, j_i\},\ q_1+q_2+q_3=0.$$

$T^{ijk}_{[q_1,q_2,q_3]}$ transform nicely under space parity

$$T^{ijk}_{[q_1,q_2,q_3]}(x_1,s_1;x_2,s_2;x_3,s_3) \to T^{ijk}_{[-q_1,-q_2,-q_3]}(x_1,s_1;x_2,s_2;x_3,s_3)$$

and similarly under other permutations.
If we want structures invariant under some transformations, we can construct them easily in q-basis.

Example : $\langle \sigma\psi\mathcal{O} \rangle$

Start from q-basis $[q_1, q_2, q_3]$, $j_1 = 0, j_2 = 1/2,\ q_3 = -q_1 - q_2$,

$$[q_1 q_2 q_3] = [0,\ 1/2,\ -1/2],\ [0, -1/2,\ 1/2].$$

Assuming that σ is odd and ψ is even under parity, then $\mathcal{O}$ and T_a have opposite parity,

$$\langle \sigma\psi\mathcal{O} \rangle \sim \lambda^a T_a$$

$$[q_1, q_2, q_3] \rightarrow [-q_1, -q_2, -q_3].$$

We then define the parity-definite structures

$$[q_1 q_2 q_3]^{\pm} = [q_1 q_2 q_3] \pm [-q_1, -q_2, -q_3].$$

$$[0,\ 1/2,\ -1/2]^{+}, \quad [0,\ 1/2,\ -1/2]^{-}.$$

$SO(3)$-basis

The $SO(3)$ basis structures are labeled by pairs (j_{12}, j_{123})

$$j_{12} \in \{|j_1 - j_2|, |j_1 - j_2| + 1, \cdots j_1 + j_2\},$$
$$j_{123} \in \{|j_3 - j_{12}|, |j_3 - j_{12}| + 1, \cdots j_3 + j_{12}\}.$$

Want to relate $T^{ijk}_{(j_{12},j_{123})}(x_1, s_1; x_2, s_2; x_3, s_3)$ with $T^{ijk}_{[q_1,q_2,q_3]}(x_1, s_1; x_2, s_2; x_3, s_3)$

Example : $\langle \sigma\psi\mathcal{O} \rangle$

Then we use CG coefficient to write them in $SO(3)$ basis. For example

$$[0,\, 1/2,\, -1/2] = \frac{(-1)^{j-1/2}}{\sqrt{2\binom{2j}{j+\frac{1}{2}}}} \left(|\frac{1}{2}, j-\frac{1}{2}\rangle + |\frac{1}{2}, j+\frac{1}{2}\rangle \right).$$

$$[0,\, -1/2,\, 1/2] = \frac{(-1)^{j-1/2}}{\sqrt{2\binom{2j}{j+\frac{1}{2}}}} \left(|\frac{1}{2}, j-\frac{1}{2}\rangle + |\frac{1}{2}, j-\frac{1}{2}\rangle \right).$$

$$[0,\, 1/2,\, -1/2]^{+} = \frac{(-1)^{j-1/2}}{\sqrt{2\binom{2j}{j+\frac{1}{2}}}} |\frac{1}{2}, j-\frac{1}{2}\rangle \Rightarrow (-1)^{j-1/2} |\frac{1}{2}, j-\frac{1}{2}\rangle$$

$$[0,\, 1/2,\, -1/2]^{-} = \frac{(-1)^{j-1/2}}{\sqrt{2\binom{2j}{j+\frac{1}{2}}}} |\frac{1}{2}, j+\frac{1}{2}\rangle \Rightarrow (-1)^{j-1/2} |\frac{1}{2}, j+\frac{1}{2}\rangle$$

Example : $\langle \sigma\psi\mathcal{O} \rangle$

Then we use CG coefficient to write them in $SO(3)$ basis. For example

$$[0,\, 1/2,\, -1/2] = \frac{(-1)^{j-1/2}}{\sqrt{2\binom{2j}{j+\frac{1}{2}}}} \left(|\frac{1}{2}, j-\frac{1}{2}\rangle + |\frac{1}{2}, j+\frac{1}{2}\rangle \right).$$

$$[0,\, -1/2,\, 1/2] = \frac{(-1)^{j-1/2}}{\sqrt{2\binom{2j}{j+\frac{1}{2}}}} \left(|\frac{1}{2}, j-\frac{1}{2}\rangle + |\frac{1}{2}, j-\frac{1}{2}\rangle \right).$$

$$[0,\, 1/2,\, -1/2]^{+} = \frac{(-1)^{j-1/2}}{\sqrt{2\binom{2j}{j+\frac{1}{2}}}} |\frac{1}{2}, j-\frac{1}{2}\rangle \Rightarrow (-1)^{j-1/2} |\frac{1}{2}, j-\frac{1}{2}\rangle$$

$$[0,\, 1/2,\, -1/2]^{-} = \frac{(-1)^{j-1/2}}{\sqrt{2\binom{2j}{j+\frac{1}{2}}}} |\frac{1}{2}, j+\frac{1}{2}\rangle \Rightarrow (-1)^{j-1/2} |\frac{1}{2}, j+\frac{1}{2}\rangle$$

Four-Point Structures: q-basis

The q-basis four-point structures are

$$T^{ijkl}_{I=[q_1 q_2 q_3 q_4]}, \quad q_i \in \{-j_i, -j_i+1, \cdots, j_i\}. \qquad (1)$$

Similar to three-point structures, we require that when

$$x_1 = (0,0,0),\ x_2 = (\tfrac{\bar{z}-z}{2}, \tfrac{\bar{z}+z}{2}, 0),\ x_3 = (0,1,0),\ x_4(L) = (0,L,0),$$

$$\lim_{L\to+\infty} L^{2\Delta_4} T^{ijkl}_{[q_1 q_2 q_3 q_4]}(x_1, s_1; x_2, s_2; x_3, s_3; x_4, s_4)$$
$$= \prod_{i=1}^{4} ((s_i)_1)^{j_i+q_i} ((s_i)_2)^{j_i-q_i}.$$

Parity: $T_{[q_1,q_2,q_3,q_4]} \to (-1)^{\sum_i j_i - q_i} T_{[q_1,q_2,q_3,q_4]}$.

Four-Point Structures: Stablizer

Given $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle$, we call a permutation a stablizer if it changes neither the ordering of the operators nor the cross-ratios.

For four identical operators, such as $\langle \psi\psi\psi\psi \rangle$, the stablizers are

$$\{e, (12)(34), (13)(24), (14)(23)\} = \mathbb{Z}_2 \times \mathbb{Z}_2$$

For a pair of identical operators, such as $\langle \sigma\sigma\psi\psi \rangle$, the stablizer group is just $\{e, (12)(34)\} = \mathbb{Z}_2$.

Given

$$\langle \psi_1 \psi_2 \psi_3 \psi_4 \rangle = \sum g^{[q_1 q_2 q_3 q_4]}_{\psi\psi\psi\psi}(z, \bar{z})\, T_{[q_1 q_2 q_3 q_4]}.$$

Under (13)(24),

$$\langle \psi_3 \psi_4 \psi_1 \psi_2 \rangle = \sum g^{[q_1 q_2 q_3 q_4]}_{\psi\psi\psi\psi}(z, \bar{z}) \cdot n(z, \bar{z})\, T_{[q_3 q_4 q_1 q_2]}.$$

Comparing with

$$\langle \psi_3 \psi_4 \psi_1 \psi_2 \rangle = \sum g^{[q_3 q_4 q_1 q_2]}_{\psi\psi\psi\psi}(z, \bar{z})\, T_{[q_3 q_4 q_1 q_2]}.$$

This implies that some function coefficients are linearly dependent of each other

$$n(z, \bar{z}) g^{[q_1 q_2 q_3 q_4]}_{\psi\psi\psi\psi}(z, \bar{z}) = g^{[q_3 q_4 q_1 q_2]}_{\psi\psi\psi\psi}$$

For example, we have $g_{[\uparrow\uparrow\uparrow\uparrow]}$, but for $g_{[\uparrow\uparrow\downarrow\downarrow]}$ and $g_{[\downarrow\downarrow\uparrow\uparrow]}$, we should choose only one of them to parameterize the four-point function.

Four-Point Structures: q-basis

The q-basis four-point structures are

$$T^{ijkl}_{I=[q_1 q_2 q_3 q_4]}, \quad q_i \in \{-j_i, -j_i+1, \cdots, j_i\}. \qquad (1)$$

Similar to three-point structures, we require that when

$$x_1 = (0,0,0),\ x_2 = (\tfrac{\bar{z}-z}{2}, \tfrac{\bar{z}+z}{2}, 0),\ x_3 = (0,1,0),\ x_4(L) = (0,L,0),$$

$$\lim_{L\to+\infty} L^{2\Delta_4} T^{ijkl}_{[q_1 q_2 q_3 q_4]}(x_1, s_1; x_2, s_2; x_3, s_3; x_4, s_4)$$
$$= \prod_{i=1}^{4} ((s_i)_1)^{j_i+q_i} ((s_i)_2)^{j_i-q_i}.$$

Parity: $T_{[q_1,q_2,q_3,q_4]} \to (-1)^{\sum_i j_i - q_i} T_{[q_1,q_2,q_3,q_4]}$.

Four-Point Structures: Example

$$\langle \sigma\sigma\psi\psi \rangle, \quad \langle \psi\sigma\sigma\psi \rangle, \quad \langle \psi\sigma\psi\sigma \rangle.$$

Parity $= (-1)^{\sum_i j_i - q_i}$, so the parity-even structures are

$$[0, 0, \frac{1}{2}, \frac{1}{2}], \quad [0, 0, -\frac{1}{2}, -\frac{1}{2}], ..$$

Besides parity and stablizers,

$$(13): \quad T_{[q_1,q_2,q_3,q_4]} \to (-1)^{q_1+q_2+q_3-q_4} T_{[q_3,q_2,q_1,q_4]}$$

$$(z \leftrightarrow \bar{z}): \quad T_{[q_1,q_2,q_3,q_4]}(z, \bar{z}) \to (-1)^{\sum_i j_i} T_{[-q_1,-q_2,-q_3,-q_4]}(\bar{z}, z)$$

Crossing Equations

$$\langle \mathcal{O}_i(x_1, s_1)\mathcal{O}_j(x_2, s_2)|\mathcal{O}_k(x_3, s_3)\mathcal{O}_l(x_4, s_4)\rangle$$
$$= \pm \langle \mathcal{O}_k(x_3, s_3)\mathcal{O}_j(x_2, s_2)|\mathcal{O}_i(x_1, s_1)\mathcal{O}_l(x_4, s_4)\rangle.$$

$$\sum_{[q_i]} g_{ijkl}^{[q_1,q_2,q_3,q_4]}(z,\bar{z})\, T^{ijkl}_{[q_1,q_2,q_3,q_4]}(x1, s1, \ldots)$$
$$= \pm \sum_{[q_i]} g_{kjil}^{[q_3,q_2,q_1,q_4]}(1-z, 1-\bar{z})\, T^{kjil}_{[q_3,q_2,q_1,q_4]}(x3, s3, \ldots)$$
$$= \pm \sum_{[q_i]} (-1)^{q_1+q_2+q_3-q_4} g_{kjil}^{[q_3,q_2,q_1,q_4]}(1-z, 1-\bar{z})\, T^{ijkl}_{[q_1,q_2,q_3,q_4]}(x1, s1, \ldots)$$

$$g_{ijkl}^{[q_1,q_2,q_3,q_4]}(z,\bar{z}) = \pm(-1)^{q_1+q_2+q_3-q_4} g_{kjil}^{[q_3,q_2,q_1,q_4]}(1-z, 1-\bar{z})$$

Crossing Equations:Example

$$\langle \psi\sigma | \psi\sigma \rangle = -\langle \psi\sigma | \psi\sigma \rangle,$$
$$\langle \sigma\sigma | \psi\psi \rangle = \langle \psi\sigma | \sigma\psi \rangle.$$

$$g_{ijkl}^{[q_1,q_2,q_3,q_4]}(z,\bar{z}) = \pm(-1)^{q_1+q_2+q_3-q_4} g_{kjil}^{[q_3,q_2,q_1,q_4]}(1-z,1-\bar{z})$$

$$g^{\psi\sigma\psi\sigma}_{[\frac{1}{2}0\frac{1}{2}0]}(z,\bar{z}) = g^{\psi\sigma\psi\sigma}_{[\frac{1}{2}0\frac{1}{2}0]}(1-z,1-\bar{z}),$$
$$g^{\psi\sigma\psi\sigma}_{[-\frac{1}{2}0-\frac{1}{2}0]}(z,\bar{z}) = g^{\psi\sigma\psi\sigma}_{[-\frac{1}{2}0-\frac{1}{2}0]}(1-z,1-\bar{z}),$$
$$g^{\sigma\sigma\psi\psi}_{[00\frac{1}{2}\frac{1}{2}]}(z,\bar{z}) = g^{\psi\sigma\sigma\psi}_{[\frac{1}{2}00\frac{1}{2}]}(1-z,1-\bar{z}),$$
$$g^{\sigma\sigma\psi\psi}_{[00-\frac{1}{2}-\frac{1}{2}]}(z,\bar{z}) = g^{\psi\sigma\sigma\psi}_{[-\frac{1}{2}00-\frac{1}{2}]}(1-z,1-\bar{z}).$$

$\Rightarrow$

Crossing even

$$g^{\psi\sigma\psi\sigma}_{[\pm\frac{1}{2}0\pm\frac{1}{2}0]}(z,\bar{z}),$$
$$g^{\sigma\sigma\psi\psi}_{[00\pm\frac{1}{2}\pm\frac{1}{2}]}(z,\bar{z}) + g^{\psi\sigma\sigma\psi}_{[\pm\frac{1}{2}00\pm\frac{1}{2}]}(z,\bar{z}),$$

Crossing odd

$$g^{\sigma\sigma\psi\psi}_{[00\pm\frac{1}{2}\pm\frac{1}{2}]}(z,\bar{z}) - g^{\psi\sigma\sigma\psi}_{[\pm\frac{1}{2}00\pm\frac{1}{2}]}(z,\bar{z}).$$

Functional: Derivative Basis $(z, \bar{z})$

$$\alpha^i[f] = \sum_{m+n \leq 2n_{\max}-1} a^i_{mn} \partial_z^m \partial_{\bar{z}}^n f(z, \bar{z}) \Big|_{(z=\frac{1}{2}, \bar{z}=\frac{1}{2})}$$

Definite parity under the transformation $z \leftrightarrow \bar{z}$,

$$(z \leftrightarrow \bar{z}): \quad T_{[q_1,q_2,q_3,q_4]}(z, \bar{z}) \to (-1)^{\sum_i j_i} T_{[-q_1,-q_2,-q_3,-q_4]}(\bar{z}, z)$$

$$T_{[q_1,q_2,q_3,q_4]^\pm} \to T_{[q_1,q_2,q_3,q_4]^\pm} \pm (-1)^{\sum_i j_i} T_{[-q_1,-q_2,-q_3,-q_4]^\pm}$$

In our examples, these are

$$[00 \pm \frac{1}{2} \pm \frac{1}{2}]^\pm = [00\frac{1}{2}\frac{1}{2}] \mp [00 - \frac{1}{2} - \frac{1}{2}]$$

Functional: Derivative Basis (x, t)

$$x = \frac{z + \bar{z} - 1}{2}, \quad y = \frac{z - \bar{z}}{2}, \quad t = (\frac{z - \bar{z}}{2})^2.$$

The crossing-symmetric point becomes $x = t = 0$.
Crossing acts by

$$z \to 1 - z, \quad \bar{z} \to 1 - \bar{z} \quad \text{or}$$

$$x \to -x, \quad y \to -y \quad \text{(xy-parity)}$$

crossing-even $g(z, \bar{z})$: xy-even.
crossing-odd $g(z, \bar{z})$: xy-odd.

crossing-even $g(x, t)$: $\quad \partial_x^m \partial_y^n g = 0, \; m + n = \text{odd}.$
crossing-odd $g(x, t)$: $\quad \partial_x^m \partial_y^n g = 0, \; m + n = \text{even}.$

Functional: Derivative Basis (x, t)

Similarly for $z \leftrightarrow \bar{z}$, $x \to x$, $y \to -y$ (t-parity),

$$g_{[q_1 q_2 q_3 q_4]^+}(z, \bar{z}) : \text{ t-parity even, y-even.}$$
$$g_{[q_1 q_2 q_3 q_4]^-}(z, \bar{z}) : \text{ t-parity odd, y-odd.}$$

Convention in Hyperion, $\partial_x^m \partial_y^n$ has the same t-parity/y-parity as g, and opposite xy-parity to g.

$$\partial_t^m \partial_x^n g^{[q_3, q_2, q_1, q_4]^+}(x, t)\Big|_{x=t=0}$$
$$\partial_t^m \partial_x^n \left(\frac{2}{(z - \bar{z})} g^{[q_3, q_2, q_1, q_4]^-}(x, t) \right) \Big|_{x=t=0}$$

$$2m + n \leq 2n_{\max} - 1.$$

Functional: Derivative Basis (x, t)

$$\partial_x^m \partial_y^n$$

$$g^{\sigma\sigma\psi\psi}_{[00\frac{1}{2}\frac{1}{2}]^+}(x,t) + g^{\psi\sigma\sigma\psi}_{[\frac{1}{2}00\frac{1}{2}]^+}(x,t), \text{xy-odd, y-even} \to \text{x-odd, y-even}$$

$$g^{\sigma\sigma\psi\psi}_{[00\frac{1}{2}\frac{1}{2}]^-}(x,t) + g^{\psi\sigma\sigma\psi}_{[\frac{1}{2}00\frac{1}{2}]^-}(x,t), \text{xy-odd, y-odd,} \to \text{x-even, y-odd}$$

$$g^{\psi\sigma\psi\sigma}_{[\frac{1}{2}0\frac{1}{2}0]^+}(x,t), \text{xy-odd, y-even,} \to \text{x-odd, y-even}$$

$$g^{\psi\sigma\psi\sigma}_{[\frac{1}{2}0\frac{1}{2}0]^-}(x,t), \text{xy-odd, y-odd} \to \text{x-even, y-odd}$$

$$g^{\sigma\sigma\psi\psi}_{[00\frac{1}{2}\frac{1}{2}]^+}(x,t) - g^{\psi\sigma\sigma\psi}_{[\frac{1}{2}00\frac{1}{2}]^+}(x,t), \text{xy-even, y-even} \to \text{x-even, y-even}$$

$$g^{\sigma\sigma\psi\psi}_{[00\frac{1}{2}\frac{1}{2}]^-}(x,t) - g^{\psi\sigma\sigma\psi}_{[\frac{1}{2}00\frac{1}{2}]^-}(x,t), \text{xy-even, y-odd} \to \text{x-odd, y-odd}$$

Smoothness of $g(t = 0)$

When we expand the four-point function in conformal blocks, we will find that the result is smooth (as a function of x_i). But it turns out that not any choice of $g_{[q_1 q_2 q_3 q_4]}(z, \bar{z})$ leads to a smooth correlator, and a finite number of boundary conditions need to be imposed on derivatives of $g_{[q_1 q_2 q_3 q_4]}(z, \bar{z})$ at $z = \bar{z}$ or $t = 0$.

$$\partial_t^{2n} \partial_x^{2m+1} g^{\psi\psi\psi\psi}_{[\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}]^+}(x, t) \quad n \geq 1, \geq 0$$

Reference: Counting Conformal Correlators

Blocks_3d Software

$$g^{ijkl}_{[q_1 q_2 q_3 q_4],\pm}(x,t) = \sum_{\Delta,j} \sum_{(j_{12}, j_{12\mathcal{O}})} \sum_{(j_{43}, j_{43\mathcal{O}})} \lambda^{(j_{12}, j_{12\mathcal{O}})}_{ij\mathcal{O}} \lambda^{(j_{43}, j_{43\mathcal{O}})}_{kl\mathcal{O}} G^{(j_{12}, j_{12\mathcal{O}}),(j_{43}, j_{43\mathcal{O}})}_{\Delta,j,[q_1 q_2 q_3 q_4],\pm}(x,t)$$

$$i^{-\#\text{fermions}} \partial_x^m \partial_t^n \left(p_\pm(x,t) G^{(j_{12}, j_{12\mathcal{O}}),(j_{43}, j_{43\mathcal{O}})}_{\Delta,j,[q_1 q_2 q_3 q_4],\pm}(x,t) \right) \Big|_{x=t=0}$$

```
./blocks_3d --help
./blocks_3d --j-external 0.5,0,0.5,0 --j-internal
0.5,1.5,2.5,3.5,4.5,5.5,6.5,7.5,8.5,9.5,10.5,11.5,12.5,13.5,14.5,15.5,16.5
--j-12 0.5 --j-43 0.5 --delta-12 0.4176 --delta-43 -0.4176
--delta-1-plus-2 1.7196 --four-pt-struct 0.5,0,0.5,0 --four-pt-sign -1
--order 60 --lambda 11 --coordinates xt --kept-pole-order 8
--num-threads 6 --debug 1 --precision 768 -o output/derivs.json
```

Functional: Derivative Basis (x, t)

Similarly for $z \leftrightarrow \bar{z}$, $x \to x$, $y \to -y$ (t-parity),

$$g_{[q_1 q_2 q_3 q_4]^+}(z, \bar{z}) : \text{ t-parity even, y-even.}$$
$$g_{[q_1 q_2 q_3 q_4]^-}(z, \bar{z}) : \text{ t-parity odd, y-odd.}$$

Convention in Hyperion, $\partial_x^m \partial_y^n$ has the same t-parity/y-parity as g, and opposite xy-parity to g.

$$\partial_t^m \partial_x^n g^{[q_3, q_2, q_1, q_4]^+}(x, t)\Big|_{x=t=0}$$
$$\partial_t^m \partial_x^n \left(\frac{2}{(z-\bar{z})} g^{[q_3, q_2, q_1, q_4]^-}(x, t) \right) \Bigg|_{x=t=0}$$

$$2m + n \leq 2n_{\max} - 1.$$

Blocks_3d Software

$$g^{ijkl}_{[q_1 q_2 q_3 q_4],\pm}(x,t) = \sum_{\Delta,j} \sum_{(j_{12},j_{12\mathcal{O}})} \sum_{(j_{43},j_{43\mathcal{O}})} \lambda^{(j_{12},j_{12\mathcal{O}})}_{ij\mathcal{O}} \lambda^{(j_{43},j_{43\mathcal{O}})}_{kl\mathcal{O}} G^{(j_{12},j_{12\mathcal{O}}),(j_{43},j_{43\mathcal{O}})}_{\Delta,j,[q_1 q_2 q_3 q_4],\pm}(x,t)$$

$$i^{-\#\text{fermions}} \partial_x^m \partial_t^n \left(p_\pm(x,t) G^{(j_{12},j_{12\mathcal{O}}),(j_{43},j_{43\mathcal{O}})}_{\Delta,j,[q_1 q_2 q_3 q_4],\pm}(x,t) \right) \Big|_{x=t=0}$$

```
./blocks_3d --help
./blocks_3d --j-external 0.5,0,0.5,0 --j-internal
0.5,1.5,2.5,3.5,4.5,5.5,6.5,7.5,8.5,9.5,10.5,11.5,12.5,13.5,14.5,15.5,16.5
--j-12 0.5 --j-43 0.5 --delta-12 0.4176 --delta-43 -0.4176
--delta-1-plus-2 1.7196 --four-pt-struct 0.5,0,0.5,0 --four-pt-sign -1
--order 60 --lambda 11 --coordinates xt --kept-pole-order 8
--num-threads 6 --debug 1 --precision 768 -o output/derivs.json
```

Blocks_3d

$$G^{ab,I}_{\Delta,j}(z,\bar{z}) \propto r^{\Delta} h^{ab,I}_{\Delta,j}(z,\bar{z})$$

$$h^{ab,I}_{\Delta,j}(z,\bar{z}) = h^{ab}_{\infty,j,I}(z,\bar{z}) + \sum_{i\in\mathcal{P}_j} \frac{(a_0 r)^{n_{j,i}}}{\Delta-\Delta_{j,i}} (\mathcal{L}_{j,i})^{a}_{a'} (\mathcal{R}_{j,i})^{b}_{b'} h^{a'b'}_{\Delta'_{j,i},j'_{j,i},I}(z,\bar{z}).$$

`Blocks_3d` computes blocks assuming that

- the three-point structures are defined in $SO(3)$ basis, $a = (j_{12}, j_{12\mathcal{O}}),\ b = (j_{43}, j_{43\mathcal{O}})$.
- the four-point structures defined in q-basis, $I = [q_1 q_2 q_3 q_4]$

The Model

The 3-dimensional $O(N)$ Gross-Neveu-Yukawa (GNY) theories with N Majorana fermions ψ_i, interacting with a pseudoscalar ϕ

$$\mathcal{L}_{\mathrm{GNY}} = \frac{1}{2}(\partial\phi)^2 + i\psi_i \not{\partial}\psi_i - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 + g\phi\psi_i\psi_i.$$

Global $O(N)$ symmetry and invariant under parity transformation.

- One of the simplest models for scalar-fermion interaction.

Condensed Matter Applications

The universality class of the GNY model is used to describe a variety of quantum phase transitions in condensed matter systems.

- $N = 8$: Quantum Phase Transitions in graphene [Phys. Rev. Lett. 85 (2000) 4940–4943].
- $N = 4$: Spinless fermions on honeycomb lattice [1703.08801].

CFT scaling dimensions $\Rightarrow$ critical exponents.

$N = 1$: Emergent Supersymmetry

- $N = 1$:

$$\mathcal{L}_{\text{GNY}} = \frac{1}{2}(\partial\phi)^2 + i\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 + g\phi\bar{\psi}\psi.$$

Emergent supersymmetry arising at criticality, $\lambda = g^2$.

Emergent supersymmetry on the boundary of topological superconductors [Science 344, 280 (2014)].

Bootstrap Setup

$$\mathcal{O}: \quad (\ell, P, \mu_{O(N)})$$

$$\mathcal{L}_{\rm GNY} = \frac{1}{2}(\partial\phi)^2 + i\psi_i \partial\!\!\!/ \psi_i - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 + g\phi\psi_i\psi_i.$$

We consider the mixed system of the lowest-dimension operators ψ_i, ϵ and σ, where their quantum numbers are,

$$\sigma \sim \phi : \quad (0, \text{odd}, \bullet)$$
$$\epsilon \sim \phi^2 : \quad (0, \text{even}, \bullet)$$
$$\psi_i : \quad (\tfrac{1}{2}, \text{even}, \Box).$$

Counting Three-point Conformal Structures

OPE	$\mathcal{O} \in (l, P, \mu)$	$\langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}_c \rangle$	Structures
$\sigma \times \sigma$	$(l \in 2\mathbb{Z}, \text{even}, \bullet)$	$\langle \sigma\sigma\mathcal{O} \rangle$	$[000]$
$\epsilon \times \epsilon$		$\langle \epsilon\epsilon\mathcal{O} \rangle$	
$\sigma \times \epsilon$	$(l \in 2\mathbb{Z}, \text{odd}, \bullet)$	$\langle \sigma\epsilon\mathcal{O} \rangle$	$[000]$
		$\langle \epsilon\sigma\mathcal{O} \rangle$	$(-1)^j[000]$
$\sigma \times \psi$	$(l \in \mathbb{Z} + \frac{1}{2}, \text{even}, \square)$	$\langle \sigma\psi^i\mathcal{O}^j \rangle$	$-\delta^{ij}[0, \frac{1}{2}, -\frac{1}{2}]^-$
		$\langle \psi^i\sigma\mathcal{O}^j \rangle$	$(-1)^{j-\frac{1}{2}}\delta^{ij}[\frac{1}{2}, 0, -\frac{1}{2}]^-$
	$(l \in \mathbb{Z} + \frac{1}{2}, \text{odd}, \square)$	$\langle \sigma\psi^i\mathcal{O}^j \rangle$	$\delta^{ij}[0, \frac{1}{2}, -\frac{1}{2}]^+$
		$\langle \psi^i\sigma\mathcal{O}^j \rangle$	$(-1)^{j-\frac{1}{2}}\delta^{ij}[\frac{1}{2}, 0, -\frac{1}{2}]^+$
$\epsilon \times \psi$	$(l \in \mathbb{Z} + \frac{1}{2}, \text{even}, \square)$	$\langle \epsilon\psi^i\mathcal{O}^j \rangle$	$\delta^{ij}[0, \frac{1}{2}, -\frac{1}{2}]^+$
		$\langle \psi^i\epsilon\mathcal{O}^j \rangle$	$(-1)^{j-\frac{1}{2}}\delta^{ij}[\frac{1}{2}, 0, -\frac{1}{2}]^+$
	$(l \in 2\mathbb{Z} + \frac{1}{2}, \text{odd}, \square)$	$\langle \epsilon\psi^i\mathcal{O}^j \rangle$	$-\delta^{ij}[0, \frac{1}{2}, -\frac{1}{2}]^-$
		$\langle \psi^i\epsilon\mathcal{O}^j \rangle$	$(-1)^{j-\frac{1}{2}}\delta^{ij}[\frac{1}{2}, 0, -\frac{1}{2}]^-$

OPE	$\mathcal{O} \in (l, P, \mu)$	$\langle \psi^i\psi^j\mathcal{O}^a \rangle$ Structures
$\psi \times \psi$	$(l \in 2\mathbb{Z}, \text{ even}, \mu \in \{\bullet, \square\square\})$	$T^{ija}_\mu[\frac{1}{2}, -\frac{1}{2}, 0]^+$
		$T^{ija}_\mu[\frac{1}{2}, \frac{1}{2}, -1]^+$
	$(l \in 2\mathbb{Z}+1, \text{ even}, \mu = \begin{smallmatrix}\square\\\square\end{smallmatrix})$	$T^{ija}_\mu[\frac{1}{2}, -\frac{1}{2}, 0]^+$
		$T^{ija}_\mu[\frac{1}{2}, \frac{1}{2}, -1]^+$
	$(l \in 2\mathbb{Z}, \text{ odd}, \mu \in \{\bullet, \square\square\})$	$T^{ija}_\mu[\frac{1}{2}, -\frac{1}{2}, 0]^-$
	$(l \in 2\mathbb{Z}+1, \text{ odd}, \mu \in \{\bullet, \square\square\})$	$T^{ija}_\mu[\frac{1}{2}, \frac{1}{2}, -1]^-$
	$(l \in (2\mathbb{Z})_{\geq 2}, \text{ odd}, \mu = \begin{smallmatrix}\square\\\square\end{smallmatrix})$	$T^{ija}_\mu[\frac{1}{2}, \frac{1}{2}, -1]^-$
	$(l \in 2\mathbb{Z}+1, \text{ odd}, \mu = \begin{smallmatrix}\square\\\square\end{smallmatrix})$	$T^{ija}_\mu[\frac{1}{2}, -\frac{1}{2}, 0]^-$

Four-point Structures

Conformal structures with nice symmetry properties,

$$\langle \psi\psi|\psi\psi\rangle = -\langle \psi\psi|\psi\psi\rangle,$$

$$(-1)^{\sum_i j_i - q_i} \Rightarrow 8 \text{ parity even structures.}$$

$$[\uparrow\uparrow\uparrow\uparrow], [\downarrow\downarrow\downarrow\downarrow], [\uparrow\uparrow\downarrow\downarrow], [\downarrow\downarrow\uparrow\uparrow], [\uparrow\downarrow\uparrow\downarrow], [\downarrow\uparrow\downarrow\uparrow], [\downarrow\uparrow\uparrow\downarrow], [\uparrow\downarrow\downarrow\uparrow]$$

Invariant under (12)(34), (13)(24), (14)(23),

$$\langle\uparrow\uparrow\uparrow\uparrow\rangle, \langle\downarrow\downarrow\downarrow\downarrow\rangle, \langle\uparrow\uparrow\downarrow\downarrow\rangle, \langle\uparrow\downarrow\uparrow\downarrow\rangle, \langle\downarrow\uparrow\uparrow\downarrow\rangle.$$

t-parity or parity under $z \to \bar{z}$ (in this case $[\uparrow] \leftrightarrow [\downarrow]$)

$$\langle\uparrow\uparrow\uparrow\uparrow\rangle^+, \langle\uparrow\uparrow\uparrow\uparrow\rangle^-, \langle\uparrow\uparrow\downarrow\downarrow\rangle^+, \langle\uparrow\downarrow\uparrow\downarrow\rangle^+, \langle\downarrow\uparrow\uparrow\downarrow\rangle^+.$$

Four-point Structures

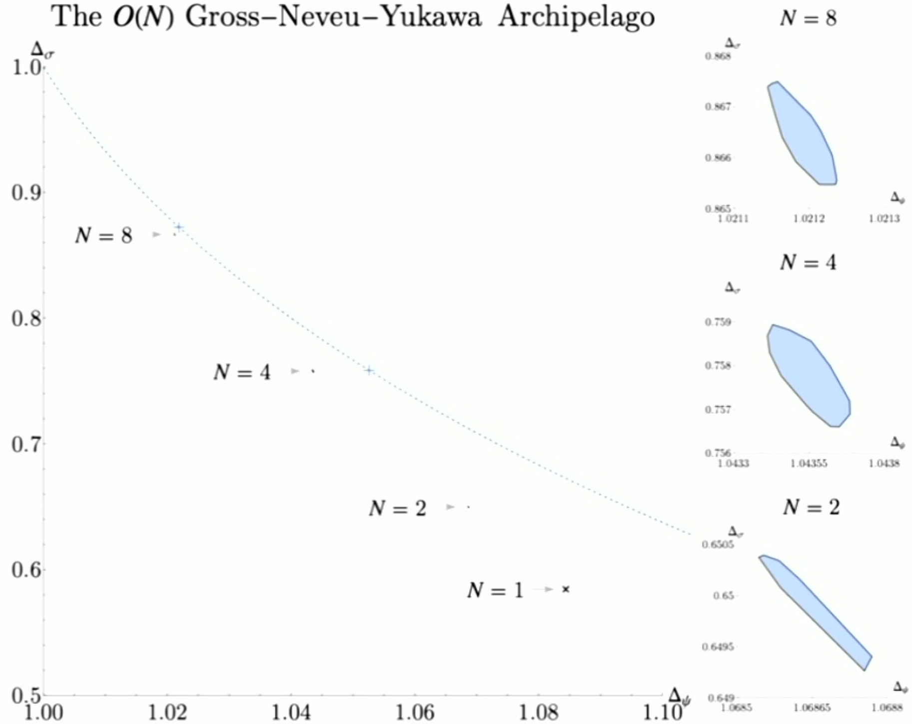
Conformal structures with nice symmetry properties,

$$\langle \epsilon\epsilon | \psi^i \psi^j \rangle = \langle \psi^i \epsilon | \epsilon \psi^j \rangle,$$
$$\langle \psi^i \epsilon | \psi^j \epsilon \rangle = -\langle \psi^j \epsilon | \psi^i \epsilon \rangle$$
$$\langle \sigma\epsilon | \psi^i \psi^j \rangle = \langle \psi^i \epsilon | \sigma \psi^j \rangle,$$
$$\langle \psi^i \sigma | \psi^j \epsilon \rangle = -\langle \psi^j \sigma | \psi^i \epsilon \rangle.$$

Parity odd structures $[\uparrow\downarrow]$, $[\downarrow\uparrow]$.
Definite t-parity structures,

$$[\uparrow\downarrow]^{\pm} = [\uparrow\downarrow] \mp [\downarrow\uparrow]$$

Results: Archipelago



Results: Archipelago